

### Sample Solutions – Assignment 6

1.) [Problem 7 on page 554.]

Solution: This is modelled by an M/M/1 queue with arrival rate  $\lambda = 2.5$  and  $\mu = 5$  (cars per hour).

- (a) Average number of cars in the system:  $L = 1$
- (b) Average time a car spends waiting for service:  $W_q = .20$  hours (12 minutes)
- (c) Average time a car spends in the system:  $W = W_q + 1/\mu = .40$  hours (24 minutes)
- (d) Probability an arriving car must wait for service:  $P_w = \lambda/\mu = .5000$  (so there is a 50/50 chance that an arriving customer must wait)

2.) [Problem 9 on p554 with  $\lambda = 2.8$ ]

Solution: This is modelled by an M/M/1 queue with arrival rate  $\lambda = 2.8$  and  $\mu = 5$  (customers per hour).

- (a) Probability that no units are in the system:  $P_0 = 0.44$
- (b) Probability of one unit in the system:  $P_1 = \frac{\lambda}{\mu} P_0 = 0.246$
- (c) Probability of two units in the system:  $P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 = 0.137$
- (d) Probability of three units in the system:  $P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0 = 0.077$
- (e) Probability of four or more units in the system:

$$\sum_{n=4}^{\infty} P_n = 1 - P_0 - P_1 - P_2 - P_3 = 0.098$$

So there is roughly a ten percent chance that more than two customers are waiting.

- (f) Average time a customer spends in the queue:  $W_q = .25$  hours (15 minutes)

3.) [Problem 22 on p556-7] Each situation is modelled by either an M/M/1 queue or an M/M/2 queue.

Here are the alternatives:

**A: Single Channel, one employee:**  $\lambda = 0.4$  cars per min. and  $\mu = 0.5$  cars per min.

**B: Single Channel, two employees:**  $\lambda = 0.4$  cars per min. and  $\mu = 0.8$  cars per min.

**C: Two Channels:**  $\lambda = 0.4$  cars per min. (overall) and  $\mu = 0.5$  cars per min. for each of the  $k = 2$  channels.

**NOTE:** Since our chosen unit of time is “minute”, our answers to parts (d) and (e) will differ from those in the back of the book, where the units is “hour”.

	A	B	C
(a)	$P_0 = 1 - \frac{\lambda}{\mu} = 0.2$	$1 - \frac{\lambda}{\mu} = 0.5$	$\frac{2\mu - \lambda}{2\mu + \lambda} = 0.43$
(b)	$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = 3.2$	$\frac{\lambda^2}{\mu(\mu - \lambda)} = 0.5$	$\frac{\lambda^3}{\mu(2\mu - \lambda)(2\mu + \lambda)} = 0.15$
(c)	$L = L_q + \frac{\lambda}{\mu} = 4.0$	$L_q + \frac{\lambda}{\mu} = 1.0$	$L_q + \frac{\lambda}{\mu} = 0.95$
(d)	$W_q = L_q / \lambda = 8.0$	$L_q / \lambda = 1.25$	$L_q / \lambda = 0.38$
(e)	$W = W_q + \frac{1}{\mu} = 10.0$	$W_q + \frac{1}{\mu} = 2.5$	$W_q + \frac{1}{\mu} = 2.38$
(f)	$P_w = \frac{\lambda}{\mu} = 0.8$	$\frac{\lambda}{\mu} = 0.5$	$\frac{\lambda^2}{\mu(2\mu + \lambda)} = 0.23$

In every measure except the first, the two-channel model performs best. So, if we can afford it, we should choose the two-channel model.

4.) [Problem 23 on p557]

Solution: We apply the cost model

$$TC = c_w L + c_s k$$

to each of the above three alternatives, where  $TC$  is total cost,  $c_w$  is the cost (per hour) for each waiting customer,  $L$  is the number of customers in the system,  $c_s$  is the cost per hour to operate each channel and  $k$  is the number of channels.

**A:** We have  $k = 1$ ,  $c_w = 25$ ,  $L = 4$ ,  $c_s = 26.5$ , giving

$$TC = 25 \times 4 + 26.5 \times 1 = \$126.50$$

per hour for the first option.

**B:** We have  $k = 1$ ,  $c_w = 25$ ,  $L = 1$ ,  $c_s = 33$ , giving

$$TC = 25 \times 1 + 33 \times 1 = \$58.00$$

per hour for the second option.

**C:** We have  $k = 2$ ,  $c_w = 25$ ,  $L = 0.95$ ,  $c_s = 26.5$ , giving

$$TC = 25 \times 0.95 + 26.5 \times 2 = \$76.75$$

per hour for the two-channel option.

So the lowest-cost option, under these assumptions, is Alternative B: a single channel with two workers.

5.) [Problem 29 on p558 with 3.5 jobs per day] This is an M/G/1 queue (p544).

(a) The arrival rate is  $\lambda = 0.4375$  jobs per hour, and the service rate is  $\mu = 0.5$  jobs per hour with a standard deviation of  $\sigma = 1.5$  hours.

(b) Operating characteristics:

$P_0 = 1 - \frac{\lambda}{\mu} = 0.125$  is the probability that there are no jobs in the system.

$L_q = \frac{\lambda^2 \sigma^2 + (\lambda/\mu)^2}{2P_0} = 4.785$  jobs in the queue, on average.

$L = L_q + \frac{\lambda}{\mu} = 5.66$  jobs in the system, on average.

$W_q = L_q/\lambda = 10.94$  hours is the average time a job spends in the queue.

$W = W_q + \frac{1}{\mu} = 12.94$  hours is the average time a job spends in the system.

$P_w = \frac{\lambda}{\mu} = .875$  is the probability that an arriving job must wait.

With  $c_w = 35$  and  $c_s = 28$  as given, the total cost per hour is

$$TC = c_w L + c_s k = 35 \times 5.66 + 28 \times 1 = \$226.10$$

(c) We now look at a revised model with all parameters the same except that the standard deviation  $\sigma$  is reduced to zero and the cost  $c_s$  of operation is increased to \$32 per hour.

Operating characteristics:

$P_0 = 1 - \frac{\lambda}{\mu} = 0.125$  is still the probability that there are no jobs in the system.

Now only  $L_q = \left(\frac{\lambda}{\mu}\right)^2 / (2P_0) = 3.0625$  jobs in the queue, on average.

And only  $L = L_q + \frac{\lambda}{\mu} = 3.9375$  total jobs in the system, on average.

$W_q = L_q/\lambda = 7$  hours is the new average time a job spends in the queue.

$W = W_q + \frac{1}{\mu} = 9$  hours is the new average time a job spends in the system.

But  $P_w = \frac{\lambda}{\mu} = .875$  is still the probability that an arriving job must wait.

With  $c_w = 35$  and  $c_s$  increased to 32, the total cost per hour is

$$TC = c_w L + c_s k = 35 \times 3.9375 + 32 \times 1 = \$169.81$$

So the new equipment will reduce overall cost by \$56.29 per hour. Most of the waiting line characteristics improve with this simple reduction of uncertainty (except  $P_0$  and  $P_w$  which did not depend on  $\sigma$ ). For example, the total time a job spends in the system is reduced by over 30%.

(d) With this cost model, the new system will save the company \$2,251.50 in a single work week. If the new equipment has a half-life of one year and costs less than \$112,575 (assuming 50 weeks of shop operation per year), then it seems economically viable to purchase the new system.