

Sample Solutions – Assignment 6

1.) Problem #17 on p259. We are given

$$\mathsf{T} : M_{n \times n}(\mathbb{R}) \rightarrow M_{n \times n}(\mathbb{R})$$

via $\mathsf{T}(A) = A^t$, the transpose of A .

(a) We show that the only eigenvalues of T are the obvious ones: $\lambda = \pm 1$.

Indeed, for any symmetric matrix A , $A^t = A$ and so $\mathsf{T}(A) = 1A$. So, for $n \geq 1$, there exists a non-zero eigenvector belonging to $\lambda = 1$. Similarly, any skew-symmetric matrix A satisfies $A^t = -A$ and so $\mathsf{T}(A) = (-1)A$. So, for $n \geq 2$, there exists a non-zero eigenvector belonging to $\lambda = -1$. (For $n = 1$, we were done with $\lambda = 1$.)

Now suppose $n \geq 2$ and suppose $\lambda \neq \pm 1$. If A satisfies $\mathsf{T}(A) = \lambda A$, we must prove that $A = O$, the zero matrix. First, let's look at the diagonal entries of $A = [a_{ij}]$. The (i, i) -entry of A^t is a_{ii} also. So the equation $A^t = \lambda A$ gives

$$a_{ii} = \lambda a_{ii} \quad (1 \leq i \leq n)$$

which forces $a_{ii} = 0$ since we get $0 = (\lambda - 1)a_{ii}$ and $\lambda \neq 1$.

Similarly, if $i \neq j$, then looking at row i , column j and row j , column i , the equation $A^t = \lambda A$ gives

$$a_{ij} = \lambda a_{ji} \quad a_{ji} = \lambda a_{ij}$$

which gives $a_{ij} = \lambda^2 a_{ij}$ or

$$0 = (\lambda^2 - 1)a_{ij},$$

which forces $a_{ij} = 0$ since $\lambda \neq \pm 1$. So, for $\lambda \neq \pm 1$, we have proved that any matrix A satisfying $\mathsf{T}(A) = \lambda A$ must be the zero matrix. So we're done.

(b) For $\lambda = 1$, the eigenspace W_λ is precisely the subspace of symmetric matrices. As shown in class, this subspace has dimension $n + \binom{n}{2} = \binom{n+1}{2}$.

For $\lambda = -1$, the eigenspace W_λ is the subspace of skew-symmetric matrices. This subspace has dimension $\binom{n}{2}$.

(c) To find a basis for $M_{2 \times 2}(\mathbb{R})$ consisting of eigenvectors for T , we take the union of a basis for the subspace of symmetric matrices and a basis for the subspace of skew-symmetric matrices. We get

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Then we have

$$[\mathbb{T}]_{\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(d) For $n > 2$, we do the same thing, except that we need to use notation and a bit of abstraction since n is arbitrary. First, we use some of the standard basis matrices:

- let E^{ii} be the $n \times n$ matrix with i^{th} diagonal entry equal to 1 and all other entries zero.

We also need similarly sparse matrices to complete the basis for the subspace of symmetric matrices and to build a basis for the subspace of skew-symmetric matrices.

- For $i < j$, let F^{ij} be the $n \times n$ matrix with (i, j) -entry equal to one, (j, i) -entry equal to one and all other entries zero;
- For $i < j$, let G^{ij} be the $n \times n$ matrix with (i, j) -entry equal to one, (j, i) -entry equal to -1 and all other entries zero.

Then our basis is

$$\beta = \{E^{11}, \dots, E^{nn}, F^{12}, \dots, F^{n-1,n}, G^{12}, \dots, G^{n-1,n}\}.$$

The matrix representing \mathbb{T} with respect to this basis has simple form

$$[\mathbb{T}]_{\beta} = \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$$

where the matrix I is $\binom{n+1}{2} \times \binom{n+1}{2}$ and the matrix $-I$ is $\binom{n}{2} \times \binom{n}{2}$.

2.) Problem #15(a) on p259.

Proof: We are given that x is an eigenvector of \mathbb{T} associated to eigenvalue λ . It follows that $x \neq 0$. So we need only show that $\mathbb{T}^m(x) = \lambda x$ for each $m \geq 1$. We prove this by induction.

Base Case: For $m = 1$, the thing we must prove is

$$\mathbb{T}(x) = \lambda x.$$

But that is exactly our assumption. So this case is done, by hypothesis.

Induction Hypothesis: Assume, for some $m \geq 1$, that $\mathbb{T}^m(x) = \lambda^m x$.

Induction Step: We now prove the result for $m + 1$.

$$\begin{aligned} \mathbb{T}^{m+1}(x) &= \mathbb{T}(\mathbb{T}^m(x)) && \text{(function composition associative)} \\ &= \mathbb{T}(\lambda^m x) && \text{(induction hypothesis)} \\ &= \lambda^m \mathbb{T}(x) && \text{(T linear)} \\ &= \lambda^m (\lambda x) && \text{(base case)} \\ &= \lambda^{m+1} x && \text{(VS 6)} \end{aligned}$$

By the Principle of Mathematical Induction, the result now holds for all positive integers m .

3.) Exercise 22(a) on p260.

Proof: We make good use of the previous result.

Write

$$g(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0.$$

Then

$$g(\mathbb{T}) = c_n \mathbb{T}^n + c_{n-1} \mathbb{T}^{n-1} + \cdots + c_1 \mathbb{T} + c_0 \mathbb{I}$$

and

$$\begin{aligned} g(\mathbb{T})(x) &= [c_n \mathbb{T}^n + c_{n-1} \mathbb{T}^{n-1} + \cdots + c_1 \mathbb{T} + c_0 \mathbb{I}](x) \\ &= [c_n \mathbb{T}^n](x) + [c_{n-1} \mathbb{T}^{n-1}](x) + \cdots + [c_1 \mathbb{T}](x) + [c_0 \mathbb{I}](x) \\ &= c_n [\mathbb{T}^n(x)] + c_{n-1} [\mathbb{T}^{n-1}(x)] + \cdots + c_1 [\mathbb{T}(x)] + c_0 [\mathbb{I}(x)] \\ &= c_n [\lambda^n x] + c_{n-1} [\lambda^{n-1} x] + \cdots + c_1 [\lambda x] + c_0 x \quad (\text{using problem 2}) \\ &= (c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0) x \quad (\text{VS 6, VS 8}) \\ &= g(\lambda)x \end{aligned}$$

4.) Parts (b), (d) and (f) of exercise #2 on p279.

(b) We can do this one by eye. We have $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. We find the eigenvalues:

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 1-t & 3 \\ 3 & 1-t \end{vmatrix} = (t-1)^2 - 9 = t^2 - 2t - 8 = (t-4)(t+2).$$

So the eigenvalues are $\lambda = 4$ and $\mu = -2$. Easily, we see that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and, after a bit more thought, that $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So we have

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$D = Q^{-1}AQ = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

(d) Now we have to do some work: we have $A = \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix}$. First, we find the eigenvalues:

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 7-t & -4 & 0 \\ 8 & -5-t & 0 \\ 6 & -6 & 3-t \end{vmatrix} =$$

$$(7-t)(-5-t)(3-t)-(-4)8(3-t) = [(t-7)(t+5)+32](3-t) = [t^2-2t-3](3-t) = -(t+1)(t-3)^2.$$

So the eigenvalues are $\lambda = -1$ with multiplicity one and $\mu = 3$ with (algebraic) multiplicity two. To get to the point, A is diagonalizable if and only if the eigenspace W_μ has dimension two. Indeed

$$W_\mu = \text{nullsp} \left(\begin{bmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix} \right) = \text{nullsp} \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

does have dimension two, with basis $\gamma = \{[1, 1, 0]^t, [0, 0, 1]^t\}$. So A is diagonalizable and we return to the eigenspace W_λ :

$$W_\lambda = \text{nullsp} \left(\begin{bmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{bmatrix} \right) = \text{nullsp} \left(\begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix} \right).$$

This is a one-dimensional space, as expected and a basis is given by $\alpha = \{[2, 4, 3]^t\}$. So, our change-of-coordinates matrix is

$$Q = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

and we compute

$$D = Q^{-1}AQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

as expected.

(f) First we find the eigenvalues of A . Since A is upper-triangular, these are just the entries along the diagonal: $\lambda = 1$ with multiplicity two and $\mu = 3$ with multiplicity one.

Next we focus on the eigenspace W_λ since trouble can only arise here. We find, through row reduction, that

$$W_\lambda = \text{nullsp}(A - I) = \text{nullsp} \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \right) = \text{nullsp} \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

which is clearly a one-dimensional space. Since the geometric multiplicity (one) is not equal to the algebraic multiplicity (two), we immediately conclude that A is not diagonalizable.

5.) Exercise #7 on p280 using the matrix given in the assignment, which is

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

To efficiently compute A^n for large integers n , we find an invertible matrix Q and a diagonal matrix D satisfying $A = QDQ^{-1}$ so that

$$\begin{aligned} A^n &= (QDQ^{-1})^n = (QDQ^{-1})(QDQ^{-1}) \cdots (QDQ^{-1}) \\ &= QD(Q^{-1}Q)D(Q^{-1}Q) \cdots (Q^{-1}Q)DQ^{-1} \\ &= QDIDI \cdots IDQ^{-1} \\ &= QD^nQ^{-1}. \end{aligned}$$

We first find the eigenvalues:

$$\chi_A(t) = \begin{vmatrix} -t & 1 \\ 2 & 1-t \end{vmatrix} = t(t-1) - 2 = (t-2)(t+1).$$

So the eigenvalues are $\lambda = 2$ and $\mu = -1$. Clearly $[1, 2]^t$ is an eigenvector associated to λ and $[1, -1]^t$ is an eigenvector associated to μ . So we have

$$Q = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$Q^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

So we have, for any positive integer n ,

$$A^n = QD^nQ^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Multiplying through, we get

$$A^n = \frac{1}{3} \begin{bmatrix} 2^n + 2(-1)^n & 2^n + (-1)^{n+1} \\ 2^{n+1} + 2(-1)^{n+1} & 2^{n+1} + (-1)^n \end{bmatrix}.$$