Math. Methods in Decision Making
D Term, Sections D01-D02
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Sample Solutions – Assignment 2

1.) (a) First we construct the dual of the following linear program:

$$\begin{array}{rl} \textbf{maximize} & 3x_1 + 2x_2 + 6x_3 \\ \textbf{subject to} & x_1 + x_2 + 2x_3 \le 4 \\ & 2x_1 + 3x_3 \le 7 \\ & 2x_1 + x_2 + 3x_3 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

Multiplying the first constraint by $y_1 \ge 0$, multiplying the second constraint by $y_2 \ge 0$ and multiplying the third constraint by $y_3 \ge 0$ and summing these inequalities, we obtain an upper bound on the objective value provided the coefficents in this expression are no less than the objective coefficients of the corresponding variables. Our problem becomes:

$$\begin{array}{rl} \textbf{minimize} & 4y_1 + 7y_2 + 5y_3 \\ \textbf{subject to} & y_1 + 2y_2 + 2y_3 \ge 3 \\ & y_1 + y_3 \ge 2 \\ & 2y_1 + 3y_2 + 3y_3 \ge 6 \\ & y_1, y_2, y_3 \ge 0 \end{array}$$

(b) Exhibit a basic feasible solution to this dual LP.

In Assignment 1, we solved the primal LP by the simplex method. We know that the final tableau for that problem contains – in the reduced cost row – a basic optimal solution to the dual LP. Here's that tableau:

```
     x1       x2       x3       x4       x5       x6   |    b
-------------------------------------------------------|---------
   -1.00     1.00     0.00     3.00     0.00    -2.00  |   2.00
   -1.00     0.00     0.00     3.00     1.00    -3.00  |   4.00
    1.00     0.00     1.00    -1.00     0.00     1.00  |   1.00
                                                       |
-------------------------------------------------------|---------
   -1.00     0.00     0.00     0.00     0.00    -2.00  | -10.00
```

So a basic feasible solution for the dual is $(y_1, y_2, y_3) = (0, 0, 2)$. We extend this by finding the values of the surplus variables y_4, y_5 and y_6. The full solution is then

$$y^* = (0, 0, 2, 0, 0, 0)$$

since all constraints in the dual are "tight". [Another basic feasible solution is $y = (3, 0, 0, 0, 1, 0)$.

(c) Exhibit a feasible solution to this dual LP which is <u>not</u> basic.

Most solutions are not basic, but how to guarantee that the solution we find is not basic? We note that every basic feasible solution is by setting some (three in this case) variables equal to zero and the others are determined by the right-hand side values in the tableau. So the following solution

$$y = (3, 0, 2, 4, 3, 6)$$

(which is easily verified to be feasible) cannot be basic since it has only one non-zero entry.

2.) *[10 points]* At the Dorf Motor Company, painting of automobile bodies is carried out by teams of two or three workers. A study of productivity measures over a two-month period yields the following daily statistics.

Team	A	B	C	D
Standard	22.3	10.7	15.1	8.7
Custom	5.1	0.2	2.4	8.1
Fails	0.2	3.3	1.9	1.3
Wages	\$960	\$480	\$700	\$640

We must determine if any of the teams is less efficient than the others using Data Envelopment Analysis.

First consider Team A. They produce more standard paint jobs than any other team. So there is no way that a combination of the other teams can outperform Team A in all categories.

A similar idea rules out Team D. They produce more custom paint jobs than any other team. So it is impossible for a convex combination of the other teams to outperform Team D in all categories.

Next look at Team B. Their wages are lower than any other team. So there is no way that a combination of the other teams can outperform Team B in all categories.

So the only case that requires a computer is Team C. Data Envelopment Analysis in this case requires us to solve the following linear program:

$$\begin{array}{llllll}
\textbf{minimize} & E & & & & \\
\textbf{subject to} & & & & & \\
 & & 22.3x_A & +10.7x_B & +8.7x_D & \\
 & & 5.1x_A & +0.2x_B & +8.1x_D & \\
 & -1.9E & +0.2x_A & +3.3x_B & +1.3x_D & \\
 & -700E & +960x_A & +480x_B & +640x_D & \\
 & & x_A & +x_B & +x_D & = \\
E, & & x_A, & x_B, & x_D & \geq
\end{array}$$

The initial set-up — with two surplus variables and two slack variables — is unfortunately not a tableau:

```
    E       xA       xB       xD       x5       x6       x7       x8   |    b
---------------------------------------------------------------------|---------
  0.00   22.30    10.70     8.70    -1.00     0.00     0.00     0.00 |  15.10
```

0.00	5.10	.20	8.10	0.00	-1.00	0.00	0.00	|	2.40
-1.90	.20	3.30	1.30	0.00	0.00	1.00	0.00	|	0.00
-700.00	960.00	480.00	640.00	0.00	0.00	0.00	1.00	|	0.00
0.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	|	1.00
-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	|	0.00

where maximizing $-E$ is equivalent to minimizing E. Now, since we don't have the Management Scientist here, we must first find a tableau, then a feasible tableau, then pivot to optimality. The tableaus are, in turn,

E	xA	xB	xD	x5	x6	x7	x8	b
0.00	0.00	11.60	13.60	1.00	0.00	0.00	0.00	7.20
0.00	0.00	4.90	-3.00	0.00	1.00	0.00	0.00	2.70
-1.90	0.00	3.10	1.10	0.00	0.00	1.00	0.00	-.20
-700.00	0.00	-480.00	-320.00	0.00	0.00	0.00	1.00	-960.00
0.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	1.00
-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

E	xA	xB	xD	x5	x6	x7	x8	b
0.00	0.00	11.60	13.60	1.00	0.00	0.00	0.00	7.20
0.00	0.00	4.90	-3.00	0.00	1.00	0.00	0.00	2.70
0.00	0.00	4.40	1.96	0.00	0.00	1.00	-.00	2.40
1.00	0.00	.68	.45	0.00	0.00	0.00	-.00	1.37
0.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	1.00
0.00	0.00	.68	.45	0.00	0.00	0.00	-.00	1.37

E	xA	xB	xD	x5	x6	x7	x8	b
0.00	0.00	0.00	1.00	.11	0.00	-.31	.00	.10
0.00	0.00	0.00	0.00	.61	1.00	-2.73	.00	.55
0.00	0.00	1.00	0.00	-.05	0.00	.36	-.00	.50
1.00	0.00	0.00	0.00	-.01	0.00	-.10	-.00	.98
0.00	1.00	0.00	0.00	-.06	0.00	-.05	.00	.39
0.00	0.00	0.00	0.00	-.01	0.00	-.10	-.00	.98

with optimal objective value $E^* = 0.98133$. So Team C is judged to be inefficient by this method, with an efficiency rating of 98% of that of the other teams.

3.) Problem 11 on page 289.

The optimal tableau is given as

```
    x1       x2       x3       x4       x5    |     b
------------------------------------------|---------
    0.00     1.00     3.33     0.00    -2.22 |  20.00
    0.00     0.00     -.66     1.00      .44 |   1.00
    1.00     0.00    -1.66     0.00     2.77 |  25.00
------------------------------------------|---------
    0.00     0.00   -33.33     0.00   -44.44 | -1600.00
```

(a) The range of optimality for c_1 is obtained by first replacing the first entry in the bottom row by t, eliminating this by subtracting t times row three from this and determining what conditions on t guarantee that all reduced costs are still non-positive. We get

```
    x1       x2           x3           x4           x5          |     b
----------------------------------------------------------------|---------
    0.00     1.00         3.33         0.00        -2.22        |  20.00
    0.00     0.00         -.66         1.00          .44        |   1.00
    1.00     0.00        -1.66         0.00         2.77        |  25.00
----------------------------------------------------------------|---------
    0.00     0.00      (5t-100)/3      0.00     -(400+25t)/9 | -(1600+25t)
```

forcing t to lie between -16 and 20. So the range of optimality for c_1 (currently at 40) is

$$24 \leq c_1 \leq 60.$$

Similarly, to study c_2, we go back to the optimal tableau, change the second entry in the bottom row to t and eliminate to get

```
    x1       x2           x3           x4           x5          |     b
----------------------------------------------------------------|---------
    0.00     1.00         3.33         0.00        -2.22        |  20.00
    0.00     0.00         -.66         1.00          .44        |   1.00
    1.00     0.00        -1.66         0.00         2.77        |  25.00
----------------------------------------------------------------|---------
    0.00     0.00    -(100+10t)/3      0.00      (20t-400)/9 | -(1600+20t)
```

so that $-10 \leq t \leq 20$ and

$$20 \leq c_2 \leq 50.$$

(b) If c_1 is reduced to 30, the above analysis shows that the current basic feasible solution is still optimal. The profit reduces to $1600 + 25t$ where $t = -10$; i.e., profit is only \$1350.

(c) The dual price for the first constraint is 100/3. This means that we should be willing to pay up to \$33.33 per unit for extra quantities of this material (up to some limit). If the price is higher than that, our net profit would go down and we should **not** purchase; if we can get Material 1 at less than \$33.33 per unit, we should buy it since net profit will increase.

(d) Material 3 is most valuable to us at this point, with dual price 400/9. So, if all three materials were available at equal prices, this is the one we would choose. We should be willing to pay any amount less than \$44.44 per unit.

4.) Problem 13 on page 289.

Initial system, with one surplus variable and one slack variable:

x1	x2	x3	x4	x5	x6	b
3.00	1.00	2.00	0.00	0.00	0.00	30.00
2.00	1.00	3.00	1.00	-1.00	0.00	15.00
0.00	2.00	0.00	3.00	0.00	1.00	25.00
3.00	1.00	5.00	3.00	0.00	0.00	0.00

To get a feasible tableau, pivot in row 2, column 5 and then row 1 column 1:

x1	x2	x3	x4	x5	x6	b
1.00	.33	.66	0.00	0.00	0.00	10.00
0.00	-.33	-1.66	-1.00	1.00	0.00	5.00
0.00	2.00	0.00	3.00	0.00	1.00	25.00
0.00	0.00	3.00	3.00	0.00	0.00	-30.00

Now pivot as usual to optimality:

x1	x2	x3	x4	x5	x6	b
1.50	.50	1.00	0.00	0.00	0.00	15.00
2.50	1.16	0.00	0.00	1.00	.33	38.33
0.00	.66	0.00	1.00	0.00	.33	8.33
-4.50	-3.50	0.00	0.00	0.00	-1.00	-100.00

(a) Optimal basis: $\{x_3, x_5, x_4\}$ Optimal solution

$$x^* = (0, 0, 15, 25/3, 115/3, 0).$$

(b) Range of optimality for c_3 turns out to be $c_3 \geq 2$. (Insert t, eliminate to find $t \geq -3$.)

(c) If c_3 is changed from 5 down to 1, then $t = -4$ and the current basis is no longer optimal. Beginning with

```
     x1       x2       x3       x4       x5       x6    |     b
--------------------------------------------------------|---------
     1.50      .50     1.00     0.00     0.00     0.00  |  15.00
     2.50     1.16     0.00     0.00     1.00      .33  |  38.33
     0.00      .66     0.00     1.00     0.00      .33  |   8.33
--------------------------------------------------------|---------
    -4.50    -3.50   (-4.00)    0.00     0.00    -1.00  | -100.00
```

we “fix” this diagram so that it is a tableau again:

```
     x1       x2       x3       x4       x5       x6    |     b
--------------------------------------------------------|---------
     1.50      .50     1.00     0.00     0.00     0.00  |  15.00
     2.50     1.16     0.00     0.00     1.00      .33  |  38.33
     0.00      .66     0.00     1.00     0.00      .33  |   8.33
--------------------------------------------------------|---------
     1.50    -1.50     0.00     0.00     0.00    -1.00  | -40.00
```

But now, as expected, it is no longer optimal. So we pivot x_1 into the basis and x_3 gets booted out:

```
     x1       x2       x3       x4       x5       x6    |     b
--------------------------------------------------------|---------
     1.00      .33      .66     0.00     0.00     0.00  |  10.00
     0.00      .33    -1.66     0.00     1.00      .33  |  13.33
     0.00      .66     0.00     1.00     0.00      .33  |   8.33
--------------------------------------------------------|---------
     0.00    -2.00    -1.00     0.00     0.00    -1.00  | -55.00
```

The new optimal solution is $x^{**} = (10, 0, 0, 25/3, 40/3, 0)$ and we've lost \$45 since z^* has dropped to 55.

(d) Going back to the original optimal tableau, it is straightforward to see that the range of optimality for c_2 is $c_2 \leq 4.5$ (easier since x_2 is non-basic at optimality).

(e) So, from part (d), if c_2 is changed from 1 to 4, this would have no effect on our optimal solution. Both the solution and the objective value remain unchanged.

5.) We compare four ways to solve the following knapsack problem: our knapsack has a capacity of 20 lbs. and here are the items.

Item	1	2	3	4	5	6	7	8	9	10
Weight (lb.)	4	1	7	3	2	5	8	1	6	3
Value	22	4	38	9	11	26	44	7	29	20

(a) What solution is obtained by the greedy approach where the items are ordered most valuable to least?

Choose, in order, items 7, 3, 6 with
Total weight: 20
Total Value: 108

(b) What solution is obtained by the greedy approach where the items are ordered lightest to heaviest?

Choose, in order, items 8, 2, 5, 10, 4, 1, 6 with
Total weight: 19
Total Value: 99
(not nearly as good).

(c) What solution is obtained by the greedy approach where the items are ordered by the ratio of "value per pound" (highest ratio to lowest)?

The ordering here is as follows:

Item	8	10	7	1	5	3	6	9	2	4
Value/lb.	7	6.7	5.5	5.5	5.5	5.4	5.2	4.8	4	3

Here, we choose items 8, 10, 7, 1, 5 for a knapsack of weight 18 lbs. So we must skip all other items until we reach item 2 (weight one).
Total weight: 19
Total Value: 108
(This solution has overall value per pound of 5.68. But there is slack.)

(d) Either by hand or using a computer, find the combination of items having maximum value subject to your given total capacity restriction. Explain your process.

I used CPLEX for this part: the optimal solution to the integer program is to choose items $\{3, 5, 7, 10\}$ for total value 113 and total weight 20.

We note that none of our heuristics obtained the same objective value, but they were all close.