Matrices & Linear Algebra II
C Term, Section C01
W. J. Martin
February 19, 2004

Sample Solutions – Assignment 5

1.) Problem #11 on p97.

The problem, in Martin's notation is:
Let V *be a vector space and let* $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ *be linear. Then* $\mathsf{T} \circ \mathsf{T} = \mathsf{T}_0$ *(the zero function which sends every vector to the zero vector) if and only if the image of* T *is a subset of the null space of* T*:* $\mathsf{im}\,\mathsf{T} \subseteq \mathsf{nullsp}\,\mathsf{T}$.

Proof: ($\Rightarrow$) Assume $\mathsf{T} \circ \mathsf{T} = \mathsf{T}_0$. Let $w \in \mathsf{im}\,\mathsf{T}$. Then there is some $v \in \mathsf{V}$ for which $\mathsf{T}(v) = w$. But then

$$\mathsf{T}(w) = \mathsf{T}(\mathsf{T}(v)) = (\mathsf{T} \circ \mathsf{T})(v) = \mathsf{T}_0(v) = 0$$

by hypothesis. So $\mathsf{T}(w) = 0$ and $w \in \mathsf{nullsp}\,\mathsf{T}$. Since w was an arbitrary element of $\mathsf{im}\,\mathsf{T}$, this proves $\mathsf{im}\,\mathsf{T} \subseteq \mathsf{nullsp}\,\mathsf{T}$.

($\Leftarrow$) Assume $\mathsf{im}\,\mathsf{T} \subseteq \mathsf{nullsp}\,\mathsf{T}$. To prove that the two functions $\mathsf{T} \circ \mathsf{T}$ and T_0 are equal, we show that they do the same thing to each vector in V. So let u be any vector in V. Then $\mathsf{T}(u)$ is obviously in $\mathsf{im}\,\mathsf{T}$. So, by hypothesis, $\mathsf{T}(u)$ also lies in $\mathsf{nullsp}\,\mathsf{T}$. That means that $\mathsf{T}(\mathsf{T}(u)) = 0$. So we have, for any u, $(\mathsf{T} \circ T)(u) = 0$ which is the same as $\mathsf{T}_0(u)$. Thus $\mathsf{T} \circ \mathsf{T} = \mathsf{T}_0$ as desired.

2.) Problem #7 on p107.
(a) We must show that, if $A^2 = O$, then A is not invertible. We will (reluctantly) use the method of proof by contradiction. We will contradict the fact that the zero matrix is not invertible. So let us suppose that $A^2 = O$ and yet A is invertible. Then, multiply both sides on the left by A^{-1}, we find

$$A = IA = (A^{-1}A)A = A^{-1}(AA) = A^{-1}A^2 = A^{-1}O = O.$$

This is a contradiction: we cannot have $A = O$ when A is invertible.
(b) We must show that, if $AB = O$ and B is not the zero matrix, then A is not invertible. We will prove this by contrapositive. Suppose that A is invertible and $AB = O$. Then, multiplying both sides on the left by A^{-1}, we find

$$B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}O = O.$$

So, given that A is invertible, either $B = O$ or $AB \neq O$.

3.) Let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear transformation from the real vector space V to itself. Let λ be a real number and consider the following subset of V:

$$\mathsf{W}_\lambda = \{v \in \mathsf{V} \mid \mathsf{T}(v) = \lambda v\}.$$

Prove that W_λ is a subspace of V.
Proof: We will apply the Subspace Test. First, since $\mathsf{T}(0) = 0 = \lambda 0$, we have $0 \in \mathsf{W}_\lambda$. To show closure under addition, let u and v be any vectors in W_λ. Then $\mathsf{T}(u) = \lambda u$ and $\mathsf{T}(v) = \lambda v$. So

$$\mathsf{T}(u+v) = \mathsf{T}(u) + \mathsf{T}(v) = \lambda u + \lambda v = \lambda(u+v)$$

using the fact that T is linear, showing that $u + v \in \mathsf{W}_\lambda$. Finally, we must establish that W_λ is closed under scalar multiplication. Let $u \in \mathsf{W}_\lambda$ and let c be any real number. Then

$$\mathsf{T}(cu) = c\mathsf{T}(u) = c(\lambda u) = \lambda(cu).$$

This proves that cu also lies in the eigenspace W_λ. So, by the Subspace Test, we conclude that W_λ is indeed a subspace of V.

4.) Parts (b), (d) and (f) of exercise #2 on p256-7.
(b) We are given the linear operator $\mathsf{T} : \mathsf{P}_1(\mathbb{R}) \to \mathsf{P}_1(\mathbb{R})$ with the rule

$$\mathsf{T}(a + bx) = (6a - 6b) + (12a - 11b)x.$$

The given basis is $\beta = \{3 + 4x,\ 2 + 3x\}$. Let us write $\beta = \{u_1, u_2\}$. We compute

$$\mathsf{T}(u_1) = \mathsf{T}(3 + 4x) = -6 - 8x = (-2)u_1 + 0u_2$$

and

$$\mathsf{T}(u_2) = \mathsf{T}(2 + 3x) = -6 - 9x = 0u_1 + (-3)u_2.$$

So

$$[\mathsf{T}]_\beta = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

and YES, β is a basis of eigenvectors for T because this is a diagonal matrix.

(d) We are given the linear operator $\mathsf{T} : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ with the rule

$$\mathsf{T}(a + bx + cx^2) = (-4a + 2b - 2c) + (-7a - 3b - 7c)x + (7a + b + 5c)x^2.$$

The given basis is $\beta = \{x - x^2,\ -1 + x^2,\ -1 - x + x^2\}$. Again, write $\beta = \{u_1, u_2, u_3\}$. We compute

$$\mathsf{T}(u_1) = \mathsf{T}(x - x^2) = 4 + 4x - 4x^2 = 0u_1 + 0u_2 + (-4)u_3,$$

$$\mathsf{T}(u_2) = \mathsf{T}(-1 + x^2) = 2 - 2x^2 = 0u_1 + (-2)u_2 + 0u_3$$

and

$$\mathsf{T}(u_3) = \mathsf{T}(-1 - x + x^2) = 3x - 3x^2 = 3u_1 + 0u_2 + 0u_3.$$

So

$$[\mathsf{T}]_\beta = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

and NO, β is NOT a basis of eigenvectors for T because this is not a diagonal matrix.

(f) We are given $\mathsf{T} : \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{M}_{2\times 2}(\mathbb{R})$ with the rule

$$\mathsf{T}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} (-7a-4b+4c-4d) & b \\ (-8a-4b+5c-4d) & d \end{bmatrix}.$$

The given basis is

$$\beta = \{u_1, \ldots\} := \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \right\}.$$

Now

$$\mathsf{T}(u_1) = \mathsf{T}\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix} = -3u_1,$$

$$\mathsf{T}(u_2) = \mathsf{T}\left(\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = u_2,$$

$$\mathsf{T}(u_3) = \mathsf{T}\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = u_3,$$

and

$$\mathsf{T}(u_4) = \mathsf{T}\left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = u_4.$$

So

$$[\mathsf{T}]_\beta = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and YES, β is a basis of eigenvectors for T.

5.) Complete parts (b) and (d) of exercise #3 on p257.
(b) We first compute the characteristic polynomial

$$\chi_A(t) = \begin{vmatrix} -t & -2 & -3 \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{vmatrix} = -t^3 + 6t^2 - 11t + 6 = (1-t)(2-t)(3-t).$$

So the eigenvalues are 1, 2 and 3. Since the eigenvalues are distinct, we will be successful in finding a basis of eigenvectors (Corollary p261).

(ii) We easily row reduce $A - \lambda I$ for $\lambda = 1, 2, 3$ to find eigenspaces

$$\mathsf{W}_1 = \mathsf{nullsp}\,(A - I) = \{(s, s, -s) \mid s \in \mathbb{R}\},$$

$$\mathsf{W}_2 = \mathsf{nullsp}\,(A - 2I) = \{(s, -s, 0) \mid s \in \mathbb{R}\},$$

$$\mathsf{W}_3 = \mathsf{nullsp}\,(A - 3I) = \{(s, 0, -s) \mid s \in \mathbb{R}\}$$

(iii) A basis of eigenvectors is given by the three columns of the matrix

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}.$$

(iv) The above matrix Q is invertible and $Q^{-1}AQ$ is the diagonal matrix D with diagonal $1, 2, 3$.

(d) We first compute the characteristic polynomial

$$\chi_A(t) = \begin{vmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{vmatrix} = -t^3 + 2t^2 - t = (1-t)(1-t)(-t).$$

So the eigenvalues are 0, 1 and 1.

(ii) We row reduce $A - \lambda I$ for $\lambda = 0, 1$ to find eigenspaces

$$\mathsf{W}_0 = \mathsf{nullsp}\,(A) = \mathsf{nullsp}\left(\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \{(s, 4s, 2s) \mid s \in \mathbb{R}\}$$

and

$$\mathsf{W}_1 = \mathsf{nullsp}\,(A - I) = \mathsf{nullsp}\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \{(s, s, s) \mid s \in \mathbb{R}\}.$$

(iii) A basis of eigenvectors does not exist because the second eigenspace is too small. is given by the three columns of the matrix

(iv) There are no such matrices Q and D.