Matrices & Linear Algebra II
C Term, Section C01
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## Sample Solutions – Assignment 3

**1.)** #4 on p74.

We are given $\mathsf{T} : \mathsf{M}_{2\times 3}(\mathbb{F}) \to \mathsf{M}_{2\times 2}(\mathbb{F})$ and we must first show that $\mathsf{T}$ is a linear transformation. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be any two matrices in $\mathsf{M}_{2\times 3}(\mathbb{F})$. Then

$$A + B = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) & (a_{13}+b_{13}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) & (a_{23}+b_{23}) \end{bmatrix}$$

and

$$\mathsf{T}(A+B) = \begin{bmatrix} 2(a_{11}+b_{11}) - (a_{12}+b_{12}) & (a_{13}+b_{13}) + 2(a_{12}+b_{12}) \\ 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2b_{11} - b_{12} & b_{13} + 2b_{12} \\ 0 & 0 \end{bmatrix} = \mathsf{T}(A) + \mathsf{T}(B).$$

Similarly, for any scalar $c$ in $\mathbb{F}$,

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix}$$

and

$$\mathsf{T}(cA) = \begin{bmatrix} 2(ca_{11}) - (ca_{12}) & (ca_{13}) + 2(ca_{12}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} c(2a_{11} - a_{12}) & c(a_{13}) + 2a_{12}) \\ c0 & c0 \end{bmatrix} = c\mathsf{T}(A).$$

Therefore $\mathsf{T}$ is linear.

Next, we must find bases for the null space and range of $\mathsf{T}$. We can do this by looking at the equation $\mathsf{T}(A) = \mathbf{0}_{2\times 2}$. This gives us a homogeneous linear system in the six unknowns $a_{ij}$:

$$\begin{aligned} 2a_{11} - a_{12} \qquad &= 0 \\ 2a_{12} + a_{13} &= 0 \end{aligned}$$

with the three entries on the second row being completely free. We row reduce this system to find the equivalent linear system

$$\begin{aligned} a_{11} \qquad + \frac{1}{4}a_{13} &= 0 \\ a_{12} + \frac{1}{2}a_{13} &= 0 \end{aligned}$$

So the nullity is four and a basis for the null space is

$$\beta = \left\{ \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

The image (or range) of $\mathsf{T}$ is clearly

$$\operatorname{im}\mathsf{T} = \left\{ \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \mid c, d \in \mathbb{F} \right\}$$

since the second row of any $\mathsf{T}(A)$ is always zero and we can obtain first row $(c, d)$ by taking $a_{11} = c/2$, $a_{12} = 0$, $a_{13} = d$. Thus the rank of $\mathsf{T}$ is two confirming the Dimension Theorem: $4 + 2 = 6$.

Finally, we note that $\mathsf{T}$ is neither one-to-one nor onto. By Theorem 2.4, we conclude $\mathsf{T}$ fails to be one-to-one since it has a non-trivial null space. Also, the codomain, $\mathsf{M}_{2\times 2}(\mathbb{F})$, has dimension four and $\mathsf{T}$ has rank only two, so $\mathsf{T}$ is not onto.

**2.)** #9 on p75.

**(a)** FAILS $\mathsf{T}(a+b) = \mathsf{T}(a) + \mathsf{T}(b)$. Let $a = b = (0, 0)$. Then

$$\mathsf{T}(a+b) = \mathsf{T}(a) = \mathsf{T}(b) = (1, 0) \neq (2, 0) = \mathsf{T}(a) + \mathsf{T}(b).$$

**(b)** FAILS $\mathsf{T}(a+b) = \mathsf{T}(a) + \mathsf{T}(b)$. Let $a = b = (3, 0)$. Then

$$\mathsf{T}(a+b) = \mathsf{T}(6, 0) = (6, 36) \neq (6, 18) = (3, 9) + (3, 9) = \mathsf{T}(a) + \mathsf{T}(b).$$

**(c)** FAILS $\mathsf{T}(a+b) = \mathsf{T}(a) + \mathsf{T}(b)$. Let $a = b = (\pi/2, 0)$. Then

$$\mathsf{T}(a+b) = \mathsf{T}(\pi, 0) = (0, 0) \neq (2, 0) = (1, 0) + (1, 0) = \mathsf{T}(a) + \mathsf{T}(b).$$

**(d)** FAILS $\mathsf{T}(a+b) = \mathsf{T}(a) + \mathsf{T}(b)$. Let $a = (1, 0)$, $b = (-1, 0)$. Then

$$\mathsf{T}(a+b) = \mathsf{T}(0, 0) = (0, 0) \neq (2, 0) = (1, 0) + (1, 0) = \mathsf{T}(a) + \mathsf{T}(b).$$

**(e)** FAILS $\mathsf{T}(a+b) = \mathsf{T}(a) + \mathsf{T}(b)$. Let $a = b = (0, 0)$. Then

$$\mathsf{T}(a+b) = \mathsf{T}(a) = \mathsf{T}(b) = (1, 0) \neq (2, 0) = \mathsf{T}(a) + \mathsf{T}(b).$$

[NOTE: Observe that each of these functions fails to satisfy $\mathsf{T}(ca) = c\mathsf{T}(a)$ as well. For example, in parts (a), (b), (c) and (e), take $c = 2$ and $a$ as given above and in part (d) take $c = -1$. However, only one failure is enough to prove that $\mathsf{T}$ is not linear.]

**3.)** #24 on p76-7.

**(a)** The formula is $\mathsf{T}(a, b) = (0, b)$. The image of any $x$ clearly lies on the $y$-axis and the line joining $x$ to $\mathsf{T}(x)$ is always parallel to the $x$-axis. The diagram would have an example, say $x = (3, 4)$, and its image, together with arrows pointing horizontally toward the $x$-axis from both directions.

**(b)** The formula is $\mathsf{T}(a, b) = (0, b - a)$. The image of any $x$ clearly lies on the $y$-axis and the line joining $x$ to $\mathsf{T}(x)$ always has slope one (add $a$ to each coordinate of $\mathsf{T}(x)$ to get back to $x$). The diagram would have an example, say $x = (3, 4)$ and its image $(0, 1)$, together with arrows pointing diagonally toward the $x$-axis from both directions.

**4.)** Consider $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathsf{T}(a_1, a_2) = (2a_2, 0, 3a_1 + a_2)$. Complete #3 on p84 using this linear transformation $\mathsf{T}$ in place of the one given there.

The function $\mathsf{T}$ maps the standard basis vectors to

$$(0, 0, 3) = (-2) \cdot (1, 1, 0) + 0 \cdot (0, 1, 1) + 1 \cdot (2, 2, 3)$$

and

$$(2, 0, 1) = 0 \cdot (1, 1, 0) + (-2) \cdot (0, 1, 1) + 1 \cdot (2, 2, 3).$$

We record these coefficients as the two columns of our matrix:

$$[\mathsf{T}]_\beta^\gamma = \begin{bmatrix} -2 & 0 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}.$$

The function $\mathsf{T}$ maps the vectors in $\alpha = \{(1, 2),\ (2, 3)\}$ to

$$(4, 0, 5) = (-2) \cdot (1, 1, 0) + (-4) \cdot (0, 1, 1) + 3 \cdot (2, 2, 3)$$

and

$$(6, 0, 9) = (-4) \cdot (1, 1, 0) + (-6) \cdot (0, 1, 1) + 5 \cdot (2, 2, 3).$$

We use these coefficients to obtain our second matrix representing $\mathsf{T}$:

$$[\mathsf{T}]_\alpha^\gamma = \begin{bmatrix} -2 & -4 \\ -4 & -6 \\ 3 & 5 \end{bmatrix}.$$

**5.)** #9 on p85.

We must prove that, when $\mathbb{C}$ is viewed as a two-dimensional real vector space, the map $z \mapsto \bar{z}$ is a linear transformation. Let $z = a + bi$ and $w = c + di$. Then $z + w = (a + c) + (b + d)i$ and we have

$$\overline{z + w} = (a + c) - (b + d)i = (a - bi) + (c - di) = \bar{z} + \bar{w}$$

and, for any real number $r$, $rz = (ra) + (rb)i$ with both $ra$ and $rb$ being real numbers so that

$$\overline{rz} = (ra) - (rb)i = r(a - bi) = r\bar{z}.$$

These two results prove that $\mathsf{T}$ is linear. Now we construct the matrix representing $\mathsf{T}$ with respect to the standard ordered basis $\beta = \{1, i\}$: $\mathsf{T}(1) = 1 = 1 \cdot 1 + 0 \cdot i$ and $\mathsf{T}(i) = -i = 0 \cdot 1 + (-1) \cdot i$ giving

$$[\mathsf{T}]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$