Math. Methods in Decision Making
D Term, Sections D01-D02
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Sample Solutions – Assignment 1

1.) We solve the following linear program via the simplex method:

$$\begin{array}{rl} \textbf{maximize} & 3x_1 + 2x_2 + 6x_3 \\ \textbf{subject to} & x_1 + x_2 + 2x_3 \le 4 \\ & 2x_1 + 3x_3 \le 7 \\ & 2x_1 + x_2 + 3x_3 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

(a) First, we introduce slack variables x_4, x_5, x_6 to obtain the equivalent linear program:

$$\begin{array}{lccccccccc} \textbf{maximize} & 3x_1 & +2x_2 & +6x_3 & & & & & \\ \textbf{subject to} & x_1 & +x_2 & +2x_3 & +x_4 & & & = & 4 \\ & 2x_1 & & +3x_3 & & +x_5 & & = & 7 \\ & 2x_1 & +x_2 & +3x_3 & & & +x_6 & = & 5 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \ge & 0 \end{array}$$

So the initial tableau for the simplex method is

x1	x2	x3	x4	x5	x6	b
1.00	1.00	2.00	1.00	0.00	0.00	4.00
2.00	0.00	3.00	0.00	1.00	0.00	7.00
2.00	1.00	3.00	0.00	0.00	1.00	5.00
3.00	2.00	6.00	0.00	0.00	0.00	0.00

(b) Now we apply the simplex method. Our initial basis is $\{x_4, x_5, x_6\}$. The initial solution is $x = (0, 0, 0, 4, 7, 5)$ with objective value $z = 0$.

1st pivot: Let x_1 enter the basis. The pivot column is column one. The ratios are $\{4/1, 7/2, 5/2\}$ with the last being the smallest. So the pivot row is row three and x_6 exits the basis. The new tableau is

x1	x2	x3	x4	x5	x6	b
0.00	.50	.50	1.00	0.00	-.50	1.50
0.00	-1.00	0.00	0.00	1.00	-1.00	2.00
1.00	.50	1.50	0.00	0.00	.50	2.50
0.00	.50	1.50	0.00	0.00	-1.50	-7.50

and the new basis is $\{x_4, x_5, x_1\}$. This tableau corresponds to the basic feasible solution $x = (2.5, 0, 0, 1.5, 2, 0)$ with objective value $z = 7.5$. This is not yet optimal, so we pivot again.

2nd pivot: Let x_2 enter the basis. The pivot column is column two. The ratios are $1.5/.5 = 3$ and $2.5/.5 = 5$. So the pivot row is row one and x_4 exits the basis. The new tableau is

x1	x2	x3	x4	x5	x6	b
0.00	1.00	1.00	2.00	0.00	-1.00	3.00
0.00	0.00	1.00	2.00	1.00	-2.00	5.00
1.00	0.00	1.00	-1.00	0.00	1.00	1.00
0.00	0.00	1.00	-1.00	0.00	-1.00	-9.00

and the new basis is $\{x_2, x_5, x_1\}$. This tableau corresponds to the basic feasible solution $x = (1, 3, 0, 0, 5, 0)$ with objective value $z = 9$. This is still not optimal, so we pivot once more.

3rd pivot: Let x_3 enter the basis. The pivot column is column three. The ratios are $\{3, 5, 1\}$. So the pivot row is row three and x_1 exits the basis. The new tableau is:

x1	x2	x3	x4	x5	x6	b
-1.00	1.00	0.00	3.00	0.00	-2.00	2.00
-1.00	0.00	0.00	3.00	1.00	-3.00	4.00
1.00	0.00	1.00	-1.00	0.00	1.00	1.00
-1.00	0.00	0.00	0.00	0.00	-2.00	-10.00

Now we have an optimal tableau, because all reduced costs are non-positive.

(c) Reading the final tableau, we see that the optimal basis is $\{x_2, x_5, x_3\}$. The optimal solution is $\mathbf{x}^* = (0, 2, 1, 0, 4, 0)$ with optimal objective value $z^* = 10$.

[**NOTE:** If we had instead chosen x_3 as the entering variable in our first iteration, we would have immediately reached optimality after one pivot. In this case we find a different optimal solution: $x = (0, 0, 5/3, 2/3, 2, 0)$ with basis $\{x_4, x_5, x_3\}$ and — of course – the same $z^* = 10$.

2.) We construct a simple example of a linear programming problem involving two decision variables and three constraints in which the optimal solution is not unique.

$$
\begin{array}{rll}
\textbf{maximize} & x_1 & \\
\textbf{subject to} & x_1 + \quad x_2 & \leq 3 \\
& x_1 & \leq 2 \\
& \qquad\quad x_2 & \leq 2 \\
& x_1, x_2 & \geq 0
\end{array}
$$

The feasible region is a square of side two with one corner shaved off. The level curves of the objective function are vertical lines ($x_1 = z$). The graphical method tells us to align a ruler vertically and slide it in the positive direction until we reach the "last point" of the feasible region. But in this case, there are infinitely many last points. There are two basic feasible solutions which are optimal: $x = (2, 0)$ and $x = (2, 1)$. There are infinitely many other optimal solutions, namely all of the points in the line segment joining these two.

3.) Problem 14 on page 262.

We first construct our linear programming problem. Let

$$\begin{aligned} x_1 &= \text{number of gallons of Heidelberg Sweet to be produced} \\ x_2 &= \text{number of gallons of Heidelberg Regular to be produced} \\ x_3 &= \text{number of gallons of Deutschland Extra Dry to be produced} \end{aligned}$$

Then our problem can be expressed as follows:

$$\begin{array}{llllllllll} \textbf{maximize} & x_1 & +1.2x_2 & +2x_3 & & & & & & \\ \textbf{s.t. } \text{(Grade A grapes)} & x_1 & +2x_2 & & +x_4 & & & & = & 150 \\ \text{(Grade B grapes)} & x_1 & & +2x_3 & & +x_5 & & & = & 150 \\ \text{(sugar)} & 2x_1 & +x_2 & & & & +x_6 & & = & 80 \\ \text{(labor)} & 2x_1 & +3x_2 & +x_3 & & & & +x_7 & = & 225 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0 \end{array}$$

where we have introduced four slack variables to obtain an LP in equality form.

(a) To find an optimal solution to the LP, we apply the simplex method using MAPLE. The initial tableau is

```
     x1       x2       x3       x4       x5       x6       x7    |     b
-----------------------------------------------------------------|---------
    1.00     2.00     0.00     1.00     0.00     0.00     0.00   | 150.00
    1.00     0.00     2.00     0.00     1.00     0.00     0.00   | 150.00
    2.00     1.00     0.00     0.00     0.00     1.00     0.00   |  80.00
    2.00     3.00     1.00     0.00     0.00     0.00     1.00   | 225.00
-----------------------------------------------------------------|---------
    1.00     1.20     2.00     0.00     0.00     0.00     0.00   |   0.00
```

and the final tableau is

```
     x1       x2       x3       x4       x5       x6       x7    |     b
-----------------------------------------------------------------|---------
    0.00     0.00     0.00     1.00      .33     0.00     -.66   |  50.00
     .50     0.00     1.00     0.00      .50     0.00     0.00   |  75.00
    1.50     0.00     0.00     0.00      .16     1.00     -.33   |  30.00
     .50     1.00     0.00     0.00     -.16     0.00      .33   |  50.00
-----------------------------------------------------------------|---------
    -.59     0.00     0.00     0.00     -.80     0.00     -.40   | -210.00
```

So an optimal solution is:

$$x^* = (0, 50, 75, 50, 0, 30, 0)$$

with optimal objective value $z* = 210$. Our optimal basis is $\{x_4, x_3, x_6, x_2\}$. So we should make no Heidelberg Sweet, 50 gallons of Heidelberg regular, and 75 gallons of Deutschland Extra Dry for a total profit of $210.

(b) The meanings and final values of the slack variables are:
x_4 = amount of unused Grade A grapes (in bushels) (= 50 in opt. soln)
x_5 = amount of unused Grade B grapes (in bushels) (= 0 in opt. soln)
x_6 = amount of unused sugar (in pounds) (= 30 in opt. soln)
x_7 = amount of unused labor (in hours) (= 0 in opt. soln)

(c) Which resources should be increased?
Grade B grapes (increased profit of 50 cents per bushel)
Labor (increased profit of 40 cents per hour)

4.) Problem 20 on page 263.

We first define our variables: let
x_1 = number of baxes of 20-gallon bags to be produced
x_2 = number of baxes of 30-gallon bags to be produced
x_3 = number of baxes of 33-gallon bags to be produced
Then our linear programming problem is

maximize	$0.1x_1$	$+0.15x_2$	$+0.2x_3$		
s.t.					
(cutting time (sec))	$2x_1$	$+3x_2$	$+3x_3$	$\leq$	7200
(sealing time (sec))	$2x_1$	$+3x_2$	$+3x_3$	$\leq$	10800
(pkging time (sec))	$2x_1$	$+3x_2$	$+3x_3$	$\leq$	14400
	$x_1,$	$x_2,$	x_3	$\geq$	0

Initial tableau:

x1	x2	x3	x4	x5	x6	b
2.00	3.00	3.00	1.00	0.00	0.00	7200.00
2.00	2.00	3.00	0.00	1.00	0.00	10800.00
3.00	4.00	5.00	0.00	0.00	1.00	14400.00
.10	.15	.20	0.00	0.00	0.00	0.00

Apply the simplex method (by computer) to obtain the following optimal tableau

x1	x2	x3	x4	x5	x6	b
.66	1.00	1.00	.33	0.00	0.00	2400.00
0.00	-1.00	0.00	-1.00	1.00	0.00	3600.00

```
     -.33   -1.00    0.00   -1.66    0.00    1.00 | 2400.00
------------------------------------------------|---------
     -.03    -.05    0.00    -.06    0.00    0.00 | -480.00
```

So the optimal mix is to make only 33-gallon bags. We can make 2400 boxes of these with the available resources. We will end up using all of our cutting time ($x_4 = 0$) but will have 1 hour of unused sealing time ($x_5 = 3600$ seconds) and 40 minutes of unused packaging time ($x_6 = 2400$ sec.). This plan yields a profit of \$480.

5.) We consider the given asphalt mixture problem. we must design a low-cost mix of one cubic yard of asphalt concrete using the following ingredients:

Component	**Skid Resistance**	**Cohesion**	**Contribution to Sed. Effect**	**Cost** (per yd^3)
Aggregate 1	44	0.1	2.0	5.00
Aggregate 2	30	0.3	5.0	4.00
Sand	10	0.4	1.0	1.00
Grade A Bitumen	0	0.7	8.0	9.00
Grade B Bitumen	0	0.8	6.0	12.00

Let

$$\begin{aligned}
x_1 &= \text{Amount (in cubic yards) of Aggregate 1 to include}\\
x_2 &= \text{Amount (in cubic yards) of Aggregate 2 to include}\\
x_3 &= \text{Amount (in cubic yards) of sand to include}\\
x_4 &= \text{Amount (in cubic yards) of Grade A bitumen to include}\\
x_5 &= \text{Amount (in cubic yards) of Grade B bitumen to include}
\end{aligned}$$

Then our linear program is

$$\begin{array}{rll}
\textbf{minimize} & 5x_1 + 4x_2 + x_3 + 9x_4 + 12x_5 & \\
\textbf{subject to} & & \\
\textit{(Skid resist.)} & 44x_1 + 30x_2 + 10x_3 & \geq 30\\
\textit{(cohesion)} & 0.1x_1 + 0.3x_2 + 0.4x_3 + 0.7x_4 + 0.8x_5 & \geq 0.3\\
\textit{(Sed. effect)} & 2x_1 + 5x_2 + x_3 + 8x_4 + 6x_5 & \leq 4\\
\textit{(total 1 yd}^3\textit{)} & x_1 + x_2 + x_3 + x_4 + x_5 & = 1\\
 & x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{array}$$

By the way, here is an initial (pseudo-)tableau and an optimal tableau for the problem: we start with

x1	x2	x3	x4	x5	x6	x7	x8	b
44.00	30.00	10.00	0.00	0.00	-1.00	0.00	0.00	30.00
.10	.30	.40	.70	.80	0.00	-1.00	0.00	.30
2.00	5.00	1.00	8.00	6.00	0.00	0.00	1.00	4.00
1.00	1.00	1.00	1.00	1.00	0.00	0.00	0.00	1.00
-5.00	-4.00	-1.00	-9.00	-12.00	0.00	0.00	0.00	0.00

where we have introduced two surplus variables and one slack variable, but we don't have an initial basis. Eventually, we reach

x1	x2	x3	x4	x5	x6	x7	x8	b
1.00	0.00	0.00	-1.25	0.00	-.09	-6.25	-.31	.31
0.00	0.00	0.00	.25	1.00	-.05	-5.18	-.12	.12
0.00	1.00	0.00	1.75	0.00	.08	8.04	.47	.53
0.00	0.00	1.00	.25	0.00	.06	3.39	-.04	.04
0.00	0.00	0.00	-5.00	0.00	-.66	-57.86	-1.11	5.11

showing that the optimal mix contains 31% Aggregate 1, 53% Aggregate 2, 12% Grade B bitumen, and 4% sand at a cost of $5.11 per cubic yard.