Combinatorics
D Term, 2008
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Combinatorics Assignment 2

DUE DATE: Monday, March 24, 4pm. Place in my mail slot in Room SH108.

N.B. Please keep n mind Dr. Martin's rules for Combinatorics assignments, reproduced for your convenience on the back of this page.

Please complete the following ten problems:

1. Let k and n be positive integers.

   (a) Find integers a, b and c such that
   $$k^3 = a\binom{k}{3} + b\binom{k}{2} + c\binom{k}{1}.$$

   (b) Using part (a) and Theorem 6.16 in the white notes, find a formula for $1^3 + 2^3 + \cdots + n^3$.

2. For n a positive integer, we have
   $$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

   (a) First prove this using the Binomial Theorem.

   (b) Now give a combinatorial proof. *[HINT: Each n-element subset of $\{1, 2, \ldots, 2n\}$ is the union of a k-element subset of $\{1, 2, \ldots, n\}$ and an $(n-k)$-element subset of $\{n+1, n+2, \ldots, 2n\}$ for some k, $0 \leq k \leq n$. Partition this set according to the value of k and apply the Addition Principle.]*

3. Let S denote the set of all subsets of $A = \{1, 2, 3, 4, 5\}$. For σ a subset of A, define the *weight* of σ to be $w(\sigma) = 0$ if σ is empty and $w(\sigma) = \max \sigma$ (the largest element of σ) otherwise. Write down the generating function $\Phi_S(x)$ of S with respect to this weight function. Explain briefly.

4. Repeat the above exercise for weight function $w(\sigma) = \max \sigma - \min \sigma$ when σ is non-empty (and $w(\emptyset) = 0$ as above).

5. Suppose two ordinary dice, one red and one green, are rolled. Then we obtain 36 possible outcomes, denoted by $S = \{(a, b) \mid 1 \leq a, b \leq 6\}$. For $\sigma = (a, b)$ in S, let $w(\sigma) = |a - b|$.

   (a) Construct the generating function $\Phi_S(x)$ for S with respect to weight function w;

   (b) Using Theorem 1.2.3, find the average value of $|a - b|$ over all 36 dice rolls.

6. Make a list of all four-letter "words" that can be formed from the "alphabet" $\{x, y\}$. If the weight of a word is defined to be the number of occurrences of "yy" in the word, find the generating function for this set of 16 words. (For example, the word $\sigma = yyyx$ has weight two since the substring "yy" occurs starting in the first position and also starting in the second position.)

7. In this problem, we consider a two-variable generating function. For a positive integer n, the "odd part" of n is the largest odd integer dividing n. Clearly each positive integer n can be uniquely expressed as $n = 2^k m$ where $k \geq 0$ and m is the odd part of n. For each such n, let $\alpha(n) = k$ and let $\beta(n) = m$.

   (a) Write down the first twenty terms of the generating function

$$\Phi_{\mathbb{N}}(x, y) = \sum_{n \geq 1} x^{\alpha(n)} y^{\beta(n)}.$$

   (b) This generating function factors into a product $\Psi(x)\Upsilon(y)$. Find Ψ and Υ, or at least give the first ten terms of each.

8. Find the first six terms, $a_0 + a_1 x + \cdots + a_5 x^5$ of the **inverse** of the generating function of $\Phi(x) = 1 - x + x^3$.

9. Repeat the previous exercise for $\Phi(x) = 1 + x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \cdots$.

10. Let A and B be sets of configurations with weight functions α and β, respectively. Now define a weight function w on $A \times B$ by

$$w(\sigma) = 2\alpha(a) + \beta(b)$$

where $\sigma = (a, b)$ is any element of $A \times B$. Prove that, with respect to this w, we have

$$\Phi_{A \times B}(x) = \Phi_A(x^2)\Phi_B(x).$$

Professor Martin's Rules for Combinatorics Assignments:

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) **Use only one side of each sheet of paper**. Write legibly using correct English;

III) Show your work. Almost all solutions will have at least two English sentences: one introducing the problem and one summarizing the solution. In all cases, use FULL SENTENCES;

IV) Late assignments will not be accepted for credit;

V) Use a staple when you submit more than one sheet and want them all back. There is a stapler for public use in the Mathematical Sciences Department Office (SH108).