Combinatorics, D Term, 2008
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Combinatorics Assignment 3

DUE DATE: Friday, March 28, 4pm. Place in my mail slot in Room SH108.

N.B. Please keep in mind Dr. Martin's rules for Combinatorics assignments.

Please complete the following five problems:

1. Let k and n be positive integers.

   (a) Find an expression, involving one simple summation, for the coefficient of x^n in the generating function $(1-3x)^{-4}(1+x)^{-k}$.

   (b) Find the exact value in part (a) for $n=4$, $k=2$.

2. Using MAPLE's command `sum( x^n, n=0..infinity);` we learn that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, a rational function. Use MAPLE to determine which of the following series is represented by a rational function. In each case, write down the simplified form given by MAPLE. **(a)** $\sum_{n=0}^{\infty} 2n\, x^{2n}$ **(b)** $\sum_{n=0}^{\infty} n^2\, x^{2n}$ **(c)** $\sum_{n=0}^{\infty} x^{n^2}$ **(d)** $\sum_{n=0}^{\infty} x^{\frac{1}{2}n}$

3. Here, we compute antiderivatives of the quantities appearing in the Binomial Theorem.
   (a) Show that, for a positive integer n,
   $$\frac{1}{n+1}(1+x)^{n+1} = \frac{1}{n+1} + \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} x^{k+1};$$
   (b) Deduce that
   $$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}.$$

4. Let k be a non-negative integer.
   (a) Expand $(1-x)^{-1}(1-x)^{-(k+1)}$ as a double sum using the Binomial Theorem.
   (b) Use part (a) to prove that $\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$.

5. In this problem, we count non-negative integer solutions to the equation
   $$x_1 + \cdots + x_r + 2x_{r+1} + \cdots + 2x_{r+s} = n$$
   where r, s and n are non-negative integers.
   (a) Express the total number of solutions as a simple sum[1] of products of binomial coefficients.
   (b) Use MAPLE to extract the coefficient in part (a) when $r=s=n=10$. (I.e., the answer here is just an integer.)

[1] The index of summation should be over a simple contiguous sequence of integers