Matrices & Linear Algebra II
C Term, Section C01
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Sample Solutions – Assignment 7

1.) Exercise #8 on p337.

(a) We have $\langle (a,b),(c,d)\rangle = ac - bd$ but then

$$\langle (0,1),(0,1)\rangle = -1 < 0$$

contradicting property (d) in the definition. So $\langle \cdot,\cdot\rangle$ is not an inner product.

(b) We have $\langle A,B\rangle = \mathrm{tr}(A+B)$ but then

$$\langle \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \rangle = \mathrm{tr}\left(\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \right) = -2$$

contradicting property (d) in the definition. So $\langle \cdot,\cdot\rangle$ is not an inner product.

(c) We have $\langle f,g\rangle = \int_0^1 f'(t)g(t)\,dt$ but then

$$\langle 1-x^2, 1-x^2\rangle = \int_0^1 (-2t)(1-t^2)dt = \int_0^1 2t^3 - 2t\,dt = \frac{1}{2}t^4 - t^2\Big|_0^1 = -\frac{1}{2} < 0$$

contradicting property (d) in the definition. So $\langle \cdot,\cdot\rangle$ is not an inner product.

2.) Exercise #2(a,f,h) on p353.

(a) We have

$$\{u_1,u_2,u_3\} = \{(1,0,1),\ (0,1,1),\ (1,3,3)\}.$$

We begin with

$$v_1 = u_1 = (1,0,1).$$

Next,

$$v_2 = u_2 - \frac{\langle u_2,v_1\rangle}{\langle v_1,v_1\rangle}v_1 = (0,1,1) - \frac{1}{2}(1,0,1) = (-\frac{1}{2},1,\frac{1}{2}).$$

Then we compute

$$v_3 = u_3 - \frac{\langle u_3,v_1\rangle}{\langle v_1,v_1\rangle}v_1 - \frac{\langle u_3,v_2\rangle}{\langle v_2,v_2\rangle}v_2$$
$$= (1,3,3) - \frac{4}{2}(1,0,1) - \frac{4}{3/2}(-\frac{1}{2},1,\frac{1}{2}) = (\frac{1}{3},\frac{1}{3},-\frac{1}{3}).$$

So the set $\{v_1,v_2,v_3\}$ is an orthogonal basis for $\mathbb{R}^3$. We normalize to obtain the following orthonormal basis:

$$\beta = \left\{ \frac{1}{\sqrt{2}}(1,0,1),\ \frac{1}{\sqrt{6}}(-1,2,1),\ \frac{1}{\sqrt{3}}(1,1,-1) \right\}.$$

Now to solve

$$x = (1,1,2) = c_1 w_1 + c_2 w_2 + c_3 w_3$$

where $\beta = \{w_1, w_2, w_3\}$, we just compute inner products:

$$\begin{aligned}
c_1 &= \langle x, w_1 \rangle = \frac{1}{\sqrt{2}}(1+0+2) = 3\sqrt{2}/2, \\
c_2 &= \langle x, w_2 \rangle = \frac{1}{\sqrt{6}}(-1+2+2) = \sqrt{6}/2, \\
c_3 &= \langle x, w_3 \rangle = \frac{1}{\sqrt{3}}(1+1-2) = 0.
\end{aligned}$$

And, indeed,

$$\frac{3\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}}(1,0,1) + \frac{\sqrt{6}}{2} \cdot \frac{1}{\sqrt{6}}(-1,2,1) = (1,1,2).$$

(f) We have

$$\{u_1, u_2, u_3\} = \{(1,-2,-1,3),\ (3,6,3,-1),\ (1,4,2,8)\}.$$

We begin with

$$v_1 = u_1 = (1,-2,-1,3).$$

Next,

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = (3,6,3,-1) - \frac{-15}{15}(1,-2,-1,3) = (4,4,2,2).$$

Finally

$$\begin{aligned}
v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \\
= (1,4,2,8) - \frac{15}{15}(1,-2,-1,3) - \frac{40}{40}(4,4,2,2) = (-4,2,1,3).
\end{aligned}$$

So the set $\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathsf{span}(S)$. We normalize to obtain the following orthonormal basis:

$$\beta = \left\{ \frac{1}{\sqrt{15}}(1,-2,-1,3),\ \frac{1}{\sqrt{40}}(4,4,2,2),\ \frac{1}{\sqrt{30}}(-4,2,1,3) \right\}.$$

Now to solve

$$x = (-1,2,1,1) = c_1 w_1 + c_2 w_2 + c_3 w_3$$

where $\beta = \{w_1, w_2, w_3\}$, we again compute inner products:

$$\begin{aligned}
c_1 &= \langle x, w_1 \rangle = \frac{1}{\sqrt{15}}(-1-2-1+3) = -\sqrt{15}/5, \\
c_2 &= \langle x, w_2 \rangle = \frac{1}{\sqrt{40}}(-4+8+2+2) = \sqrt{40}/5, \\
c_3 &= \langle x, w_3 \rangle = \frac{1}{\sqrt{30}}(4+4+1+3) = 2\sqrt{30}/5.
\end{aligned}$$

And, indeed,

$$-\frac{\sqrt{15}}{5}\cdot\frac{1}{\sqrt{15}}(1,-2,-1,3)+\frac{\sqrt{40}}{5}\cdot\frac{1}{\sqrt{40}}(4,4,2,2)+\frac{2\sqrt{30}}{5}\cdot\frac{1}{\sqrt{30}}(-4,2,1,3)=(-1,2,1,1)$$

confirming Theorem 6.5 in this case.

(h) Our third vector space is $\mathsf{M}_{2\times 2}(\mathbb{R})$ with a subspace W spanned by

$$S=\left\{\begin{bmatrix}2&2\\2&1\end{bmatrix},\begin{bmatrix}11&4\\2&5\end{bmatrix},\begin{bmatrix}4&-12\\3&-16\end{bmatrix}\right\}.$$

We apply Gram-Schmidt:

$$v_1=\begin{bmatrix}2&2\\2&1\end{bmatrix}.$$

Note that $\langle v_1,v_1\rangle=\mathsf{tr}(v_1^t v_1)=\mathsf{tr}\left(\begin{bmatrix}8&6\\6&5\end{bmatrix}\right)=13$. Now

$$v_2=u_2-\frac{\langle u_2,v_1\rangle}{\langle v_1,v_1\rangle}v_1=\begin{bmatrix}11&4\\2&5\end{bmatrix}-\frac{39}{13}\begin{bmatrix}2&2\\2&1\end{bmatrix}=\begin{bmatrix}5&-2\\-4&2\end{bmatrix},$$

$$\langle v_2,v_2\rangle=\mathsf{tr}(v_2^t v_2)=\mathsf{tr}\left(\begin{bmatrix}41&-18\\-18&8\end{bmatrix}\right)=49,$$

$$v_3=u_3-\frac{\langle u_3,v_1\rangle}{\langle v_1,v_1\rangle}v_1-\frac{\langle u_3,v_2\rangle}{\langle v_2,v_2\rangle}v_2$$

$$=\begin{bmatrix}4&-12\\3&-16\end{bmatrix}-\frac{-26}{13}\begin{bmatrix}2&2\\2&1\end{bmatrix}-\frac{0}{49}\begin{bmatrix}5&-2\\-4&2\end{bmatrix}=\begin{bmatrix}8&-8\\7&-14\end{bmatrix}.$$

So we finally have our orthogonal basis. Here is the orthonormal basis

$$\beta=\{w_1,w_2,w_3\}=\left\{\frac{1}{\sqrt{13}}\begin{bmatrix}2&2\\2&1\end{bmatrix},\frac{1}{7}\begin{bmatrix}5&-2\\-4&2\end{bmatrix},\frac{1}{\sqrt{373}}\begin{bmatrix}8&-8\\7&-14\end{bmatrix}\right\}.$$

Now we check that Theorem 6.5 holds by computing the "Fourier coefficients" of

$$A=\begin{bmatrix}8&6\\25&-13\end{bmatrix}$$

relative to the basis β. We find that

$$A=c_1v_1+c_2v_2+c_3v_3$$

where

$$\begin{aligned}c_1&=\langle A,v_1\rangle=65/=5\sqrt{13},\\c_2&=\langle A,v_2\rangle=-98/7=-14,\\c_3&=\langle A,v_3\rangle=373/\sqrt{373}=\sqrt{373}.\end{aligned}$$

Check:

$$\frac{5\sqrt{13}}{\sqrt{13}}\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} + \frac{-14}{7}\begin{bmatrix} 5 & -2 \\ -4 & 2 \end{bmatrix} + \frac{\sqrt{373}}{\sqrt{373}}\begin{bmatrix} 8 & -8 \\ 7 & -14 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 25 & -13 \end{bmatrix} = A.$$

3.) Let $\mathsf{V} = \mathsf{C}([-1, 1])$ with inner product

$$\langle f, g\rangle = \frac{2}{\pi}\int_{-1}^{1} f(t)g(t)\sqrt{1-t^2}\,dt.$$

Apply the Gram-Schmidt orthogonalization process to the ordered set $\{1, t, t^2\}$ to find an orthonormal basis $\beta = \{v_0, v_1, v_2\}$ for $\mathsf{span}(1, t, t^2)$.

Solution: We first take

$$v_0 = u_0 = 1.$$

We compute

$$\langle v_0, v_0\rangle = \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-t^2}\,dt = 1.$$

We will also need

$$\langle u_1, v_0\rangle = \frac{2}{\pi}\int_{-1}^{1} t\sqrt{1-t^2}\,dt = 0.$$

Next, we take

$$v_1 = u_1 - \frac{\langle u_1, v_0\rangle}{\langle v_0, v_0\rangle}v_0 = t - \frac{0}{1}\cdot 1 = t.$$

We will now need

$$\langle v_1, v_1\rangle = \frac{2}{\pi}\int_{-1}^{1} t^2\sqrt{1-t^2}\,dt = 1/4.$$

Next,

$$v_2 = u_2 - \frac{\langle u_2, v_0\rangle}{\langle v_0, v_0\rangle}v_0 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle}v_1 = t^2 - \frac{1/4}{1}1 - \frac{0}{1/4}t = t^2 - \frac{1}{4}.$$

To normalize these, we compute $\langle v_0, v_0\rangle = 1$,

$$\langle v_1, v_1\rangle = \frac{2}{\pi}\int_{-1}^{1} t^2\sqrt{1-t^2}\,dt = \frac{1}{4},$$

$$\langle v_2, v_2\rangle = \frac{2}{\pi}\int_{-1}^{1} (t^2 - \frac{1}{4})^2\sqrt{1-t^2}\,dt = \frac{1}{16}.$$

So the desired orthonormal basis is $\beta = \{1, 2t, 4t^2 - 1\}$. These are the first few Chebyshev polynomials of the second kind.

4.) Parts (b) and (d) of exercise #2 on p495.

(b) We have $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. We compute

$$\chi_A(t) = \begin{vmatrix} 1-t & 2 \\ 3 & 2-t \end{vmatrix} = (1-t)(2-t) - 6 = t^2 - 3t - 4 = (t-4)(t+1).$$

So the eigenvalues are $\lambda = 4$ and $\mu = -1$. They are distinct! So A is diagonalizable. $Ax = 4x$ is solved by taking $x = (2,3)^t$ and $Ay = -y$ is solved by taking $y = (1,-1)^t$. So we take

$$Q = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \qquad J = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

so that $A = QJQ^{-1}$.

(d) We have $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}$. We compute

$$\chi_A(t) = \begin{vmatrix} 2-t & 1 & 0 & 0 \\ 0 & 2-t & 1 & 0 \\ 0 & 0 & 3-t & 0 \\ 0 & 1 & -1 & 3-t \end{vmatrix} = (3-t)^2(2-t)^2.$$

So the eigenvalues are $\lambda = 3$ and $\mu = 2$. They both have algebraic multiplicity two. We observe

$$A - \lambda I = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and determine

$$\mathsf{W}_\lambda = \mathsf{nullsp}\,(A - 3I) = \mathsf{nullsp}\left(\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} r \\ r \\ r \\ s \end{bmatrix} : r, s \in \mathbb{R} \right\}.$$

So $\lambda = 3$ has geometric multiplicity two and we have no problem finding two linearly indepdent eigenvectors. For example, we can take

$$v_1 = (1,1,1,0)^t, \qquad v_2 = (0,0,0,1)^t.$$

We now look at

$$A - \mu I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and find that

$$\mathsf{W}_\mu = \mathsf{nullsp}\,(A-2I) = \mathsf{nullsp}\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix} \; : \; r \in \mathbb{R}\right\}.$$

So this eigenvalue is defective and we must look at the generalized eigenspace

$$\mathsf{K}_\mu = \mathsf{nullsp}\,((A-2I)^2) = \mathsf{nullsp}\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} r \\ s \\ 0 \\ -s \end{bmatrix} \; : \; r, s \in \mathbb{R}\right\}.$$

We take $v_4 = (0,1,0,-1)^t$ and $v_3 = (A-2I)v_4 = (1,0,0,0)^t$.

We now have the ingredients we need. Taking

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \qquad J = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

we have $A = QJQ^{-1}$ as desired.

5.) We look at parts (b) and (d) of exercise #3 on p495.

(b) We first find the matrix $[\mathsf{T}]_\beta$ where β is the given basis:

$$A = [\mathsf{T}]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix is upper triangular, so we know that the eigenvalues are the entries along the diagonal: $\lambda = 1$ with multiplicity two and $\mu = 0$ with multiplicity three. The eigenspaces and generalized eigenspaces are

$$\mathsf{W}_\lambda = \mathsf{nullsp}\,(A-I) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ r \\ 0 \end{bmatrix} \; : \; r \in \mathbb{R}\right\},$$

$$\mathsf{K}_\lambda = \mathsf{nullsp}\,((A-I)^2) = \mathsf{nullsp}\left(\begin{bmatrix} 1 & -2 & 2 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ r \\ s \end{bmatrix} \; : \; r, s \in \mathbb{R}\right\},$$

$$\mathsf{W}_\mu = \mathsf{nullsp}\,(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ r \\ 0 \end{bmatrix} \;:\; r \in \mathbb{R} \right\}$$

$$\mathsf{nullsp}\,(A^2) = \mathsf{nullsp}\left(\begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) = \left\{ \begin{bmatrix} r \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} \;:\; r, s \in \mathbb{R} \right\},$$

and

$$\mathsf{K}_\mu = \mathsf{nullsp}\,(A^3) = \mathsf{nullsp}\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) = \left\{ \begin{bmatrix} r \\ s \\ t \\ 0 \\ 0 \end{bmatrix} \;:\; r, s, t \in \mathbb{R} \right\}.$$

So we take

$$\begin{gathered} v_2 = (0,0,0,0,1)^t, \qquad v_5 = (0,0,1,0,0)^t, \\ v_1 = (A - I)v_2 = (0,0,0,1,0)^t, \\ v_4 = (A - 0I)v_5 = (0,2,0,0,0)^t, \\ v_3 = (A - 0I)v_4 = (2,0,0,0,0)^t. \end{gathered}$$

Our matrix Q of generalized eigenvectors has $v_1, \ldots, v_5$ as its columns and our matrix J has two Jordan blocks:

$$Q = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) Let's make this a problem of matrices and first find the matrix $[\mathsf{T}]_\beta$ where β is the standard basis:

$$A = [\mathsf{T}]_\beta = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

We find the characteristic polynomial:

$$\chi_T(t) = (3-t)^2(2-t)^2 - (3-t)^2 = (3-t)^3(1-t).$$

So the eigenvalues are $\lambda = 3$ with multiplicity three and $\mu = 1$ with multiplicity one.

(In brief!) We quickly find that A is diagonalizable and we obtain a basis of eigenvectors giving us

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$