Combinatorics Assignment 2

DUE DATE: Monday, March 24, 4pm. Place in my mail slot in Room SH108.

N.B. Please keep in mind Dr. Martin’s rules for Combinatorics assignments, reproduced for your convenience on the back of this page.

Please complete the following ten problems:

1. Let \( k \) and \( n \) be positive integers.
   
   (a) Find integers \( a, b, \) and \( c \) such that
   
   \[ k^3 = a \binom{k}{3} + b \binom{k}{2} + c \binom{k}{1}. \]

   (b) Using part (a) and Theorem 6.16 in the white notes, find a formula for \( 1^3 + 2^3 + \cdots + n^3. \)

2. For \( n \) a positive integer, we have

   \[ \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}. \]

   (a) First prove this using the Binomial Theorem.

   (b) Now give a combinatorial proof. [HINT: Each \( n \)-element subset of \( \{1, 2, \ldots, 2n\} \) is the union of a \( k \)-element subset of \( \{1, 2, \ldots, n\} \) and an \( (n-k) \)-element subset of \( \{n+1, n+2, \ldots, 2n\} \) for some \( k, 0 \leq k \leq n \). Partition this set according to the value of \( k \) and apply the Addition Principle.]

3. Let \( S \) denote the set of all subsets of \( A = \{1, 2, 3, 4, 5\} \). For \( \sigma \) a subset of \( A \), define the weight of \( \sigma \) to be \( w(\sigma) = 0 \) if \( \sigma \) is empty and \( w(\sigma) = \max \sigma \) (the largest element of \( \sigma \)) otherwise. Write down the generating function \( \Phi_S(x) \) of \( S \) with respect to this weight function. Explain briefly.

4. Repeat the above exercise for weight function \( w(\sigma) = \max \sigma - \min \sigma \) when \( \sigma \) is non-empty (and \( w(\emptyset) = 0 \) as above).

5. Suppose two ordinary dice, one red and one green, are rolled. Then we obtain 36 possible outcomes, denoted by \( S = \{(a, b) \mid 1 \leq a, b \leq 6\} \). For \( \sigma = (a, b) \) in \( S \), let \( w(\sigma) = |a - b| \).

   (a) Construct the generating function \( \Phi_S(x) \) for \( S \) with respect to weight function \( w; \)

   (b) Using Theorem 1.2.3, find the average value of \( |a - b| \) over all 36 dice rolls.
6. Make a list of all four-letter “words” that can be formed from the “alphabet” \{x, y\}. If the weight of a word is defined to be the number of occurrences of “yy” in the word, find the generating function for this set of 16 words. (For example, the word \(\sigma = yyyx\) has weight two since the substring “yy” occurs starting in the first position and also starting in the second position.)

7. In this problem, we consider a two-variable generating function. For a positive integer \(n\), the “odd part” of \(n\) is the largest odd integer dividing \(n\). Clearly each positive integer \(n\) can be uniquely expressed as \(n = 2^k m\) where \(k \geq 0\) and \(m\) is the odd part of \(n\). For each such \(n\), let \(\alpha(n) = k\) and let \(\beta(n) = m\).

(a) Write down the first twenty terms of the generating function

\[
\Phi_N(x, y) = \sum_{n \geq 1} x^{\alpha(n)} y^{\beta(n)}.
\]

(b) This generating function factors into a product \(\Psi(x)\Upsilon(y)\). Find \(\Psi\) and \(\Upsilon\), or at least give the first ten terms of each.

8. Find the first six terms, \(a_0 + a_1 x + \cdots + a_5 x^5\) of the inverse of the generating function of \(\Phi(x) = 1 - x + x^3\).

9. Repeat the previous exercise for \(\Phi(x) = 1 + x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \cdots\).

10. Let \(A\) and \(B\) be sets of configurations with weight functions \(\alpha\) and \(\beta\), respectively. Now define a weight function \(w\) on \(A \times B\) by

\[
w(\sigma) = 2\alpha(a) + \beta(b)
\]

where \(\sigma = (a, b)\) is any element of \(A \times B\). Prove that, with respect to this \(w\), we have

\[
\Phi_{A \times B}(x) = \Phi_A(x^2)\Phi_B(x).
\]

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**Professor Martin’s Rules for Combinatorics Assignments:**

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) **Use only one side of each sheet of paper.** Write legibly using correct English;

III) Show your work. Almost all solutions will have at least two English sentences: one introducing the problem and one summarizing the solution. In all cases, use FULL SENTENCES;

IV) Late assignments will not be accepted for credit;

V) Use a staple when you submit more than one sheet and want them all back. There is a stapler for public use in the Mathematical Sciences Department Office (SH108).