Cometric Association Schemes

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UVM Combo Seminar
Outline

Motivating example
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Definitions and Parameters

   Introduction
   The Bannai/Ito Conjectures
   The known examples
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Structure of Imprimitive Cometric Schemes
   Dismantlability
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Structure of Imprimitive Cometric Schemes
  Dismantlability

Recent Work
  The Ideal
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  Proof
The Leech lattice

- even unimodular lattice in $\mathbb{R}^{24}$
- kissing number 196,560 (optimal)
- automorphism group is a double cover of $Co_1$ (order $2^9 \cdot 8,315,553,613,086,720,000$)

We will focus on the spherical code consisting of the 196,560 (scaled) shortest vectors.
Shortest vectors

The 196,560 norm two vectors:

- \( \frac{1}{\sqrt{8}} (\pm 2^8, 0^{16}) \) — support is also support of a Golay codeword, even num of \(-\) signs \((2^7 \cdot 759)\)
- \( \frac{1}{\sqrt{8}} (\mp 3, \pm 1^{23}) \) — upper signs taken on support of a Golay codeword \((2^{12} \cdot 24)\)
- \( \frac{1}{\sqrt{8}} (\pm 4^2, 0^{22}) \) — any two positions, any signs \((\binom{24}{2} \cdot 2^2)\)

Scale these to unit vectors to get \( X \subset S^{23} \). Among these vectors, there are only 6 non-zero angles. These determine six highly symmetric graphs on \( X \).
Association Schemes

A (symmetric) association scheme consists of a set \( \{A_0, \ldots, A_d\} \) of symmetric 01-matrices with

- \( A_0 = I \)
- \( \sum_i A_i = J \) (the all-ones matrix)
- \( A_i A_j \) is a linear combination of \( A_0, \ldots, A_d \)

Rows and columns are indexed by base set \( X \) of size \( v \).
Bose-Mesner algebra

\[ \mathcal{A} = \text{span}\{A_0, \ldots, A_d\} \]

is a commutative semisimple matrix algebra containing \( I \). It is also closed under entrywise multiplication \( \circ \) (also called “Schur mult.” or “Hadamard mult.”) and contains the identity \( J \) for this multiplication.

Second basis of minimal idempotents:

\[ \mathcal{A} = \text{span}\{E_0, \ldots, E_d\} \]
Orthogonality relations

\[
A_i = \sum_{j=0}^{d} P_{ji} E_j \quad \quad \quad E_j = \frac{1}{v} \sum_{i=0}^{d} Q_{ij} A_i
\]

The change-of-basis matrices \( P \) and \( Q \) are called the “first and second eigenmatrices” of the scheme. A scaled version of \( P \) is called the “character table”:

\[
PQ = vl
\]

\[
MP = Q^\top K
\]

where \( M \) is a diagonal matrix of multiplicities \( m_j = \text{rank } E_j \) and \( K \) is a diagonal matrix of valencies \( v_i = \text{rowsum } A_i \).
A taste of duality

\[ A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \quad E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q_{ij}^k E_k \]

\[ A_i \circ A_j = \delta_{ij} A_i \quad E_i E_j = \delta_{ij} E_i \]

\[ A_i E_j = P_{ji} E_j \quad A_i \circ E_j = \frac{1}{v} Q_{ij} A_i \]

\[ \sum_{i=0}^{d} A_i = J \quad \sum_{j=0}^{d} E_j = I \]

\[ A_0 = I \quad E_0 = \frac{1}{v} J \]
Metric and Cometric Schemes

The scheme is *metric* (or *P-polynomial*) if there is an ordering of the $A_i$ for which

- $p_{ij}^k = 0$ whenever $k > i + j$
- $p_{ij}^{i+j} > 0$ whenever $i + j \leq d$

The scheme is *cometric* (or *Q-polynomial*) if there is an ordering of the $E_j$ for which

- $q_{ij}^k = 0$ whenever $k > i + j$
- $q_{ij}^{i+j} > 0$ whenever $i + j \leq d$
Distance-Regular Graphs

- For a graph $G$, how big can $\text{Aut}(G)$ be?
- Does combinatorial regularity imply the presence of a group?
- Includes graphs fundamental to coding theory: $H(n, q)$, $J(n, k)$, coset graphs of Golay codes, and more
- Massive effort under way to classify all metric-cometric schemes:
  - (Leonard, 1982) all parameters $P_{ji}$ are given by evaluations of $q$-Racah polynomials and their limiting cases
  - (Terwilliger, et al., 1990-present) aim to classify all irreducible modules for the subconstituent algebra
  - (many authors) prove uniqueness of known families based on parameters alone, or together with local information
- Small diameter examples plentiful, not well-structured (strongly regular graphs, Hadamard matrices, projective planes, etc.)
Distance-Regular Graphs

We have a rough classification of distance-regular graphs into:

- primitive (all distance-$i$ graphs are connected)
- bipartite (distance $i$ disconnected for $i$ even)
- antipodal (distance $d$ – max. distance – graph disconn.)
- both bipartite and antipodal (e.g., cubes)
The Conjectures of Bannai and Ito

Conjecture (Bannai & Ito)

For each \( k > 2 \), there are only finitely distance-regular graphs of valency \( v_1 = k \).
The Conjectures of Bannai and Ito

Conjecture (Bannai & Ito)

For each \( k > 2 \), there are only finitely distance-regular graphs of valency \( v_1 = k \).

Let \( V_j = \text{colsp}E_j \) denote the \( j^{\text{th}} \) eigenspace of the cometric scheme.

Conjecture (unpublished)

For each \( m > 2 \), there are only finitely many cometric association schemes (up to isomorphism) with \( \dim V_1 = m \).
The Conjectures of Bannai and Ito

Conjecture (Bannai & Ito)

Every primitive cometric scheme of sufficiently large diameter $d$ is metric as well.
The Conjectures of Bannai and Ito

Conjecture (Bannai & Ito)

The multiplicities $m_0, m_1, \ldots, m_d$ of a cometric association scheme, given by $m_j = \dim V_j$ form a unimodal sequence:

$$m_0 < m_1 \leq m_2 \leq \cdots \leq m_r \geq m_{r+1} \geq \cdots \geq m_d.$$

Conjecture (D. Stanton)

For $j < d/2$,

$$m_j \leq m_{j+1}, \quad m_j \leq m_{d-j}.$$

Theorem (Caughman & Sagan, 2001)

If $(X, A)$ is also “dual thin”, then Stanton’s conjecture holds.
Spherical Designs

**Spherical $t$-Design:** Finite subset $X \subset S^{m-1}$ for which

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{m-1}|} \int f(x) dx$$

for all polynomials $f$ in $m$ variables of total degree at most $t$.

**Example:** The 196,560 shortest vectors of the Leech lattice form a spherical 11-design in $\mathbb{R}^{24}$.

**Seymour and Zaslavsky (1984):** Such finite sets $X$ exist for all $t$ in each dimension $m$. 
Cometric schemes from spherical designs

Theorem (Delsarte, Goethals, Seidel (1977))

The number $s$ of non-zero angles in a spherical $t$-design is at least $t/2$. If $t \geq 2s - 2$, then $X$ carries a cometric association scheme.

Examples: 24-cell ($t = 5, s = 4$); $E_6$ ($t = 5, s = 4$); $E_8$ ($t = 7, s = 4$); Leech ($t = 11, s = 6$).
Cometric schemes from combinatorial designs

**Defn:** A *Delsarte t-design* in a cometric scheme $(X, A)$ is any non-trivial subset $Y$ of $X$ whose characteristic vector $\chi_Y$ is orthogonal to $V_1, \ldots, V_t$.

**Examples:** orthogonal arrays ("dual codes"), block designs.

**Theorem (Delsarte (1973))**

*If s non-zero relations occur among pairs of elements of $Y$, then $t \leq 2s$. If $t \geq 2s - 2$, then $Y$ carries a cometric association scheme.*
Cometric schemes from semilattices

**Defn:** The *dual width* $w^*$ of $Y \subseteq X$ is the maximum $j$ in the $Q$-polynomial ordering for which $E_j \chi_Y \neq 0$.

**Theorem (Brouwer, Godsil, Koolen, WJM (2003))**

For any $Y$ in a $d$-class cometric scheme, $w^* \geq d - s$. If equality holds, then $Y$ carries a cometric association scheme.
Group schemes

Every finite group $G$ yields an association scheme via the center of the group algebra of its right regular representation $g \mapsto R_g$.

**Conjugacy classes:** $C_0 = \{e\}, C_1, \ldots, C_n$

$$A_i = \sum_{g \in C_i} R_g$$

**Extended conjugacy classes:** $C'_0 = \{e\}, C'_i = C_i \cup (C_i)^{-1}$

Symmetrized scheme:

$$A_i = \sum_{g \in C'_i} R_g$$
Cometric group schemes

Theorem (Kiyota and Suzuki (2000))

The symmetrized group scheme is cometric if and only if \( G \) is one of the following groups:

- \( \mathbb{Z}_n \)
- \( S_3 \)
- \( A_4 \)
- \( SL(2, 3) \)
- \( F_{21} = \mathbb{Z}_7 \rtimes \mathbb{Z}_3 \)
A Census

The following cometric association schemes are known:

- $Q$-polynomial distance-regular graphs (i.e., metric and cometric)
- duals of metric translation schemes
- bipartite doubles of Hermitian forms dual polar spaces $[^2 A_{2d-1}(r)]$ (Bannai & Ito)
- schemes arising from linked systems of symmetric designs (3-class, $Q$-antipodal) [Cameron & Seidel]
- extended $Q$-bipartite doubles of linked systems (4-class, $Q$-bipartite and $Q$-antipodal) [Muzychuk, Williford, WJM]
- real MUBS [Bannai & Bannai]
Census

census of cometric schemes, continued:

- the block schemes of the Witt designs 4-(11,5,1), 5-(24,8,1) and a 4-(47,11,8) design (Delsarte) (primitive 3-class schemes on 66, 759 and 4324 vertices respectively)

- the block schemes of the 5-(12,6,1) design and the 5-(24,12,48) design (Q-bipartite 4-class schemes on 132 and 2576 vertices, respectively)

- shortest vectors in lattices $E_6$, $E_7$, $E_8$ (4-class, Q-bipartite)

- the scheme on the vertices of the 24-cell (4-class, Q-bipartite, Q-antipodal, 24 vertices)
Census

- the scheme on the shortest vectors in the Leech lattice (6-class, $Q$-bipartite, 196560 vertices)
- 5 schemes arising from derived designs of this:

<table>
<thead>
<tr>
<th>Class</th>
<th>Vertices</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-class</td>
<td>2025</td>
<td>primitive</td>
</tr>
<tr>
<td>4-class</td>
<td>2816</td>
<td>$Q$-bipartite</td>
</tr>
<tr>
<td>4-class</td>
<td>4600</td>
<td>$Q$-bipartite</td>
</tr>
<tr>
<td>4-class</td>
<td>7128</td>
<td>primitive</td>
</tr>
<tr>
<td>5-class</td>
<td>47104</td>
<td>primitive</td>
</tr>
</tbody>
</table>

- $Q$-bipartite quotient of Leech lattice example (3-class, primitive)
- three more schemes arising from lattices (4-, 5-, 11-class, $Q$-bipartite)
- three schemes from dismantling dual schemes of metric translation schemes (4-, 5-, and 6-class, all $Q$-antipodal)
An association scheme is *imprimitive* if there is a subset $A_{i_0}, A_{i_1}, \ldots, A_{i_e}$ of the associate matrices $A_i$ satisfying

$$\sum_h A_{i_h} = I_w \otimes J_r$$

for some $1 < w, r < v$.

Any imprimitive distance-regular graph is either bipartite or antipodal or both.
Imprimitivity

Theorem (Suzuki, 1998)

Any imprimitive cometric association scheme is either $Q$-bipartite or $Q$-antipodal or both, with possible exceptions if the number of classes is four or six.

Theorem (Cerzo and Suzuki, 2006)

The exception with $d = 4$ does not occur.
Q-bipartite Schemes

Today, let me simply say that these correspond to very symmetric sets of lines through the origin in $\mathbb{R}^m$. 
Q-antipodal Structure

- Gardiner, 1970s: a $P$-antipodal scheme has $r \leq k$
Q-antipodal Structure

- **Gardiner, 1970s**: a $P$-antipodal scheme has $r \leq k$
- **Theorem**: A $Q$-antipodal scheme with $d$ odd has $w \leq m_1$ ($d$ even case not complete)
**Q-antipodal Structure**

- **Gardiner, 1970s:** a \( P \)-antipodal scheme has \( r \leq k \)
- **Theorem:** A \( Q \)-antipodal scheme with \( d \) odd has \( w \leq m_1 \) (\( d \) even case not complete)
- With natural ordering, \( Q_{0d} = Q_{2d} = \cdots = m_d \) and \( Q_{1d} = Q_{3d} = \cdots = -1 \)
- \( p_{ij}^k = 0 \) unless \( i + j + k \) is even or \( ijk \) odd.
Theorem (Muzychuk, Williford, WJM (2007))

Every $Q$-antipodal scheme is dismantlable:
the subscheme induced on any non-trivial collection of $w'$ $Q$-antipodal classes is cometric for $w' \geq 1$ and $Q$-antipodal with $d$ classes for $w' > 1$. 
Dismantlability

Y_1

Y_2

Y_3

Y_4
Dismantlability

\[ Y_4 \quad Y_2 \quad Y_3 \]
Trivial cases

- halved graph of a bipartite $Q$-polynomial distance-regular graph
- linked systems of symmetric designs (by defn.)

Corollary (van Dam)

Every $Q$-antipodal 3-class cometric association scheme arises from a linked system of symmetric designs.
A new example via dismantling

Coset graph of the shortened ternary Golay code:

- intersection array \( \{20, 18, 4, 1; 1, 2, 18, 20\} \)
A new example via dismantling

Coset graph of the shortened ternary Golay code:

- intersection array \( \{20, 18, 4, 1; 1, 2, 18, 20\} \)
- antipodal distance-regular graph belonging to a translation scheme
A new example via dismantling

Coset graph of the shortened ternary Golay code:
- intersection array \{20, 18, 4, 1; 1, 2, 18, 20\}
- antipodal distance-regular graph belonging to a translation scheme
- dual association scheme is $Q$-antipodal on $\nu = 243$ vertices with $w = 3$ $Q$-antipodal classes
A new example via dismantling

Coset graph of the shortened ternary Golay code:

- intersection array \{20, 18, 4, 1; 1, 2, 18, 20\}
- antipodal distance-regular graph belonging to a translation scheme
- dual association scheme is $Q$-antipodal on $v = 243$ vertices with $w = 3$ $Q$-antipodal classes
- Remove one of these to obtain a $Q$-antipodal scheme on 162 vertices having $w = 2$ $Q$-antipodal classes which is not metric
A new example via dismantling

Coset graph of the shortened ternary Golay code:

- intersection array \{20, 18, 4, 1; 1, 2, 18, 20\}
- antipodal distance-regular graph belonging to a translation scheme
- dual association scheme is $Q$-antipodal on $v = 243$ vertices with $w = 3$ $Q$-antipodal classes
- Remove one of these to obtain a $Q$-antipodal scheme on 162 vertices having $w = 2$ $Q$-antipodal classes which is not metric
- parameters

\[ d = 4, \ n = 162, \ i^*(X, A) = \{20, 18, 3, 1; 1, 3, 18, 20\}\]

formally dual to those of an unknown diameter four bipartite distance-regular graph.
Dismantling the dual of a coset graph

- Two more distance-regular coset graphs yield $Q$-antipodal schemes with five and six classes.
Dismantling the dual of a coset graph

- Two more distance-regular coset graphs yield $Q$-antipodal schemes with five and six classes.
- Parameters

  \[ \begin{array}{c}
  \begin{aligned}
  d &= 5, \quad v = 486, \\
  \iota^* (X, A) &= \{22, 20, \frac{27}{2}, 2, 1; 1, 2, \frac{27}{2}, 20, 22\}, \quad w = 2
  \end{aligned}
  \\
  d &= 6, \quad v = 1536, \\
  \iota^* (X, A) &= \{21, 20, 16, 8, 2, 1; 1, 2, 4, 16, 20, 21\}, \quad w = 3.
  \end{array} \]

This last scheme is formally dual to a distance-regular graph which was proven not to exist by Brouwer, Cohen and Neumaier.
Dismantling the dual of a coset graph

- Two more distance-regular coset graphs yield $Q$-antipodal schemes with five and six classes.
- Parameters
  
  $d = 5, \; v = 486,$
  
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  $d = 6, \; v = 1536,$
  
  $i^*(X, A) = \{21, 20, 16, 8, 2, 1; 1, 2, 4, 16, 20, 21\}, \; w = 3.$

- This last scheme is formally dual to a distance-regular graph which was proven not to exist by Brouwer, Cohen and Neumaier.
Building Up?

- **Theorem (M,W,M):** The first multiplicity does not depend on $w$
- **Question:** Which $Q$-polynomial bipartite drg’s can be extended to $w > 2$?
- Works for 2-cube, 3-cube, 4-cube
An elementary ring homomorphism

Let \((X, A)\) be a cometric association scheme on \(v\) vertices with first primitive idempotent \(E_1\).
Let \(\gamma : \mathbb{R}[Z_1, \ldots, Z_v] \to \mathbb{R}^X\) via

\[
Z_a \mapsto \bar{a}
\]

(the \(a\)-column of \(E1\)) and extending linearly and via the Schur product \(\circ\).
E.g., \(Z_a Z_b^2 - 3Z_a \mapsto (\bar{a} \circ \bar{b} \circ \bar{b}) - 3\bar{a}\)

We are interested in \(I = \ker \gamma\).
The ideal of a cometric scheme

**Easy to prove:** $I$ contains exactly those polynomials in $v$ variables which vanish on each column of $E_1$. 
Simpler viewpoint for computation

Our ideal can be generated by $\nu - m_1$ linearly independent polynomials all of degree one, together with higher degree polynomials in $m_1$ variables representing functions on the eigenspace $V_1$. 
Example

For the $m$-cube, $V_1$ has dimension $m$ and our ideal is generated by

$$ (Y_i + Y_j)(Y_i - Y_j) $$

$$ \sum_i Y_i^2 - 1 $$

and some linear polynomials.
Strange example

For the $n$-gon, $V_1$ has dimension 2 and our ideal is generated by

$$X^2 + Y^2 - 1$$

and

$$(X - 1)(X - \eta_1) \cdots (X - \eta_{\lfloor n/2 \rfloor})$$

where $\eta_k = \cos(2\pi k/n)$. 
Example

For the $E_8$ association scheme (240 vertices in $\mathbb{R}^8$), our ideal is generated by polynomials of degree two and four, namely

$$Z_i Z_j (Z_i^2 - Z_j^2)$$

($i \neq j$) and

$$Z_h Z_i Z_j Z_k - Z_{h'} Z_{i'} Z_{j'} Z_{k'}$$

for any bipartition of $\{1, \ldots, 8\}$ into two sets of size four, together with

$$Z_1^2 + \cdots + Z_8^2 - 1.$$
A Conjecture?

Conjecture

For each $m > 2$, there is a constant $K(m)$, independent of the choice of cometric association scheme, such that whenever $\dim V_1 = m$, the ideal $I$ is generated by a set of polynomials all of total degree at most $K(m)$. 

William J. Martin
Cometric Association Schemes
**Equivalence**

This conjecture is equivalent to the dual of the Bannai-Ito Conjecture.

**Theorem**

The following are equivalent:

- For each $m > 2$, there are only finitely many cometric association schemes (up to isomorphism) with $\dim V_1 = m$;
- For each $m > 2$, there is a constant $K(m)$, independent of the choice of cometric association scheme, such that whenever $\dim V_1 = m$, the ideal $I$ is generated by a set of polynomials all of total degree at most $K(m)$. 
Goal for Last Part

In the last part of the talk, I will outline a proof of the dual of the Bannai-Ito Conjecture. This is joint work with Jason Williford.

**Theorem (Williford & WJM):** For any fixed $m_1 > 2$, there are only finitely many cometric association schemes with first multiplicity $m_1$. 
Splitting Fields

The *splitting field* of an association scheme is the smallest extension of the rationals containing all of the entries of the first eigenmatrix \( P \).

Our first theorem concerns actions of automorphisms of the splitting field on the \( E_i \), and is true for any association scheme.
Splitting Fields

Theorem
Let \((X, \mathcal{R})\) be an association scheme and let \(\mathcal{A}\) be its Bose-Mesner algebra. Let \(F\) be the splitting field of this scheme and \(G = \text{Gal}(F/Q)\). Let \(G\) act on the matrices of the Bose-Mesner algebra \(\mathcal{A}\) entrywise. Then this induces a faithful action of \(G\) on the primitive idempotents \(E_0, \ldots, E_d\).
The Splitting Field of a Cometric Scheme

The following result has been obtained independently by Cerzo and Suzuki (2006).

**Theorem**

The splitting field of any cometric association scheme with \( m_1 > 2 \) is at most a degree two extension of the rationals.
Proof.

Let \((X, \mathcal{R})\) be a cometric association scheme of diameter \(d\), and let \(E_0, E_1, \ldots, E_d\) be a \(Q\)-polynomial ordering of the primitive idempotents. Let \(F\) be the splitting field of the scheme, generated by the entries \(Q_{ij}\) of the matrix \(Q\), and suppose \([F : \mathbb{Q}] = n\).

Note that if \(\sigma\) is in \(G\), then \(E_0^\sigma, E_1^\sigma, \ldots, E_d^\sigma\) is also a \(Q\)-polynomial ordering. Since \(|G| = n\), and the action of \(G\) is faithful on the \(E_j\), there must then be at least \(n\) different \(Q\)-polynomial orderings of the \(E_j\). By a result of Suzuki (1998) there can be at most two \(Q\)-polynomial orderings of the \(E_j\), so \(|G| \leq 2\), therefore, \([F : \mathbb{Q}] \leq 2\). \(\square\)
Spherical Codes

For a subset $A \subset [-1, 1)$ of the possible inner products among unit vectors, a *spherical A-code* in $\mathbb{R}^m$ is a subset $Y$ of the unit sphere $S^{m-1}$ having the property that $x \cdot y \in A$ for any distinct $x, y \in Y$ where $\cdot$ denotes the ordinary dot product.
Bounds on Spherical Codes

If $A$ is bounded away from 1 — the case of interest is $A = [-1, \eta]$ for some fixed $\eta < 1$ — then $Y$ must be finite. We can obtain an upper bound

$$U(m, \eta),$$

on the size of such a set $Y$ just using a sphere-packing argument.
Kissing Numbers

An interesting special case is that of spherical $A$-codes where

$$A = [-1, \frac{1}{2}].$$

The optimal size of such a spherical code in $\mathbb{R}^m$ is called the kissing number $\tau_m$.

Kabatianski and Levenshtein (1978):

$$\tau_m \leq 2^{0.401m(1+o(m))}.$$
Repeat Today’s Result

Theorem

For any fixed $m > 2$, there are only finitely many cometric association schemes $(X, \mathcal{R})$ with some Q-polynomial ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents satisfying $\text{rank } E_1 = m$. \qed
Overview of the Proof

The proof of this theorem is broken into three steps.

- geometry of the first eigenspace $\text{colsp}E_1$: focus on the relation $R_1$ selected so that $m > Q_{1j} > Q_{ij}$ for all $i > 1$. We prove that this valency $v_1$ is bounded above by some function of $m$.
- there are only finitely many possible eigenvalues for $A_1$ in such an association scheme with $\text{rank} E_1 = m$.
- derive a contradiction using these tools.
Valency Lemma

**Lemma**

Let \((X, R)\) be an association scheme and let \(E_j\) be a primitive idempotent with rank \(m_j\). Suppose \(m_j > Q_{1j} \geq Q_{ij}\) for all \(i > 1\). Then \(v_1 \leq K\) for some \(K\) depending only on \(m_j\).
Proving the Valency Lemma

**Proof:** Fix $a \in X$ and consider the configuration

$$Y' = \{ \bar{b} : (a, b) \in R_1 \}$$

where $\bar{b}$ denotes the $b^{th}$ column of $E_j$. In $\mathbb{R}^{mj}$, the Euclidean distance from $\bar{a}$ is

$$d(\bar{a}, \bar{b}) = d_1 := \sqrt{2(m_j - Q_{1j})/\nu}$$

for each such $b$. 
Proving the Valency Lemma

**Proof:** Fix $a \in X$ and consider the configuration

$$Y' = \{ \bar{b} : (a, b) \in R_1 \}$$

where $\bar{b}$ denotes the $b^{th}$ column of $E_j$. In $\mathbb{R}^{m_j}$, the Euclidean distance from $\bar{a}$ is

$$d(\bar{a}, \bar{b}) = d_1 := \sqrt{2(m_j - Q_{1j})/v}$$

for each such $b$.

Since this is the smallest distance between any two distinct columns of $E_j$, we have $d(\bar{b}, \bar{c}) \geq d_1$ for any distinct $b, c \in R_1(a)$. Now (after a translation and re-normalization) $Y'$ forms a spherical code in $\mathbb{R}^{m_j-1}$ with center $\frac{Q_{1j}}{m_j} \bar{a}$ since each vector in $Y'$ lies in the hyperplane $\{ \bar{x} : \bar{x} \cdot \bar{a} = Q_{1j}/v \}$ inside $\text{colsp} E_j$. 
Proving the Valency Lemma

We next show that the minimum angle formed by distinct vectors in this spherical code is at least 60°.
We next show that the minimum angle formed by distinct vectors in this spherical code is at least $60^\circ$. Denote the point $\bar{a}$ by $A$ and the center of the sphere $\frac{Q_{1j}}{m_j} \bar{a}$ by $O$. 
Proving the Valency Lemma

We next show that the minimum angle formed by distinct vectors in this spherical code is at least $60^\circ$. Denote the point $\overline{a}$ by $A$ and the center of the sphere $\frac{Q_{1j}}{m_j} \overline{a}$ by $O$. Now if $B$ and $C$ are distinct points from $Y'$, then $\angle BAC \geq 60^\circ$ since $d(B, C) \geq d(A, B) = d(A, C)$. 
Proving the Valency Lemma

We next show that the minimum angle formed by distinct vectors in this spherical code is at least $60^\circ$. Denote the point $\bar{a}$ by $A$ and the center of the sphere $\frac{Q_1}{m_j}\bar{a}$ by $O$. Now if $B$ and $C$ are distinct points from $Y'$, then $\angle BAC \geq 60^\circ$ since $d(B, C) \geq d(A, B) = d(A, C)$. So, since $\angle AOB = \angle AOC = 90^\circ$, we must have $\angle BOC > 60^\circ$. 
At least 60 degrees
Proving the Valency Lemma

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Our hypothesis that $m_j > Q_{1j} \geq Q_{ij}$ for all $j > 1$ guarantees that the columns of $E_j$ are all distinct. So we have $v_1 = |Y'|$. But, from the above observation, $Y'$ can be scaled to a spherical $[-1, \frac{1}{2}]$-code in $\mathbb{R}^{m_j-1}$.
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Thus the size of \( Y' \) is bounded by the kissing number \( \tau_{m_j-1} \) in dimension \( m_j - 1 \). \( \square \)
Bounding the Number of Algebraic Integers

Theorem
Let $K > 0$ and $S$ the set of all monic polynomials with degree 2 over the integers, both of whose roots lie in $[-K, K]$. Then $S$ is finite.
Counting Polynomials

Proof.
Let \( f \in S \) have degree 2, and write \( f = x^2 + f_1 x + f_0 \).
Let \( s \) be the maximum absolute value of the roots of \( f \).
Then the coefficients satisfy

\[-2K \leq -2s \leq f_1 \leq 2s \leq 2K\]

and

\[-K^2 \leq -s^2 \leq f_0 \leq s^2 \leq K^2.\]

Since these bounds depend only on \( K \), there are only finitely many possible values for the integer coefficients \( f_0 \) and \( f_1 \), so the set of polynomials \( S \) must be finite. □
Back to the Main Proof

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Back to the Main Proof

Now we are ready to complete our proof. Suppose, by way of contradiction, that for some fixed $m > 2$, there are infinitely many non-isomorphic cometric association schemes with $\text{rank } E_1 = m$. We know that, for each of these schemes, all $P_{1i}$ have minimal polynomials over $\mathbb{Q}$ of degree one or two. Let $\mathcal{F}$ denote this family of association schemes and henceforth for each scheme in this family, order the relations $R_0, R_1, \ldots, R_d$ so that

$$m = Q_{01} > Q_{11} > Q_{21} > \cdots > Q_{d1}.$$
Proof of Main Result, Cont’d

Let $K$ be the bound of the Valency Lemma ($K$ depends only on $m$).
Proof of Main Result, Cont’d

Let \( K \) be the bound of the Valency Lemma (\( K \) depends only on \( m \)). Let \( \mathcal{B} \) denote the ring of algebraic integers and, for \( b \in \mathcal{B} \), write \( m_b(t) \) for the minimal polynomial of \( b \) over the rationals.
Proof of Main Result, Cont’d

Let $K$ be the bound of the Valency Lemma ($K$ depends only on $m$). Let $\mathcal{B}$ denote the ring of algebraic integers and, for $b \in \mathcal{B}$, write $m_b(t)$ for the minimal polynomial of $b$ over the rationals. Now consider the set

$$S = \left\{ b \in \mathcal{B} \mid \deg m_b = 2, \quad -K \leq \xi \leq K \quad \text{whenever} \quad m_b(\xi) = 0 \right\}$$

and let

$$r = \max \left( \frac{K-1}{K}, \max_{s \in S} \frac{s}{|s|} \right).$$
Proof of Main Result, Cont’d

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But this is impossible since either $P_{11} \in \mathbb{Z}$ (in which case $\frac{P_{11}}{v_1} \leq \frac{K-1}{K} \leq r$) or $P_{11}$ belongs to the set $S$: it is an algebraic integer in some extension $\mathbb{Q}[\zeta]$ of degree two and any conjugate $\xi$ of $P_{11}$ is an eigenvalue of $A_1$ so satisfies

$$-K \leq -v_1 < \xi < v_1 \leq K.$$ 

So we have arrived at the desired contradiction. □
Open Questions

- Find a good bound on the number of schemes with fixed $m_1$;
- Find the list for $m_1 = 3$ (I think Suzuki has done this) and $m_1 = 4$;
- Generate feasible parameter sets!
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- Prove $w \leq m_1$ when $d$ is even;
- Determine when a set of lines through the origin determines a $Q$-bipartite scheme;
- approach old solved and unsolved questions on tight designs using these new tools.
Thank You!
A curious property

There exists a function (a bilinear map)

\[ \star : \mathbb{R}[t] \times \mathbb{R}[t] \to \mathbb{R}[t] \]

for which

\[ \sum_{z \in \mathbb{X}} f(\langle x, z \rangle) g(\langle z, y \rangle) = (f \star g)(\langle x, y \rangle) \]

for all polynomials \( f \) and \( g \) and for all \( x, y \in \mathbb{X} \).
## Multiplication table

<table>
<thead>
<tr>
<th>*</th>
<th>1</th>
<th>$t$</th>
<th>$t^2$</th>
<th>$t^3$</th>
<th>$t^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>196560</td>
<td>8190$t$</td>
<td>630$t^2$ + 315</td>
<td>$\frac{135}{2}t^3 + \frac{405}{4}t$</td>
<td>$9t^4 + 27t^2 + \frac{27}{8}$</td>
</tr>
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<td>$\frac{135}{32}t^4 + \frac{405}{64}t^2 + \frac{135}{256}$</td>
</tr>
<tr>
<td>$t^2$</td>
<td>945$t$</td>
<td>135$t^2$ + $\frac{135}{4}$</td>
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</tr>
<tr>
<td>$t^3$</td>
<td>945</td>
<td>$\frac{675}{4}t$</td>
<td>$\frac{135}{4}t^2 + \frac{45}{8}$</td>
<td>$\frac{45}{2}t^3 + \frac{135}{8}t$</td>
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</tr>
<tr>
<td>$t^4$</td>
<td>$\frac{675}{4}t$</td>
<td>$\frac{135}{4}t^2 + \frac{45}{8}$</td>
<td>$\frac{135}{2}t^3 + \frac{405}{4}t$</td>
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</tr>
<tr>
<td>$t^5$</td>
<td>$\frac{675}{4}t$</td>
<td>$\frac{135}{4}t^2 + \frac{45}{8}$</td>
<td>$\frac{135}{2}t^3 + \frac{405}{4}t$</td>
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<td>$\frac{135}{32}t^4 + \frac{405}{64}t^2 + \frac{135}{256}$</td>
</tr>
<tr>
<td>$t^6$</td>
<td>$\frac{675}{4}$</td>
<td>$\frac{135}{4}t^2 + \frac{45}{8}$</td>
<td>$\frac{135}{2}t^3 + \frac{405}{4}t$</td>
<td>$9t^4 + 27t^2 + \frac{27}{8}$</td>
<td>$\frac{135}{32}t^4 + \frac{405}{64}t^2 + \frac{135}{256}$</td>
</tr>
</tbody>
</table>
Multiplication table, cont’d

\[ t^5 \star t^5 = \frac{45}{32} t^5 + \frac{225}{32} t^3 + \frac{675}{256} t \]

\[ t^6 \star t^6 = \frac{45}{32} t^6 + \frac{675}{512} t^4 + \frac{1485}{1024} t^2 + \frac{315}{4096} \]
Non-standard definition

A finite set $X$ of points on the unit sphere in $\mathbb{R}^m$ is an *association scheme* if there exists a function

$$\star : \mathbb{R}[t] \times \mathbb{R}[t] \to \mathbb{R}[t]$$

for which

$$\sum_{z \in X} f(\langle x, z \rangle) g(\langle z, y \rangle) = (f \star g)(\langle x, y \rangle)$$

for all polynomials $f$ and $g$ and for all $x, y \in X$. 
Suppose the finite set $X$ of points on the unit sphere in $\mathbb{R}^m$ is an association scheme according to the above definition. The scheme is said to be *cometric* (or “$Q$-polynomial”) if, for all polynomials $f$ and $g$, 

$$\deg f \star g \leq \deg f, \quad \deg g$$
Example

The $m$-cube is a cometric association scheme in $\mathbb{R}^m$. For $m = 3$, we have

\[
\begin{align*}
1 \star 1 &= 8 \\
t \star t &= \frac{8}{3} t \\
t^2 \star 1 &= \frac{8}{3} \\
t^2 \star t^2 &= \frac{16}{9} t^2 + \frac{8}{27} \\
t^3 \star t &= \frac{56}{27} t \\
t^3 \star t^3 &= \frac{16}{9} t^3 + \frac{56}{243} t
\end{align*}
\]