Matrices & Linear Algebra II
C Term, Section C01
W. J. Martin
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Sample Solutions – Assignment 1

1.) The given matrix and its unique reduced row echelon form are

$$A = \begin{bmatrix} 2 & -1 & -4 & 0 & 4 & 3 \\ -1 & 0 & 1 & 3 & 1 & -3 \\ 0 & 2 & 4 & 1 & 1 & -7 \\ 1 & 1 & 1 & 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the nullspace of A.
The solution set to the corresponding homogeneous linear system, therefore, is given by

$$\begin{aligned} x_1 &= r - 2s \\ x_2 &= -2r + 3t \\ x_3 &= r \\ x_4 &= -s + t \\ x_5 &= s \\ x_6 &= t \end{aligned}$$

where r, s and t can be any real numbers. A basis for this solution space can then be obtained by taking each free parameter equal to one and setting the others to zero:

$$\beta = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for the row space of A.
For this, we simply grab the non-zero rows in the reduced row echelon form:

$$\beta = \left\{ \begin{bmatrix} 1 & 0 & -1 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \right\}.$$

(c) Find a basis for the column space of A. Show your work.
Such a basis can be found by locating the pivots (leading ones) in the rref and taking the corresponding columns (in this case, columns 1, 2 and 4) of the original matrix:

$$\beta = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

2.) Let A be an $m \times n$ matrix with real entries.

(a) The nullspace of A is a subspace of $\mathbb{R}^n$.

Proof: We apply the Subspace Test. First let

$$\mathsf{W} = \mathsf{nullspace}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

and observe that $0 \in \mathsf{W}$ since $A0 = 0$. Next assume that x and y belong to W. Then $Ax = 0$ and $Ay = 0$. So

$$A(x + y) = Ax + Ay = 0 + 0 = 0$$

showing that $x + y$ also belongs to W. Finally, suppose x belongs to W and c is any scalar. Then $Ax = 0$, so

$$A(cx) = c(Ax) = c0 = 0$$

showing that cx also belongs to W. This completes the proof.

(b) The row space of A is a subspace of $\mathbb{R}^n$.

Proof: We apply the Subspace Test. First let

$$\mathsf{W} = \mathsf{rowspace}(A) = \{y^t A : y \in \mathbb{R}^m\}$$

and observe that $0 \in \mathsf{W}$ since $0^t A = 0$. Next assume that x and w belong to W. Then there exist y and z in $\mathbb{R}^m$ satisfying $y^t A = x$ and $z^t A = w$. So

$$(y + z)^t A = (y^t + z^t)A = (y^t A) + (z^t A) = x + w$$

showing that $x + w$ also belongs to W. Finally, suppose x belongs to W and c is any scalar. Then $y^t A = x$ for some y and

$$(cy)^t A = c(y^t A) = cx$$

showing that cx also belongs to W. So W is indeed a subspace.

(c) Prove that the column space of A is a subspace of $\mathbb{R}^m$.

Proof: We again apply the Subspace Test. First let

$$\mathsf{W} = \mathsf{colspace}(A) = \{Ax : x \in \mathbb{R}^n\}$$

and observe that $0 \in \mathsf{W}$ since $A0 = 0$. Next assume that z and w belong to W. Then there exist x and y in $\mathbb{R}^n$ satisfying $Ax = z$ and $Ay = w$. So

$$A(x + y) = Ax + Ay = z + w$$

proving that $z + w$ belongs to W. Now suppose z belongs to W and c is any scalar. Then $Ax = z$ for some x and

$$A(cx) = c(Ax) = cz$$

showing that cz belongs to W.

3.) (#19,p16) This is **NOT** a vector space. Let $x = (1, 1)$, let $a = 2$ and let $b = 3$. Then

$$(a + b) \odot x = 5 \odot (1, 1) = (5, \frac{1}{5}).$$

But

$$(a \odot x) \oplus (b \odot x) = (2 \odot (1,1)) + (3 \odot (1,1)) = (2, \frac{1}{2}) + (3, \frac{1}{3}) = (5, \frac{5}{6}).$$

So axiom (VS 8) fails — $(a+b)x \neq ax + bx$ —- and V is not a vector space.

4.) (#21, p16)

Proof: We simply verify the eight axioms for Z, knowing that the corresponding rules hold for each of the component vector spaces V and W:
(VS 1): Let $x = (x_v, x_w)$ and $y = (y_v, y_w)$ be any two vectors in Z. Then

$$x + y = (x_v, x_w) + (y_v, y_w) = (x_v + y_v, x_w + y_w) = (y_v + x_v, y_w + x_w) = y + x$$

using the same property in each component.
(VS 2): Let $x = (x_v, x_w)$, $y = (y_v, y_w)$ and $z = (z_v, z_w)$ be any vectors in Z. Then

$$(x+y)+z = [(x_v, x_w) + (y_v, y_w)] + (z_v, z_w) = (x_v + y_v, x_w + y_w) + (z_v, z_w) =$$

$$((x_v + y_v) + z_v, (x_w + y_w) + z_w) = (x_v + (y_v + z_v), x_w + (y_w + z_w)) =$$

$$(x_v, x_w) + (y_v + z_v, y_w + z_w) = (x_v, x_w) = [(y_v, y_w) + (z_v, z_w)] = x + (y+z).$$

(VS 3): If 0_v is the zero vector in V and 0_w is the zero vector in W, then our zero vector should be $0 = (0_v, 0_w)$. Indeed, for any $x = (x_v, x_w)$,

$$0 + x = (0_v, 0_w) + (x_v, x_w) = (0_v + x_v, 0_w + x_w) = (x_v, x_w) = x$$

and this is the same as $x + 0$ using (VS 1).
(VS 4): Take any $x = (x_v, x_w)$. If $-x_v$ is the inverse of x_v in V and $-x_w$ is the inverse of x_w in W, then define $-x = (-x_v, -x_w)$ so that

$$x + (-x) = (x_v + (-x_v), x_w + (-x_w)) = (0_v, 0_w) = 0.$$

(VS 5): Take any $x = (x_v, x_w)$. By definition,

$$1x = (1x_v, 1x_w) = (x_v, x_w) = x.$$

(VS 6): Let $x = (x_v, x_w)$ and let a and b be any scalars. Then

$$(ab)x = ab(x_v, x_w) = ((ab)x_v, (ab)x_w) = (a(bx_v), a(bx_w)) =$$

$$a(bx_v, bx_w) = a[b(x_v, x_w)] = a[bx]$$

(VS 7): Let $x = (x_v, x_w)$, $y = (y_v, y_w)$ and let a be any scalar. Then

$$a(x+y) = a[(x_v, x_w) + (y_v, y_w)] = a(x_v + y_v, x_w + y_w) =$$

$$(a(x_v + y_v), a(x_w + y_w)) = (ax_v + ay_v, ax_w + ay_w) = (ax_v, ax_w) + (ay_v, ay_w) =$$

$$a(x_v, x_w) + a(y_v, y_w) = ax + ay.$$

(VS 8): Let $x = (x_v, x_w)$ and let a and b be any scalars. Then

$$(a+b)x = (a+b)(x_v, x_w) = ((a+b)x_v, (a+b)x_w) =$$

$$(ax_v + bx_v, ax_w + bx_w) = (ax_v, ax_w) + (bx_v, bx_w) = a(x_v, x_w) + b(x_v, x_w)$$

$$= ax + bx.$$

5.) Consider the vector space over $\mathbb{R}$ with vectors

$$\mathsf{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \quad x, y > 0 \right\}$$

and operations

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} xz \\ yw \end{bmatrix}, \qquad a \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^a \\ y^a \end{bmatrix}.$$

Find a bijective linear transformation L from V to the vector space $\mathbb{R}^2$.

Consider the function

$$\mathsf{L} : \mathsf{V} \to \mathbb{R}^2$$

given by

$$\mathsf{L}\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \ln x \\ \ln y \end{bmatrix}.$$

Then L is indeed defined for all vectors in V since these tuples have positive entries only. Moreover,

$$\mathsf{L}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} \right) = \mathsf{L}\left(\begin{bmatrix} xz \\ yw \end{bmatrix} \right) = \begin{bmatrix} \ln(xz) \\ \ln(yw) \end{bmatrix}$$

$$= \begin{bmatrix} \ln x + \ln z \\ \ln y + \ln w \end{bmatrix} = \begin{bmatrix} \ln x \\ \ln y \end{bmatrix} + \begin{bmatrix} \ln z \\ \ln w \end{bmatrix} = \mathsf{L}\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) + \mathsf{L}\left(\begin{bmatrix} z \\ w \end{bmatrix} \right)$$

and

$$\mathsf{L}\left(a \begin{bmatrix} x \\ y \end{bmatrix} \right) = \mathsf{L}\left(\begin{bmatrix} x^a \\ y^a \end{bmatrix} \right) = \begin{bmatrix} \ln(x^a) \\ \ln(y^a) \end{bmatrix}$$

$$= \begin{bmatrix} a \ln x \\ a \ln y \end{bmatrix} = a \begin{bmatrix} \ln x \\ \ln y \end{bmatrix} = a\mathsf{L}\left(\begin{bmatrix} x \\ y \end{bmatrix} \right).$$

This shows that L is a linear transformation. Now let

$$\begin{bmatrix} s \\ t \end{bmatrix}$$

be any vector in $\mathbb{R}^2$ (i.e., s and t are any real numbers). Then let $x = \exp(s)$ and let $y = \exp(t)$. Since $\ln x = s$ and $\ln y = t$ and the logarithm is one-to-one, there is precisely one vector $v \in \mathsf{V}$ with $\mathsf{L}(v) = [s, t]^t$, namely $v = [x, y]^t$. Therefore, L is a bijection, a vector space *isomorphism*.