Combinatorics
D Term, 2008
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Combinatorics Assignment 1

DUE DATE: Monday, March 17, 4pm. Place in my mail slot in Room SH108.

N.B. Typically, no late assignments will be accepted for credit. Sample solutions will often be available soon after the deadline passes. After this, any work not yet turned in will receive a grade of zero.

PROFESSOR MARTIN'S RULES FOR COMBINATORICS ASSIGNMENTS:

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) Use only one side of each sheet of paper. Write legibly using correct English;

III) Show your work. Almost all solutions will have at least two English sentences: one introducing the problem and one summarizing the solution. In all cases, use FULL SENTENCES;

IV) Use a staple when you submit more than one sheet and want them all back. There is a stapler for public use in the Mathematical Sciences Department Office (SH108).

Please complete the following ten problems:

1. In honor of π Day, please find (e.g., from Wikipedia) the hypervolume of an n-dimensional sphere of radius r (i.e., the n-dimensional volume of its interior) and its $(n-1)$-dimensional surface volume, for $n \leq 8$.

   Now do the same for the n-dimensional cube of side length s, by noting that the differentiation trick only works if we write the hypervolume in terms of the "radius" $r = \frac{1}{2}s$. For example, in two-dimensional space, the cube has volume (i.e., area) $s^2 = 4r^2$ and surface measure (i.e., perimeter) $8r$.

   Your solution to this problem will be a table with eight rows (dimensions 1–8) and four columns (measure of the interior, and then the surface, of both the ball and the cube).

2. How many license plates of the form $\ell\ell\#\#\#\#$ are there with exactly one '9'?

3. A *lattice point* in the plane is a point with integer coordinates. Five distinct points in the plane naturally determine 10 line segments. Prove that, no matter how five lattice points are chosen in the plane, one of these line segments has its midpoint at another lattice point.

4. In any permutation of $\{1,\ldots,10\}$ there exists either an increasing subsequence of at least four numbers or a decreasing subsequence of at least four numbers (not necessarily appearing consecutively, in either case). [For example, the permutation 1 7 3 4 10 8 9 2 6 5 has 13489 as an increasing subsequence and 10 965 as a decreasing subsequence.]

5. In any subset of 52 numbers from $\{1,\ldots,100\}$ there exists a pair which differ by exactly three. (BONUS: What happens with 51 numbers?)

6. Each of 16 students has a positive whole number of dollars and together the students have 30 dollars. For each n, $1 \le n \le 29$, show that some students together have n dollars.

7. For which positive integers k is the binomial coefficient $\binom{\pi^2}{k}$ greater than zero?

8. If 101 points are chosen at random inside a square of side length one, then some two of these points are at most distance .15 apart. (BONUS: Prove the same result for 100 points.)

9. If 25 points are chosen at random inside a circle of radius $\sqrt{3}$, then some two of these are at a distance at most one apart.

10. Flying back from Toronto to Providence this weekend, I got caught in a snowstorm. The trip back involved three airports and five boarding passes over two days. (I'll leave out the part about the rental car.) The problem is to determine how many stories can be told involving three airports, five boarding passes and two days. But step one is to determine the precise statement of the problem. So your homework for Thursday is to prepare questions for me (to be asked in class) which will make the problem answerable in a well-defined way. Then you'll need to complete the enumeration for Friday.