Matrices & Linear Algebra II
C Term, Section C01
W. J. Martin
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Sample Solutions – Assignment 4

1.) (#4, p84) We are given $\mathsf{T} : \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a+b) + (2d)x + bx^2.$$

In order to find the matrix representing T with respect to the given bases, we simply compute $\mathsf{T}(E_{ij})$ for each matrix E_{ij} in basis β and express this as a linear combination of 1, x and x^2. We have

$$\begin{aligned}
\mathsf{T}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= 1\cdot 1 + 0\cdot x + 0\cdot x^2, \\
\mathsf{T}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= 1\cdot 1 + 0\cdot x + 1\cdot x^2, \\
\mathsf{T}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) &= 0\cdot 1 + 0\cdot x + 0\cdot x^2, \\
\mathsf{T}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0\cdot 1 + 2\cdot x + 0\cdot x^2.
\end{aligned}$$

So the four coordinate vectors are

$$[\mathsf{T}(E_{11})] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathsf{T}(E_{12})] = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathsf{T}(E_{11})] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathsf{T}(E_{11})] = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

giving

$$[\mathsf{T}]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

2.) Let $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation about the origin by 150 degrees counterclockwise. Find the matrix representing T

(a) with respect to the standard ordered basis β

We know that $\mathsf{T}(1,0) = (\cos\frac{5\pi}{6}, \sin\frac{5\pi}{6})$ and $\mathsf{T}(0,1) = (\cos\frac{8\pi}{6}, \sin\frac{8\pi}{6})$. So

$$[\mathsf{T}]_\beta = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

(b) with respect to the ordered basis $\gamma = \{u_1, u_2\} = \{(2,1), (1,-2)\}$.

We must first apply T to these vectors in the standard coordinate system:

$$[\mathsf{T}(u_1)]_\beta = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(-1-2\sqrt{3}) \\ \frac{1}{2}(2-\sqrt{3}) \end{bmatrix},$$

$$[\mathsf{T}(u_2)]_\beta = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(2-\sqrt{3}) \\ \frac{1}{2}(1+2\sqrt{3}) \end{bmatrix}.$$

Since

$$\mathsf{T}(u_1) = -\frac{\sqrt{3}}{2}u_1 + \frac{1}{2}u_2, \qquad \mathsf{T}(u_2) = -\frac{1}{2}u_1 - \frac{\sqrt{3}}{2}u_2,$$

we find

$$[\mathsf{T}]_\gamma = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

Another way to get to the same answer is to use the identity

$$[\mathsf{T}]_\gamma = Q^{-1}[\mathsf{T}]_\beta Q$$

where Q is the change-of-coordinates matrix which changes γ coordinates into β coordinates. Clearly $Q = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ and $Q^{-1} = \frac{1}{5}Q$ (since the columns of Q are orthogonal and have the same length). So we get the same result after multiplying the three matrices.

3.) In $\mathbb{R}^3$, let T be the linear transformation representing rotation about the axis joining the origin to $(1,1,0)$ by an angle of 30 degrees counterclockwise. (We have to define counterclockwise: say the viewer is at $(10,10,0)$ looking toward the origin.) Find the matrix representing T

(a) with respect to the ordered basis $\gamma = \{(\sqrt{2},\sqrt{2},0),(\sqrt{2},-\sqrt{2},0),(0,0,1)\}$.

First, scale the third basis vector u_3 so that u_2 and u_3' lie on the same circle — i.e., have the same length. So replace u_3 by $u_3' = (0,0,2)$. Next $\mathsf{T}(u_1) = u_1$ since u_1 lies on the axis of rotation. Now, after drawing a diagram, we see that T moves u_3' towards u_2 and moves u_2 away from u_3', each by an amount of $\pi/6$ (rotationally). Thus

$$\mathsf{T}(u_3') = \frac{\sqrt{3}}{2}u_3' + \frac{1}{2}u_2$$

and

$$\mathsf{T}(u_2) = \frac{\sqrt{3}}{2}u_2 - \frac{1}{2}u_3'.$$

Therefore, the matrix representing this transformation with respect to this basis is

$$[\mathsf{T}]_{\gamma'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}.$$

Back to the original problem:

Now, if $\mathsf{T}(u_2) = \frac{\sqrt{3}}{2}u_2 - \frac{1}{2}(0,0,2)$, then $\mathsf{T}(u_2) = \frac{\sqrt{3}}{2}u_2 - 1(0,0,1)$. Similarly, since we know

$$\mathsf{T}(0,0,2) = \frac{1}{2}u_2 + \frac{\sqrt{3}}{2}(0,0,2)$$

we use linearity to write

$$\mathsf{T}(0,0,1) = \frac{1}{4}u_2 + \frac{\sqrt{3}}{2}(0,0,1).$$

$$[\mathsf{T}]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/4 \\ 0 & -1 & \sqrt{3}/2 \end{bmatrix}.$$

(b) with respect to the standard ordered basis

We will do this by applying Theorem 2.23: If β denotes the standard ordered basis and Q is the change of coordinate basis that changes β coordinates into γ' coordinates, then

$$[\mathsf{T}]_\beta = Q^{-1}\,[\mathsf{T}]_{\gamma'}\,Q.$$

Of course, it is easier to write down Q^{-1} since β is the standard basis:

$$Q^{-1} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We row reduce the augmented matrix $[Q^{-1}|I_3]$ to find Q:

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & \sqrt{2} & 0 & 1 & 0 & 0 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \sqrt{2}/4 & \sqrt{2}/4 & 0 \\ 0 & 1 & 0 & \sqrt{2}/4 & -\sqrt{2}/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right].$$

So $Q = \frac{1}{4}Q^{-1}$ and

$$[\mathsf{T}]_\beta = Q^{-1}\,[\mathsf{T}]_{\gamma'}\,Q = \frac{1}{4}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 4+2\sqrt{3} & 4-2\sqrt{3} & 4\sqrt{2} \\ 4-2\sqrt{3} & 4+2\sqrt{3} & -4\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 4\sqrt{3} \end{bmatrix} = \begin{bmatrix} .9330 & .0670 & .3536 \\ .0670 & .9330 & -.3536 \\ -.3536 & .3536 & .8660 \end{bmatrix}.$$

4.) An engineer tracking stress in a steel beam observes that markers placed at time zero at positions $(4,4)$ and $(4,3)$ have moved to positions $(3.9,4.2)$ and $(3.95,3.1)$ respectively at time one (six months later). Assuming that this motion is a shear along some axis (and with respect to some origin), find an expression for the corresponding linear transformation T.

Define $x = (4,3)$ and $y = (4,4)$. Then $\mathsf{T}(x) = (3.95,3.1)$ and $\mathsf{T}(y) = (3.9,4.2)$. As proven in class, we may take as origin of our system the intersection of the two lines ℓ and

m where ℓ joins x to y and m joins $\mathsf{T}(x)$ to $\mathsf{T}(y)$. A simple calculation gives $\mathbf{0} = (4,2)$. The fixed axis is parallel to the line joining x to $\mathsf{T}(x)$. So we can take one basis vector $u = (-1,2)$ knowing that $\mathsf{T}(u) = u$. The second basis vector can be $v = (0,1)$.

Now with this origin and with this basis $\gamma = \{u, v\}$, we must describe the vectors of interest. First considering only the shift, we have

$$\bar{x} = (0,1), \quad \bar{y} = (0,2), \quad \overline{\mathsf{T}(x)} = (-.05, 1.1), \quad \overline{\mathsf{T}(y)} = (-.1, 2.2).$$

So the coordinate vectors are

$$[\bar{x}]_\gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [\overline{\mathsf{T}(x)}]_\gamma = \begin{bmatrix} .05 \\ 1 \end{bmatrix},$$

etc. So, noting that $v = \bar{x}$ in this shifted vector space, we have

$$[\mathsf{T}(u)]_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad [\mathsf{T}(v)]_\gamma = \begin{bmatrix} .05 \\ 1 \end{bmatrix}.$$

So

$$[\mathsf{T}]_\gamma = \begin{bmatrix} 1 & .05 \\ 0 & 1 \end{bmatrix}.$$

5.) Continuing with the scenario in Problem 4, Suppose now that the steel beam will fail when the point $(1,1)$ collides with the line $3x - y = 9$. At what time will this happen? Assume that every six months, the system evolution is described by the linear operator T.

In our shifted coordinate system, this point is $w = (-3,-1)$ and, with respect to the standard basis $\beta = \{(1,0),(0,1)\}$, we have

$$[\mathsf{T}]_\beta = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & .05 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} .9 & -.05 \\ .2 & 1.1 \end{bmatrix}.$$

We also express the given line in terms of these shifted coordinates as $y = 3x + 1$. So we have

$$[\mathsf{T}(w)]_\beta = [\mathsf{T}]_\beta [w]_\beta = \begin{bmatrix} .9 & -.05 \\ .2 & 1.1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2.65 \\ -1.7 \end{bmatrix}$$

and similarly

$$[\mathsf{T} \circ \mathsf{T}(w)]_\beta = [\mathsf{T}]_\beta [\mathsf{T}(w)]_\beta = \begin{bmatrix} .9 & -.05 \\ .2 & 1.1 \end{bmatrix} \begin{bmatrix} -2.65 \\ -1.7 \end{bmatrix} = \begin{bmatrix} -2.3 \\ -2.4 \end{bmatrix},$$

$$[\mathsf{T}^3(w)]_\beta = \begin{bmatrix} .9 & -.05 \\ .2 & 1.1 \end{bmatrix} \begin{bmatrix} -2.3 \\ -2.4 \end{bmatrix}, = \begin{bmatrix} -1.95 \\ -3.1 \end{bmatrix},$$

$$[\mathsf{T}^4(w)]_\beta = \begin{bmatrix} .9 & -.05 \\ .2 & 1.1 \end{bmatrix} \begin{bmatrix} -1.95 \\ -3.1 \end{bmatrix}, = \begin{bmatrix} -1.6 \\ -3.8 \end{bmatrix}.$$

This last vector satisfies $y = 3x + 1$ and therefore lies on the critical line. So the collision occurs after four time periods, i.e., after two years.