Combinatorics, D Term, 2008
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April 1, 2008

Combinatorics Assignment 4

Due Date: Monday, April 7, 4pm. Place in my mail slot in Room SH108.

N.B. Please keep in mind Dr. Martin's rules for Combinatorics assignments.

Please complete the following five problems:

1. Use generating functions to determine the number of positive integers less than a million whose digits sum to 23.
2. Problem 9 on page 41.
3. In the real plane, consider paths from $(0,0)$ to the line $x = n-1$ in which each step is either $U = (1,1)$, $D = (1,-1)$ or $H = (2,0)$. [For instance, the "word" $HUUD$ starts at $(0,0)$ and ends at $(5,1)$ via $(2,0),(3,1),(4,2)$.] Let b_n be the number of such paths.

   (a) By elementary means, determine b_0, b_1, b_2, b_3.

   (b) Find integers r and s for which the linear recurrence

   $$b_n = rb_{n-1} + sb_{n-2}$$

   holds for all $n \geq 2$.
4. The *Pell numbers* are defined by the linear recurrence relation

   $$P_n = 2P_{n-1} + P_{n-2}$$

   for $n \geq 2$ with initial conditions $P_0 = 0$, $P_1 = 1$. Find an expression for the generating function $\sum_n P_n x^n$ as a rational function. (As always, show your work.)
5. Problem 3 on page 46.

   10 BONUS POINTS (optional):
6. The *triangular numbers* T_n satisfy the recurrence relation

   $$T_n = T_{n-1} + n$$

   for all $n \geq 1$ with initial condition $T_0 = 0$. Of course, we know that $T_n = \binom{n+1}{2}$ from class, but here we wish to find a generating function for T_n using the recurrence. Be sure to briefly explain the steps of your solution.