Matrices & Linear Algebra II
C Term, Section C01
W. J. Martin
March 3, 2004

Sample Exercises – March 3

Since our last lesson was rushed, I thought it might help some of you to have this set of notes — which you should read as if it were being presented in class — which simply computes the generalized eigenvectors and the Jordan canonical form of two small matrices. I hope this helps you prepare for tomorrow's test and for the assignment due today. But more importantly I hope this gives you a deeper understanding of this important concept.

Here are the two matrices we will analyze:

$$A = \begin{bmatrix} 4 & 0 \\ -2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix}.$$

1.) Our first matrix is

$$A = \begin{bmatrix} 4 & 0 \\ -2 & 4 \end{bmatrix}.$$

The characteristic polynomial is $(4-t)(4-t)$. So there is only one eigenvalue, $\lambda = 4$, with algebraic multiplicity two.

The next step is to find the eigenspace W_λ. The row reduction is easy. We have

$$\mathsf{W}_\lambda = \mathsf{nullsp}\,(A-4I) = \mathsf{nullsp}\left(\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \right) = \mathsf{nullsp}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

which is clearly the span of $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

So the geometric multiplicity is less than the algebraic multiplicity and we expect the Jordan canonical form to be

$$J = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

In order to find a basis β of generalized eigenvectors so that $J = [\mathsf{L}_A]_\beta$, we first need the null space of $(A-4I)^2$. But this is clearly all of $\mathbb{R}^2$. Then we choose v_2 in $\mathsf{nullsp}\,((A-4I)^2)$ but not in $\mathsf{nullsp}\,(A-4I)$. For example, $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ will do. Then we compute

$$v_1 = (A-4I)v_2 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

It automatically holds that $v_1 \in \mathsf{W}_\lambda$ since $(A-\lambda I)^2 v_2 = 0$. So we have

$$Av_1 = 4v_1, \qquad Av_2 = v_1 + 4v_2.$$

Thus, for

$$Q = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix},$$

we have

$$A = QJQ^{-1}$$

as desired.

2.) Our second matrix is B above.

<u>*Step 1:*</u> Find the eigenvalues:

$$\chi_B(t) = \begin{vmatrix} -t & -\frac{1}{2} & \frac{1}{2} \\ 1 & -t & 2 \\ -1 & 2 & -t \end{vmatrix} = -t^3 + 1 + 1 + 4t - \frac{1}{2}t - \frac{1}{2}t = -(t^3 - 3t - 2) = -(t-2)(t+1)^2.$$

So the eigenvalues are $\lambda = 2$ with multiplicity one and $\mu = -1$ with (algebraic) multiplicity two.

<u>*Step 2:*</u> Let's look at the eigenspace W_μ first since the other one will be easy. We have

$$B + I = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$\mathsf{W}_\mu = \mathsf{nullsp}\,(B+I) = \mathsf{nullsp}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} -s \\ -s \\ s \end{bmatrix} \ : \ s \in \mathbb{R}\right\}.$$

Now we know that B is not diagonalizable, so we compute the generalized eigenspace

$$\mathsf{K}_\mu = \mathsf{nullsp}\,((B+I)^2) = \mathsf{nullsp}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 9/2 & 9/2 \\ 0 & 9/2 & 9/2 \end{bmatrix}\right) = \mathsf{nullsp}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} r \\ -s \\ s \end{bmatrix} \ : \ r, s \in \mathbb{R}\right\}.$$

We take v in K_μ but **not in** W_μ. Let's take $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

<u>*Step 3:*</u> We have the first vector in our chain of length two. Now let us compute

$$(B - \mu I)v = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Set $v_1 = (B - \mu I)v$ (this clearly lies in W_μ) and $v_2 = v$. Then

$$Bv_1 = \mu v_1, \qquad Bv_2 = v_1 + \mu v_2.$$

Step 4: We return to the easy eigenvalue: $\lambda = 2$. We find

$$B - 2I = \begin{bmatrix} -2 & -\frac{1}{2} & \frac{1}{2} \\ 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$\mathsf{W}_\lambda = \mathsf{nullsp}\,(B - \lambda I) = \mathsf{nullsp}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 \\ r \\ r \end{bmatrix} \;:\; r \in \mathbb{R}\right\}.$$

So we can take $v_3 = (0, 1, 1)^t$.

Step 5: Now we are ready to write down Q and J. The columns of Q are the vectors v_1, v_2, v_3 in our basis of generalized eigenvectors:

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since $B = QJQ^{-1}$, we are done.