

Characterizing completely regular codes from an algebraic viewpoint

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Abstract

Completely regular codes are rich substructures in distance-regular graphs and have been studied extensively over the last two decades. The class includes highly structured and beautiful examples such as perfect and uniformly packed codes while the rich properties of these codes allow for both combinatorial and algebraic analysis. In fact, these codes are fundamental to the study of distance-regular graphs themselves.

In a companion paper, we study products of completely regular codes and codes whose parameters form arithmetic progressions. This family of completely regular codes, while quite special in one sense, contains some very important examples and exhibits some of the nicest features of the larger class. Here, we approach these features from an algebraic viewpoint, exploring Q -polynomial properties of completely regular codes.

We first summarize the basic structure of the outer distribution module of a completely regular code. Then, employing a simple lemma concerning eigenvectors in association schemes, we propose to study the tightest case, where the indices of the eigenspace that appear in the outer distribution module are equally spaced. In addition to the arithmetic codes of the companion paper, this highly structured class

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includes other beautiful examples and we propose the classification of Q -polynomial completely regular codes in the Hamming graphs. A key result is Theorem 3.10 which finds that the Q -polynomial condition is equivalent to the presence of a certain Leonard pair. This connection has impact in two directions. First, the Leonard pairs are classified and we gain quite a bit of information about the algebraic structure of any code in our class. But also this gives a new setting for the study of Leonard pairs, one closely related to the classical one where a Leonard pair arises from each thin/dual-thin irreducible module of a Terwilliger algebra of some P - and Q -polynomial association scheme, yet not previously studied. It is particularly interesting that the Leonard pair associated to some code C may belong to one family in the Askey scheme while the distance-regular graph in which the code is found may belong to another.

1 Introduction

The study of digital error-correcting codes includes as an important and intriguing sub-topic the analysis and classification of highly regular codes. These include the perfect codes as well as several phenomenal families such as the Kerdock codes, the Delsarte-Goethals codes, and the Reed-Muller codes. One motivation for this branch of coding theory has always been a well-studied but mysterious connection to finite groups. Optimal codes tend to have a great deal of symmetry (as is often true in optimization problems which themselves are defined in a symmetric way), and several finite simple groups – namely the Mathieu groups – play an important role in the classification of perfect codes.

But the class of completely regular codes, which properly contains both the class of perfect codes and the class of uniformly packed codes but also, for example, the Preparata and Kasami codes, has not received a great deal of attention in recent years. Our view is that these codes deserve further study, not only because of their connection to highly symmetric codes and codes with large minimum distance, but also because of a key role that completely regular codes play in the study of distance-regular graphs. A theorem of Brouwer, et al. [2, p353] states that every distance-regular graph on a prime power number of vertices admitting an elementary abelian group of automorphisms which acts transitively on its vertices is a coset graph of some additive completely regular code in some Hamming graph (with some

conference graphs as exceptions). This gives another reason why a careful study of completely regular codes in Hamming graphs (and, more generally, in distance-regular graphs) is central to the study of association schemes.

In a companion paper, we study products of completely regular codes and codes whose parameters form arithmetic progressions. This family of completely regular codes, while quite special in one sense, contains some very important examples and exhibits some of the nicest features of the larger class. Here, we approach these features from an algebraic viewpoint, exploring Q -polynomial properties of completely regular codes.

We first summarize the basic structure of the outer distribution module of a completely regular code. Then, employing a simple lemma concerning eigenvectors in association schemes, we propose to study the tightest case, where the indices of the eigenspace that appear in the outer distribution module are equally spaced. In addition to the arithmetic codes of the companion paper, this highly structured class includes other beautiful examples and we propose the classification of Q -polynomial completely regular codes in the Hamming graphs. A key result is Theorem 3.10 which finds that the Q -polynomial condition is equivalent to the presence of a certain Leonard pair. This connection has impact in two directions. First, the Leonard pairs are classified and we gain quite a bit of information about the algebraic structure of any code in our class. But also this gives a new setting for the study of Leonard pairs, one closely related to the classical one where a Leonard pair arises from each thin/dual-thin irreducible module of a Terwilliger algebra of some P - and Q -polynomial association scheme, yet not previously studied. It is particularly interesting that the Leonard pair associated to some code C may belong to one family in the Askey scheme while the distance-regular graph in which the code is found may belong to another.

2 Preliminaries and definitions

2.1 Distance-regular graphs

Suppose that Γ is a finite, undirected, connected graph with vertex set $V\Gamma$. For vertices x and y in $V\Gamma$, let $d(x, y)$ denote the distance between x and y , i.e., the length of a shortest path connecting x and y in Γ . Let D denote the diameter of Γ ; i.e., the maximal distance between any two vertices in $V\Gamma$. For $0 \leq i \leq D$ and $x \in V\Gamma$, let $\Gamma_i(x) := \{y \in V\Gamma \mid d(x, y) = i\}$ and put

$\Gamma_{-1}(x) := \emptyset, \Gamma_{D+1}(x) := \emptyset$. The graph Γ is called *distance-regular* whenever it is regular of valency k , and there are integers b_i, c_i ($0 \leq i \leq D$) so that for any two vertices x and y in $V\Gamma$ at distance i , there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. It follows that there are exactly $a_i = k - b_i - c_i$ neighbors of y in $\Gamma_i(x)$. The numbers c_i, b_i and a_i are called the *intersection numbers* of Γ and we observe that $c_0 = 0, b_D = 0, a_0 = 0, c_1 = 1$ and $b_0 = k$. The array $\iota(\Gamma) := \{b_0, b_1, \dots, b_{D-1}; c_1, c_2, \dots, c_D\}$ is called the *intersection array* of Γ . Set the tridiagonal matrix

$$L(\Gamma) := \begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & b_2 & \\ & & \ddots & \ddots & \ddots \\ & & & c_D & a_D \end{pmatrix}.$$

From now on, assume Γ is a distance-regular graph of valency $k \geq 2$ and diameter $D \geq 2$. Define A_i to be the square matrix of size $|V\Gamma|$ whose rows and columns are indexed by $V\Gamma$ with entries

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{otherwise} \end{cases} \quad (0 \leq i \leq D, x, y \in V\Gamma).$$

We refer to A_i as the i^{th} *distance matrix* of Γ . We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ . Since Γ is distance-regular, we have for $2 \leq i \leq D$

$$AA_{i-1} = b_{i-2}A_{i-2} + a_{i-1}A_{i-1} + c_iA_i$$

so that $A_i = p_i(A)$ for some polynomial $p_i(t)$ of degree i . Let \mathbb{A} be the *Bose-Mesner algebra*, the matrix algebra over \mathbb{C} generated by A . Then $\dim \mathbb{A} = D + 1$ and $\{A_i \mid 0 \leq i \leq D\}$ is a basis for \mathbb{A} . As \mathbb{A} is semi-simple and commutative, \mathbb{A} has also a basis of pairwise orthogonal idempotents $\{E_0 = \frac{1}{|V\Gamma|}J, E_1, \dots, E_D\}$. We call these matrices the *primitive idempotents* of Γ . As \mathbb{A} is closed under the entry-wise (or Hadamard) product \circ , there exist real numbers q_{ij}^ℓ , called the *Krein parameters*, such that

$$E_i \circ E_j = \frac{1}{|V\Gamma|} \sum_{\ell=0}^D q_{ij}^\ell E_\ell \quad (0 \leq i, j \leq D) \quad (1)$$

We say the distance-regular graph Γ is *Q-polynomial* with respect to a given ordering E_0, E_1, \dots, E_D of its primitive idempotents provided its Krein parameters satisfy

- $q_{ij}^\ell = 0$ unless $|j - i| \leq \ell \leq i + j$;
- $q_{ij}^\ell \neq 0$ whenever $\ell = |j - i|$ or $\ell = i + j \leq D$.

By an *eigenvalue* of Γ , we mean an eigenvalue of $A = A_1$. Since Γ has diameter D , it has at least $D + 1$ eigenvalues; but since Γ is distance-regular, it has exactly $D + 1$ eigenvalues¹, and they are exactly the eigenvalues of $L(\Gamma)$.

We denote these eigenvalues by $\theta_0, \dots, \theta_D$ and, aside from the convention that $\theta_0 = k$, the valency of Γ , we make no further assumptions at this point about the eigenvalues except that they are distinct. We note that, with an appropriate ordering of the eigenvalues, the i^{th} primitive idempotent E_i is precisely the matrix representing orthogonal projection onto V_i , the eigenspace of A associated to θ_i . In fact, when $\theta = \theta_i$ for some i , we will sometimes write $E(\theta)$ in place of E_i when it is convenient to omit the subscript.

The following fundamental result will be very useful in this paper; it is originally due to Cameron, Goethals, and Seidel [3].

Theorem 2.1 ([3, Theorem 5.1]) *If $\mathbf{u} \in V_i$ and $\mathbf{v} \in V_j$ and $q_{ij}^\ell = 0$, then $\mathbf{u} \circ \mathbf{v}$ is orthogonal to V_ℓ where $\mathbf{u} \circ \mathbf{v}$ denotes the entry-wise product of vectors \mathbf{u} and \mathbf{v} .* ■

An elementary proof of this fact can be found in [7].

For each eigenvalue θ of Γ and for each $x \in V\Gamma$, there is a unique normalized eigenvector in V_i which is constant over each vertex subset $\Gamma_i(x)$. The entries of this eigenvector, which we shall denote by $u_i(\theta)$ ($0 \leq i \leq D$, θ an eigenvalue of Γ) are determined entirely by the intersection array, independent of the choice of x .

Suppose Γ has intersection array $\iota(\Gamma) := \{b_0, b_1, \dots, b_{D-1}; c_1, c_2, \dots, c_D\}$. and let θ be an eigenvalue of Γ . The corresponding *standard right eigenvector* $[u_0(\theta) = 1, u_1(\theta), \dots, u_D(\theta)]^\top$ of Γ with respect to θ is defined by the following

¹See, for example Lemma 11.4.1 in [5].

initial conditions and recurrence relation:

$$\begin{aligned} u_0(\theta) &= 1, \quad u_1(\theta) = \theta/k, \\ c_i u_{i-1}(\theta) + a_i u_i(\theta) + b_i u_{i+1}(\theta) &= \theta u_i(\theta) \quad (0 \leq i \leq D), \\ \text{where } u_{-1} &= u_{D+1} = 0. \end{aligned} \tag{2}$$

One easily checks that the vector \mathbf{u} of length $|V\Gamma|$ satisfying $\mathbf{u}_y = u_i(\theta)$ whenever $y \in \Gamma_i(x)$ satisfies $A\mathbf{u} = \theta\mathbf{u}$. Let $x \in V\Gamma$ and let \mathbf{e}_x denote the elementary basis vector in V corresponding to x . For $\theta = \theta_j$ ($0 \leq j \leq D$) we easily see that

$$E_j \mathbf{e}_x = \frac{m_j}{|V\Gamma|} \mathbf{u}$$

where $m_j := \text{rank } E_j$. It follows from this and (1) that, for $0 \leq h, i, j \leq D$,

$$m_i m_j u_h(\theta_i) u_h(\theta_j) = \sum_{\ell=0}^D q_{ij}^\ell m_\ell u_h(\theta_\ell). \tag{3}$$

So we can detect whether or not Γ is Q -polynomial just by looking at its standard right eigenvectors.

2.2 Codes in distance-regular graphs

Let Γ be a distance-regular graph with distinct eigenvalues $\theta_0 = k, \theta_1, \dots, \theta_D$. By a *code* in Γ , we simply mean any nonempty subset C of $V\Gamma$. We call C *trivial* if $|C| \leq 1$ or $C = V\Gamma$ and *non-trivial* otherwise. For $|C| > 1$, the *minimum distance* of C , $\delta(C)$, is defined as

$$\delta(C) := \min\{ d(x, y) \mid x, y \in C, x \neq y \}$$

and for any $x \in V\Gamma$ the distance $d(x, C)$ from x to C is defined as

$$d(x, C) := \min\{ d(x, y) \mid y \in C \}.$$

The number

$$\rho(C) := \max\{ d(x, C) \mid x \in V\Gamma \}$$

is called the *covering radius* of C .

For C a nonempty subset of $V\Gamma$ and for $0 \leq i \leq \rho$, define

$$C_i = \{ x \in V\Gamma \mid d(x, C) = i \}.$$

Then $\Pi(C) = \{C_0 = C, C_1, \dots, C_\rho\}$ is the *distance partition* of $V\Gamma$ with respect to code C .

A partition $\Pi = \{P_0, P_1, \dots, P_k\}$ of $V\Gamma$ is called *equitable* if, for all i and j , the number of neighbors a vertex in P_i has in P_j is independent of the choice of vertex in P_i . We say a code C in Γ is *completely regular* if this distance partition $\Pi(C)$ is equitable². In this case the following quantities are well-defined:

$$\gamma_i = |\{y \in C_{i-1} \mid d(x, y) = 1\}|, \quad (4)$$

$$\alpha_i = |\{y \in C_i \mid d(x, y) = 1\}|, \quad (5)$$

$$\beta_i = |\{y \in C_{i+1} \mid d(x, y) = 1\}| \quad (6)$$

where x is chosen from C_i . The numbers $\gamma_i, \alpha_i, \beta_i$ are called the *intersection numbers* of code C . Observe that a graph Γ is distance-regular if and only if each vertex is a completely regular code and these $|V\Gamma|$ codes all have the same intersection numbers. An equitable partition $\Pi = \{P_1, \dots, P_m\}$ of $V\Gamma$ is called a *completely regular partition* if all P_i are completely regular codes and any two of these have the same parameters.

If \mathbf{x} is the characteristic vector of C as a subset of $V\Gamma$, then the *outer distribution module* of C is defined as

$$\mathbb{A}\mathbf{x} = \{M\mathbf{x} \mid M \in \mathbb{A}\}.$$

Clearly, this is an \mathbb{A} -invariant subspace of the *standard module* $V := \mathbb{C}^{V\Gamma}$. Our next goal is to describe two nice bases for $\mathbb{A}\mathbf{x}$.

For $0 \leq i \leq \rho$, let \mathbf{x}_i denote the characteristic vector of C_i .

Lemma 2.2 *Let Γ be a distance-regular graph and C a completely regular code in Γ . With notation as above, we have*

- (a) *the vectors $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\rho\}$ form a basis for the outer distribution module $\mathbb{A}\mathbf{x}$ of C ;*

²This definition of a completely regular code is due to Neumaier [8]. When Γ is distance-regular, it is equivalent to the original definition, due to Delsarte [4], which we now mention. If \mathbf{x} is the characteristic vector of C , construct a $|V\Gamma| \times (D+1)$ matrix with columns $A_i\mathbf{x}$ ($0 \leq i \leq D$). Delsarte declares C to be completely regular if this *outer distribution matrix* has only $\rho + 1$ distinct rows.

(b) relative to this basis, the matrix representing the action of A on $\mathbb{A}\mathbf{x}$ is given by the tridiagonal matrix

$$U := U(C) = \begin{pmatrix} \alpha_0 & \beta_0 & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & \\ & \gamma_2 & \alpha_2 & \beta_2 & \\ & & \ddots & \ddots & \ddots \\ & & & \gamma_\rho & \alpha_\rho \end{pmatrix};$$

(c) $\dim \mathbb{A}\mathbf{x} = \rho + 1$.

Proof: From Equations (4), (5) and (6) above, we have

$$\mathbf{A}\mathbf{x}_i = \beta_{i-1}\mathbf{x}_{i-1} + \alpha_i\mathbf{x}_i + \gamma_{i+1}\mathbf{x}_{i+1} \quad (7)$$

for $0 \leq i \leq \rho$ where, for convenience, we set $\mathbf{x}_{-1} = \mathbf{0}$ and $\mathbf{x}_{\rho+1} = \mathbf{0}$. So a simple inductive argument shows that each \mathbf{x}_i lies in the outer distribution module of C . These vectors are trivially linear independent, so we need only verify that they span $\mathbb{A}\mathbf{x}$. By (7), these vectors span an A -invariant subspace of V containing the characteristic vector \mathbf{x} of C ; since $\mathbb{A}\mathbf{x}$ is defined to be the smallest such subspace, the two spaces must coincide. \blacksquare

Corollary 2.3 *Let Γ be a distance-regular graph. For any completely regular code C in Γ with characteristic vector \mathbf{x} , the outer distribution module $\mathbb{A}\mathbf{x}$ of C is closed under entrywise multiplication.*

Proof: Simply observe that the basis vectors \mathbf{x}_i satisfy $\mathbf{x}_i \circ \mathbf{x}_j = \delta_{i,j}\mathbf{x}_i$. \blacksquare

The tridiagonal matrix U appearing in the lemma is called the *quotient matrix* of Γ with respect to C .

Now note that, for $0 \leq j \leq D$, if the the vector $E_j\mathbf{x}$ is not the zero vector, then it is an eigenvector for A with eigenvalue θ_j . This motivates us to define

$$S^*(C) = \{j \mid 1 \leq j \leq D, E_j\mathbf{x} \neq \mathbf{0}\}.$$

Lemma 2.4 *Let Γ be a distance-regular graph and C a completely regular code in Γ . With notation as above, we have*

- (a) the nonzero vectors among the set $\{E_j\mathbf{x} \mid 0 \leq j \leq D\}$ form a basis for the outer distribution module $\mathbb{A}\mathbf{x}$ of C ;
- (b) relative to this basis, the matrix representing the action of A on $\mathbb{A}\mathbf{x}$ is a diagonal matrix with diagonal entries $\{\theta_j \mid j \in S^*(C) \cup \{0\}\}$;
- (c) $|S^*(C)| = \rho$.

Proof: Since \mathbb{A} is spanned both by $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$, we see that $\mathbb{A}\mathbf{x}$ is spanned by both $\{A_i\mathbf{x}\}_{i=0}^D$ and $\{E_i\mathbf{x}\}_{i=0}^D$. Since the nonzero vectors in this latter set are linearly independent, they form a basis for $\mathbb{A}\mathbf{x}$. From Lemma 2.2(c), we see that there must be exactly $\rho + 1$ nonzero vectors in this set, so $|S^*(C)| = \rho$. Finally, we have $AE_j\mathbf{x} = \theta_j E_j\mathbf{x}$ showing that the matrix representing the action of A on $\mathbb{A}\mathbf{x}$ relative to this basis is a diagonal matrix with diagonal entries as claimed. \blacksquare

Corollary 2.5 *Let Γ be a distance-regular graph and let C be a completely regular code in Γ . With notation as above, the quotient matrix U has $\rho + 1$ distinct eigenvalues, namely $\{\theta_j \mid j \in S^*(C) \cup \{0\}\}$.*

Proof: Suppose $S^*(C) = \{i_1, \dots, i_\rho\}$. Since both U and the diagonal matrix $\text{diag}(k, \theta_{i_1}, \dots, \theta_{i_\rho})$ represent the same linear transformation, A , on the module $\mathbb{A}\mathbf{x}$ with respect to different bases, these two matrices must have the same eigenvalues. \blacksquare

For C a completely regular code in Γ , we say that η is an *eigenvalue* of C if η is an eigenvalue of the quotient matrix U defined above. By $\text{Spec}(C)$, we denote the set of eigenvalues of C . The above corollary is often called ‘‘Lloyd’s Theorem’’ in coding theory. The condition that each eigenvalue of C must be an eigenvalue of Γ is a powerful condition on the existence of completely regular codes, and perfect codes in particular³.

Note that, since $\gamma_i + \alpha_i + \beta_i = k$ for all i , $\theta_0 = k$ belongs to $\text{Spec}(C)$. So

$$\text{Spec}(C) = \{k\} \cup \{\theta_j \mid j \in S^*(C)\}.$$

³A code C in a distance-regular graph is *perfect* if $|C| = 1$ or $\delta(C) = 2\rho(C) + 1$. All perfect codes are completely regular.

Set $\text{Spec}^*(C) := \text{Spec}(C) - \{k\}$. For eigenvalue η of C , there is a unique right eigenvector

$$\mathbf{u}(\eta) := [u_0 = 1, u_1, \dots, u_\rho]^\top \quad (8)$$

of U associated to η ; in analogy with the standard right eigenvectors of graph Γ , we refer to this vector as the *standard (right) eigenvector* of C associated with η . Note that this vector satisfies the following recurrence relation:

$$\begin{aligned} u_0 = 1, \quad u_1 = \frac{\eta - \alpha_0}{\beta_0}, \\ \gamma_i u_{i-1} + \alpha_i u_i + \beta_i u_{i+1} = \eta u_i \quad (0 \leq i \leq \rho), \\ \text{where } u_{-1} = u_{\rho+1} = 0. \end{aligned} \quad (9)$$

For each standard right eigenvector of C , there is an eigenvector of Γ in $\mathbb{A}\mathbf{x}$ with the same eigenvalue which is unique up to scalar multiplication. For eigenvalue θ_j of C , we refer to this *eigenvector belonging to C* either as $E_j\mathbf{x}$ or as

$$\mathbf{u}(\theta_j) = \sum_{i=0}^{\rho} u_i \mathbf{x}_i \quad (10)$$

where u is defined above, these two definitions differing only in their magnitude. Note that $\mathbf{u}(\theta_j) \in \mathbb{A}\mathbf{x} \cap V_j$.

Lemma 2.6 *Assume that Γ is Q -polynomial with Q -polynomial ordering $\theta_0 = k, \theta_1, \dots, \theta_D$ of its eigenvalues. Let C be a completely regular code with $\text{Spec}^*(C) = \{\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_\rho} \mid i_1 < i_2 < \dots < i_\rho\}$. Let $\mathbf{u}(\theta_{i_j})$ be the eigenvector with eigenvalue θ_{i_j} belonging to C . If $\mathbf{u}(\theta_{i_1})$ has $\rho + 1$ different entries, then $i_j - i_{j-1} \leq i_1$ for all $j \in \{1, \dots, \rho\}$.*

Proof: By Lemma 2.2(c), the outer distribution module $\mathbb{A}\mathbf{x}$ of C has dimension $\rho + 1$ and by Lemma 2.4(a), $\{E_j\mathbf{x} : \theta_j \in \text{Spec}(C)\}$ is a basis for it. We now consider the entrywise product $\mathbf{u}^{(p)}$ of p copies of the vector $\mathbf{u} = E_{i_1}\mathbf{x}$. Note that $\mathbf{u}^{(p)} \in \mathbb{A}\mathbf{x}$ and that $\Lambda := \{\mathbf{u}^{(p)} : 0 \leq p \leq \rho\}$ is a linearly independent set of size $\rho + 1$ by the Vandermonde property. So Λ spans $\mathbb{A}\mathbf{x}$. Suppose that $i_h - i_{h-1} \leq i_1$ for $h < j$ but $i_j > i_{j-1} + i_1$. Set

$$\mathcal{W}' = \text{span} \{E_0\mathbf{x}, E_{i_1}\mathbf{x}, \dots, E_{i_{j-1}}\mathbf{x}\}.$$

As $\mathbb{A}\mathbf{x}$ is closed under the Hadamard product, $\mathbf{u} \circ \mathcal{W}' \subseteq \mathbb{A}\mathbf{x}$ and $\mathbf{u} \circ \mathcal{W}' \subseteq V_0 + V_{i_1} + \dots + V_{i_{j-1}+i_1}$. Hence

$$\mathbf{u} \circ \mathcal{W}' \subseteq \mathbb{A}\mathbf{x} \cap (V_0 + V_{i_1} + \dots + V_{i_{j-1}+i_1}).$$

But as $q_{i_1, h}^l = 0$ for $h \leq i_{j-1}$ and $l \geq i_j$, it follows $\mathbf{u} \circ \mathcal{W}' \subseteq \mathcal{W}'$ and so $\mathbf{u}^{(p)} \circ \mathcal{W}' \subseteq \mathcal{W}'$ for $p \geq 1$ contradicting the fact that Λ spans $\mathbb{A}\mathbf{x}$. \blacksquare

Corollary 2.7 *Let Γ be a distance-regular graph and assume Γ is Q -polynomial with respect to the natural ordering $\theta_0 = k > \theta_1 > \dots > \theta_D$ of its eigenvalues. Let C be a completely regular code in Γ with $S^*(C) = \{i_1, \dots, i_\rho\}$ where $i_1 < \dots < i_\rho$ and $\rho = \rho(C)$. Then $i_j - i_{j-1} \leq i_1$ for all $j \in \{1, \dots, \rho\}$.*

Proof: A standard argument involving Sturm sequences (see, e.g., [2, p.130] and [5, Lemma 8.5.2]) shows that, if θ_{i_1} is the second largest eigenvalue of the tridiagonal matrix U , then the entries of the standard right eigenvector of C with respect to θ_{i_1} are strictly decreasing. So the eigenvector $\mathbf{u}(\theta_{i_1})$ has $\rho + 1$ distinct entries as required. \blacksquare

Our computational work suggests that Corollary 2.7 is often a strong feasibility condition for completely regular codes in the Hamming graphs.

Let Γ be a distance-regular graph with diameter $D \geq 2$. We say Γ is an antipodal 2-cover whenever for all $x \in V\Gamma$, there exists a unique vertex $y \in V\Gamma$ such that $d(x, y) = D$. We denote this vertex by $\pi(x)$ and note that the mapping $\pi : V\Gamma \rightarrow V\Gamma$ is an automorphism of Γ . It is known (cf. [2, Prop. 4.2.3(ii)]) that the subspace stabilized by this mapping is

$$\{\mathbf{v} \in V \mid \mathbf{v}_x = \mathbf{v}_{\pi(x)} \ \forall (x \in V\Gamma)\} = V_0 + V_2 + \dots + V_{2\lfloor \frac{D}{2} \rfloor}$$

and is therefore an \mathbb{A} -submodule of the standard module.

Lemma 2.8 *Let Γ be an antipodal 2-cover distance-regular graph and let $\theta_0 > \theta_1 > \dots > \theta_D$ be the distinct eigenvalues of Γ . Let C be a completely regular code with $S^*(C) = \{i_0 = 0 < i_1 < \dots < i_\rho\}$ where $\rho = \rho(C)$. Let π be the automorphism defined above. Then either*

$$\pi(C) = C \text{ and } i_j \equiv 0 \pmod{2} \ \forall (j \in \{0, \dots, \rho\})$$

or

$$\pi(C) = C_\rho \text{ and } i_j \equiv j \pmod{2} \ \forall (j \in \{0, \dots, \rho\}).$$

Proof: We know that $\mathbb{A}\mathbf{x}$ is invariant under any A_i . So

$$A_D \mathbf{x} = \tau_0 \mathbf{x}_0 + \dots + \tau_\rho \mathbf{x}_\rho$$

for some scalars τ_0, \dots, τ_ρ . Let $x \in C$ and assume $\pi(x) \in C_i$ for some i . Then $\tau_i \neq 0$ and so for any vertex $y \in C_i$, $|\{z \in C \mid d(y, z) = D\}| = 1$. This gives $C_i \subseteq \pi(C)$. Since $\rho(\pi(C)) = \rho(C)$, the code $\pi(C)$ is either C or C_ρ .

Let us first consider the case: $\pi(C) = C$. In this case, the characteristic vector of C belongs to the A -submodule $V_0 + V_2 + \dots$ as outlined above, so for each j , $E_{i_j} \mathbf{x}$ belongs to this submodule as well. Thus $i_j \equiv 0 \pmod{2}$ for all $0 \leq j \leq \rho$.

In the other case, $\pi(C) = C_\rho$ and we use a Sturm sequence argument. We know that $E_{i_j} \mathbf{x}$ is a scalar multiple of

$$u_0 \mathbf{x} + u_1 \mathbf{x}_1 + \dots + u_\rho \mathbf{x}_\rho$$

where $[u_0, u_1, \dots, u_\rho]^\top$ is the standard eigenvector of C associated with eigenvalue θ_{i_j} . But, by hypothesis, θ_{i_j} is the j^{th} largest eigenvalue of the tridiagonal quotient matrix U defined in the statement of Lemma 2.2. So by [5, Lemma 8.5.2], the sequence u_0, u_1, \dots, u_ρ has j sign changes. Since $u_0 > 0$, we find u_ρ is positive for j even and negative for j odd. But it is well-known that if \mathbf{v} is an eigenvector of an antipodal 2-cover Γ , $\mathbf{v} \in V_i$, then $\mathbf{v}_{\pi(x)} = \mathbf{v}_x$ for each $x \in V\Gamma$ when i is even and $\mathbf{v}_{\pi(x)} = -\mathbf{v}_x$ for each $x \in V\Gamma$ when i is odd. From this we obtain our result. \blacksquare

3 Q-polynomial Properties of a Code

In this section, we will define Q -polynomial and Leonard completely regular codes and establish a relation between them.

Definition 3.1 *Let Γ be a distance-regular graph with diameter D and $\text{Spec}(\Gamma) = \{\theta_0, \dots, \theta_{i_D}\}$. Let C be a completely regular code with covering radius ρ in Γ . Then C is called Q -polynomial if we have an ordering $\text{Spec}(C) = \{\theta_0, \theta_{i_1}, \dots, \theta_{i_\rho}\}$ of the eigenvalues of C such that, for each $0 \leq p \leq \rho$, $\mathbf{u}^{(p)} := \underbrace{\mathbf{u} \circ \mathbf{u} \circ \dots \circ \mathbf{u}}_{p \text{ times}} \in \text{span}\{V_{i_0}, \dots, V_{i_p}\}$ where $\mathbf{u} = E_{i_1} \mathbf{x} \in V_{i_1}$.*

In this case, we say C is Q -polynomial with respect to θ_{i_1} .

Remark 3.2 *Let Γ be a distance-regular graph and $x \in V\Gamma$. Then $C = \{x\}$ is completely regular and C is Q -polynomial with respect to the ordering $\theta_0, \theta_{i_1}, \dots, \theta_{i_D}$ of $\text{Spec}(C)$ if and only if Γ is Q -polynomial with respect to the ordering $E_0, E_{i_1}, \dots, E_{i_D}$ of its primitive idempotents.*

Note that any completely regular code with covering radius at most 2 is Q -polynomial. Also if we take for C an antipodal pair in a doubled Odd graph Γ (see, for example [2, Sec. 9.1D]) then C is Q -polynomial but Γ is not Q -polynomial if its valency is at least 3.

Let X be a finite abelian group. A *translation distance-regular graph* on X is a distance-regular graph Γ with vertex set X such that if x and y are adjacent then $x+z$ and $y+z$ are adjacent for all $x, y, z \in X$. A code $C \subseteq X$ is called *additive* for all $x, y \in C$, also $x-y \in C$; i.e., C is a subgroup of X . If C is an additive code in a translation distance-regular graph on X , then we obtain the usual coset partition $\Delta(C) := \{C+x \mid x \in X\}$ of X ; whenever C is a completely regular code, it is easy to see that $\Delta(C)$ is a completely regular partition. For any additive code C in a translation distance-regular graph Γ on vertex set X , the *coset graph* of C in Γ is the graph with vertex set X/C and an edge joining coset C' to coset C'' whenever Γ has an edge with one end in C' and the other in C'' . It follows from Theorem 11.1.6 in [2] that this coset graph is distance-regular whenever C is an additive completely regular code in a translation distance-regular graph.

Proposition 3.3 *Let X be a finite abelian group and let Γ be a translation distance-regular graph on X . Let C be an additive completely regular code in Γ and let $\Delta(C)$ be the partition of X into cosets of C . Then C is Q -polynomial if and only if $\Gamma/\Delta(C)$ is a Q -polynomial distance-regular graph.*

Proof: Let C be an additive completely regular code in Γ whose intersection numbers are γ_i, α_i and β_i ($0 \leq i \leq \rho$). Then by [2, p.352,353], eigenvalues of $\Gamma/\Delta(C)$ are $\frac{\eta_i - \alpha_0}{\gamma_1}$ for $\eta_i \in \text{Spec}(C)$. We see that $L(\Gamma/\Delta(C)) = \frac{1}{\gamma_1}(U - \alpha_0 I)$. Now the result follows easily. \blacksquare

Definition 3.4 *Let Γ be a distance-regular graph. Let η be an eigenvalue of a completely regular code C in Γ and let $u = [u_0 = 1, \dots, u_\rho]^\top$ be the standard eigenvector of η . Then the η is called non-degenerate if $u_{i-1} \neq u_i$ ($1 \leq i \leq \rho$) and $u_{i-1} \neq u_{i+1}$ ($1 \leq i \leq \rho - 1$).*

Note that the second largest eigenvalue of a completely regular code is always non-degenerate. Likewise, if a code C is Q -polynomial with respect to the ordering $\{\eta_0, \eta_1, \dots, \eta_\rho\}$ of its eigenvalues, then η_1 is non-degenerate for C . This follows from Definition 3.1, which implies that the entrywise powers of $\mathbf{u} = E(\eta_1)\mathbf{x}$ are linearly independent and Equation (10) which then tells us that the $\rho + 1$ entries of the standard eigenvector for η_1 are all distinct.

Proposition 3.5 *Let Γ be a distance-regular graph with valency k . Let C be a completely regular code with covering radius ρ and $\text{Spec}(C) = \{\eta_i \mid 0 \leq i \leq \rho\}$ in Γ . Let $u(\eta_i) := [u_0 = 1, u_1(\eta_i), \dots, u_\rho(\eta_i)]^T$ be the standard eigenvector corresponding to eigenvalue η_i of C ($0 \leq i \leq \rho$). Then there are (unique) $\lambda_i, \tau_i \in \mathbb{R}$ such that $\sum_i \lambda_i = 1, \sum_i \tau_i = 1$ and the following two hold:*

$$u^{(2)}(\eta_1) = \sum_{i=0}^{\rho} \lambda_i u(\eta_i) \quad (11)$$

and

$$u^{(3)}(\eta_1) = \sum_{i=0}^{\rho} \tau_i u(\eta_i) \quad (12)$$

In particular, if η_1 is non-degenerate then the intersection numbers of C are determined by the set of values

$$\{\eta_0, \eta_1\} \cup \{\eta_i \mid \lambda_i \neq 0 \text{ or } \tau_i \neq 0\} \cup \{\lambda_0, \dots, \lambda_\rho\} \cup \{\tau_0, \dots, \tau_\rho\}.$$

Proof: Let $u(\eta_i)$ be the standard eigenvector of η_i . The set $\{u(\eta_0), \dots, u(\eta_\rho)\}$ forms a basis of $\mathbb{R}^{\rho+1}$. Hence scalars λ_i and τ_i each summing to one and satisfying (11) and (12) exist.

As $\gamma_j u_{j-1}(\eta_i) + \alpha_j u_j(\eta_i) + \beta_j u_{j+1}(\eta_i) = \eta_i u_j(\eta_i)$, (11) and (12) can be rewritten as

$$\gamma_j u_{j-1}^2(\eta_1) + \alpha_j u_j^2(\eta_1) + \beta_j u_{j+1}^2(\eta_1) = \sum_{i=0}^{\rho} \lambda_i \eta_i u_j(\eta_i)$$

and

$$\gamma_j u_{j-1}^3(\eta_1) + \alpha_j u_j^3(\eta_1) + \beta_j u_{j+1}^3(\eta_1) = \sum_{i=0}^{\rho} \tau_i \eta_i u_j(\eta_i).$$

Assume that we know the set $\{\eta_i \mid \lambda_i \neq 0 \text{ or } \tau_i \neq 0 \text{ or } i = 0, 1\}$ and all the λ_i and τ_i . We use induction on j to recover $\gamma_j, \alpha_j, \beta_j$ as well as $u_{j+1}(\eta_i)$ for $1 \leq i \leq \rho$. For $j = 0$, the equations

$$\alpha_0 + \beta_0 = k,$$

$$\alpha_0 + \beta_0 u_1(\eta_i) = \eta_i \quad \text{for } 0 \leq i \leq \rho$$

and

$$\alpha_0 + \beta_0 u_1^2(\eta_1) = \sum_{i=0}^{\rho} \lambda_i \eta_i.$$

easily allow us to obtain⁴ $\alpha_0, \beta_0, u_1(\eta_i)$ for $0 \leq i \leq \rho$. Suppose that, for all $j \leq m$, the numbers $\gamma_j, \alpha_j, \beta_j$, and $u_{j+1}(\eta_i)$ ($0 \leq i \leq \rho$) are known. Now consider the case $j = m + 1$; we have four equations:

$$\gamma_{m+1} + \alpha_{m+1} + \beta_{m+1} = k, \quad (13)$$

$$\gamma_{m+1}u_m(\eta_1) + \alpha_{m+1}u_{m+1}(\eta_1) + \beta_{m+1}u_{m+2}(\eta_1) = \eta_1 u_{m+1}(\eta_1), \quad (14)$$

$$\gamma_{m+1}u_m^2(\eta_1) + \alpha_{m+1}u_{m+1}^2(\eta_1) + \beta_{m+1}u_{m+2}^2(\eta_1) = \sum_{i=0}^{\rho} \lambda_i \eta_i u_{m+1}(\eta_i) \quad (15)$$

and

$$\gamma_{m+1}u_m^3(\eta_1) + \alpha_{m+1}u_{m+1}^3(\eta_1) + \beta_{m+1}u_{m+2}^3(\eta_1) = \sum_{i=0}^{\rho} \tau_i \eta_i u_{m+1}(\eta_i). \quad (16)$$

As η_1 is non-degenerate, we obtain by Equations (13)–(16):

$$\begin{aligned} u_{m+2}(\eta_1) &= \frac{R_\tau - R_\lambda (u_{m+1}(\eta_1) + u_m(\eta_1)) + \eta_1 u_{m+1}^2(\eta_1) u_m(\eta_1)}{R_\lambda + k u_{m+1}(\eta_1) u_m(\eta_1) - \eta_1 u_{m+1}(\eta_1) (u_{m+1}(\eta_1) + u_m(\eta_1))}, \\ \gamma_{m+1} &= \frac{R_\lambda + k u_{m+2}(\eta_1) u_{m+1}(\eta_1) - \eta_1 u_{m+1}(\eta_1) (u_{m+2}(\eta_1) + u_{m+1}(\eta_1))}{(u_m(\eta_1) - u_{m+2}(\eta_1))(u_m(\eta_1) - u_{m+1}(\eta_1))}, \\ \alpha_{m+1} &= \frac{R_\lambda + k u_{m+2}(\eta_1) u_m(\eta_1) - \eta_1 u_{m+1}(\eta_1) (u_{m+2}(\eta_1) + u_m(\eta_1))}{(u_{m+1}(\eta_1) - u_{m+2}(\eta_1))(u_{m+1}(\eta_1) - u_m(\eta_1))}, \\ \beta_{m+1} &= \frac{R_\lambda + k u_{m+1}(\eta_1) u_m(\eta_1) - \eta_1 u_{m+1}(\eta_1) (u_{m+1}(\eta_1) + u_m(\eta_1))}{(u_{m+2}(\eta_1) - u_{m+1}(\eta_1))(u_{m+2}(\eta_1) - u_m(\eta_1))}, \end{aligned}$$

where R_λ and R_τ are shorthand for the expressions on the right-hand sides of Equations (15) and (16), respectively; these quantities are presumed known by the induction hypothesis.

But we also have, for $0 \leq i \leq \rho$,

$$\gamma_{m+1}u_m(\eta_i) + \alpha_{m+1}u_{m+1}(\eta_i) + \beta_{m+1}u_{m+2}(\eta_i) = \eta_i u_{m+1}(\eta_i) \quad (17)$$

⁴Indeed, $\beta_0 \neq 0$. If we denote by S the sum on the right-hand side of the last equation, the simultaneous equations $k + \beta_0(u_1(\eta_1) - 1) = \eta_1$ and $k + \beta_0(u_1(\eta_1)^2 - 1) = S$ allow us to solve for $u_1(\eta_1) + 1$ and then for β_0 so that all the remaining equations become linear.

with (13) and (14) as special cases; from these, we now obtain $u_{m+2}(\eta_i)$ for $2 \leq i \leq \rho$. \blacksquare

Lemma 3.6 *Let λ_j and τ_j be the constants defined in Proposition 3.5 above. Suppose that $\text{Spec}^*(C) = \{\theta_{i_1}, \dots, \theta_{i_\rho}\}$. If $\lambda_j \neq 0$, then $q_{i_1, i_1}^{i_j} \neq 0$ and if $\tau_j \neq 0$, then there exists i_ℓ such that $q_{i_1, i_1}^{i_\ell} \neq 0$ and $q_{i_\ell, i_1}^{i_j} \neq 0$.*

Proof: Put $\mathbf{u}(\theta_{i_j}) := \sum_{h=0}^{\rho} u_h(\theta_{i_j}) \mathbf{x}_h$. Then $\mathbf{u}^{(2)}(\theta_{i_1}) = \sum_{j=0}^{\rho} \lambda_j \mathbf{u}(\theta_{i_j})$, $\mathbf{u}^{(3)}(\theta_{i_1}) = \sum_{j=0}^{\rho} \tau_j \mathbf{u}(\theta_{i_j})$ and $\mathbf{u}(\theta_{i_j}) \in V_{i_j}$. If $\lambda_j \neq 0$ then as $\mathbf{u}^{(2)}(\theta_{i_1})$ is not orthogonal to V_{i_j} , by Theorem 2.1, $q_{i_1, i_1}^{i_j} \neq 0$. Since $\mathbf{u}^{(3)}(\theta_{i_1}) = \sum_{\ell=0}^{\rho} \lambda_\ell \mathbf{u}(\theta_{i_\ell}) \circ \mathbf{u}(\theta_{i_1})$, if $\tau_j \neq 0$ then there exists ℓ such that $\lambda_\ell \neq 0$ and $\mathbf{u}(\theta_{i_\ell}) \circ \mathbf{u}(\theta_{i_1})$ is not orthogonal to V_{i_j} , by Theorem 2.1, there exists i_ℓ such that $q_{i_1, i_1}^{i_\ell} \neq 0$ and $q_{i_\ell, i_1}^{i_j} \neq 0$. \blacksquare

Let Γ be a distance-regular graph with adjacency matrix A and let $C \subseteq V\Gamma$ be a completely regular code with covering radius ρ , $\text{Spec}^*(C) = \{\theta_{i_1}, \dots, \theta_{i_\rho}\}$ and distance partition $\{C_0, C_1, \dots, C_\rho\}$. For $0 \leq i \leq \rho$, let \mathbf{x}_i denote the characteristic vector of subconstituent C_i . Let $\mathcal{B}^* := \{\mathbf{x}_i \mid i = 0, \dots, \rho\}$ and $\mathcal{B} := \{E_{i_j} \mathbf{x}_0 \mid j = 0, \dots, \rho\}$. Then both \mathcal{B}^* and \mathcal{B} are bases for the outer distribution module $\mathbb{A}\mathbf{x}$ of C . Now consider first the linear transformation \mathcal{A} on $\mathbb{A}\mathbf{x}$ which is defined by $\mathcal{A}(\mathbf{y}) = A\mathbf{y}$ for $\mathbf{y} \in \mathbb{A}\mathbf{x}$. For any nontrivial eigenvalue θ of C , define the linear transformation $\mathcal{A}^*(\theta)$ on $\mathbb{A}\mathbf{x}$ by $\mathcal{A}^*(\theta)(\mathbf{y}) = (E(\theta)\mathbf{x}_0) \circ \mathbf{y}$ for $\mathbf{y} \in \mathbb{A}\mathbf{x}$. Since $A\mathbf{x}_i = \beta_{i-1}\mathbf{x}_{i-1} + \alpha_i\mathbf{x}_i + \gamma_{i+1}\mathbf{x}_{i+1}$, the matrix representing \mathcal{A} with respect to the basis \mathcal{B}^* is irreducible tridiagonal (i.e. each entry on the subdiagonal and each entry on the superdiagonal are nonzero) and the matrix representing \mathcal{A} with respect to the basis \mathcal{B} is diagonal. We can easily check that the matrix representing $\mathcal{A}^*(\theta)$ with respect to the basis \mathcal{B}^* is diagonal as $(E(\theta)\mathbf{x}_0) \circ \mathbf{x}_i = (E(\theta)\mathbf{x}_0)_y \mathbf{x}_i$ where $y \in C_i$. We now define a Leonard completely regular code.

Definition 3.7 *With above notation, a completely regular code C is called Leonard if there exists a nontrivial eigenvalue θ of C such that the matrix representing $\mathcal{A}^* = \mathcal{A}^*(\theta)$ with respect to \mathcal{B} is irreducible tridiagonal. When this happens for a particular eigenvalue θ , we will say that C is Leonard with respect to θ . Note that, following Terwilliger [10, p.150], the pair $\mathcal{A}, \mathcal{A}^*$ is a Leonard pair on $\mathbb{A}\mathbf{x}$ for a Leonard completely regular code.*

Proposition 3.8 *Let Γ be a distance-regular graph. Then any Leonard completely regular code of Γ is a Q -polynomial completely regular code.*

Proof: Let C be a completely regular code with covering radius ρ and characteristic vector \mathbf{x} . Suppose C is Leonard with respect to the nontrivial eigenvalue θ of C . Since the matrix representing $\mathcal{A}^*(\theta)$ is irreducible tridiagonal with respect to some ordering of the basis \mathcal{B} , we may index $\text{Spec}^*(C) = \{\theta_{i_1}, \dots, \theta_{i_\rho}\}$ so that, for $0 \leq j \leq \rho$ we have $E_{i_1}\mathbf{x} \circ E_{i_j}\mathbf{x} = \epsilon_j E_{i_{j-1}}\mathbf{x} + \varphi_j E_{i_j}\mathbf{x} + \psi_j E_{i_{j+1}}\mathbf{x}$ for some scalars $\epsilon_j, \varphi_j, \psi_j$ (ϵ_j and ψ_j being nonzero) where $E_{i_{-1}}\mathbf{x} = E_{i_{\rho+1}}\mathbf{x} = 0$. The result follows. \blacksquare

Definition 3.9 *We say a Leonard code is of type Krawtchouk if the corresponding Leonard pair is of type Krawtchouk as defined in Terwilliger [12]. In a similar fashion, we define Leonard codes of type Hahn, dual Hahn, Racah and so on.*

Sometimes we also say that a Leonard code is of class (I), (IA), (IB), (II), (IIA), (IIB), (IIC), (IID) and (III) if the corresponding Leonard pair is of class (I), (IA), (IB), (II), (IIA), (IIB), (IIC), (IID) and (III), respectively, where we use the notation of Bannai and Ito [1].

It is a natural problem to choose one of these families and to classify all Leonard codes of that type. It is interesting to note that a Leonard code of a given type may appear within a classical distance-regular graph of some other type. For example, the n -cube is obviously a Q -polynomial distance-regular graph of Krawtchouk type, and it contains the binary repetition code, which is not of Krawtchouk type. Below, in Example 3.13, we describe additive binary completely regular codes found by Rifa and Zinoviev which are of dual Hahn type.

Let θ be an eigenvalue of C and $\mathcal{A}^* := \mathcal{A}^*(\theta)$. For $0 \leq i \leq \rho$, as $\mathcal{A}^*\mathbf{x}_i = (E(\theta)\mathbf{x})_y\mathbf{x}_i$ where $y \in C_i$, the vector \mathbf{x}_i is an eigenvector for \mathcal{A}^* . Let F_j^* and F_j denote the primitive idempotent corresponding to \mathbf{x}_j and $E_{i_j}\mathbf{x}$, respectively. In [11, Lemma 5.7], Terwilliger shows that if at least three of the following four conditions hold then $\mathcal{A}, \mathcal{A}^*$ is a Leonard pair.

$$F_h^* \mathcal{A} F_j^* \begin{cases} = 0 & \text{if } h - j > 1 \\ \neq 0 & \text{if } h - j = 1 \end{cases} \quad (0 \leq h, j \leq \rho), \quad (18)$$

$$F_h^* \mathcal{A} F_j^* \begin{cases} = 0 & \text{if } j - h > 1 \\ \neq 0 & \text{if } j - h = 1 \end{cases} \quad (0 \leq h, j \leq \rho), \quad (19)$$

$$F_h \mathcal{A}^* F_j \begin{cases} = 0 & \text{if } h - j > 1 \\ \neq 0 & \text{if } h - j = 1 \end{cases} \quad (0 \leq h, j \leq \rho), \quad (20)$$

$$F_h \mathcal{A}^* F_j \begin{cases} = 0 & \text{if } j - h > 1 \\ \neq 0 & \text{if } j - h = 1 \end{cases} \quad (0 \leq h, j \leq \rho). \quad (21)$$

Note that Equations (18) and (19) together imply that the matrix representing \mathcal{A} with respect to \mathcal{B}^* is irreducible tridiagonal.

Theorem 3.10 *Let Γ be a distance-regular graph with diameter D and $\text{Spec}(\Gamma) = \{\theta_0, \dots, \theta_D\}$. Let C be a completely regular code in Γ . Then C is Leonard if and only if C is Q -polynomial.*

Proof: The ‘only if’ part is done by Proposition 3.8, so we only need to show the ‘if’ part. Let C be a completely regular code with covering radius ρ which is Q -polynomial with respect to eigenvalue θ . Definition 3.1 then gives us a natural ordering $\text{Spec}^*(C) = \{\theta_{i_1}, \dots, \theta_{i_\rho}\}$ where $\theta_{i_1} = \theta$. Let $\mathcal{A}^* := \mathcal{A}^*(\theta)$ and we now consider the products $F_h \mathcal{A}^* F_j$ for $0 \leq h, j \leq \rho$. As $\mathcal{A}^* E_{i_j} \mathbf{x} = E_{i_1} \mathbf{x} \circ E_{i_j} \mathbf{x}$ and as C is Q -polynomial, there exists a polynomial p_{j+1} of degree exactly $j + 1$ such that $\mathcal{A}^* E_{i_j} \mathbf{x} = p_{j+1}(E_{i_1} \mathbf{x})$. Since \mathcal{B} is a basis for $\mathbb{A} \mathbf{x}$ and $\mathcal{A}^* E_{i_j} \mathbf{x} \in \mathbb{A} \mathbf{x}$, we can write $\mathcal{A}^* E_{i_j} \mathbf{x} = \sum_{l=0}^{\rho} \xi_l E_{i_l} \mathbf{x}$ where $\xi_l \in \mathbb{R}$ ($0 \leq l \leq \rho$) satisfy the following condition:

$$\xi_l \begin{cases} = 0 & \text{if } l > j + 1 \\ \neq 0 & \text{if } l = j + 1 \end{cases}.$$

Observe $F_h E_{i_l} \mathbf{x} = \delta_{h,l} E_{i_l} \mathbf{x}$ for $0 \leq h, l \leq \rho$. By this, we find

$$F_h \mathcal{A}^* E_{i_j} \mathbf{x} \begin{cases} = 0 & \text{if } h > j + 1 \\ \neq 0 & \text{if } h = j + 1 \end{cases}.$$

So

$$F_h \mathcal{A}^* F_j \begin{cases} = 0 & \text{if } h - j > 1 \\ \neq 0 & \text{if } h - j = 1 \end{cases}.$$

and the result follows. ■

In [12], Terwilliger gave a parametrization of any Leonard pair. It follows that, for any Leonard pair, there are at most seven free parameters. (Allowing for equivalence under affine transformations, this may be reduced to five.) We now show that the Leonard pair associated to a Q -polynomial completely regular code in a known distance-regular graph has all its parameters determined by just six free parameters.

Corollary 3.11 *Let Γ be a distance-regular graph of valency k and diameter D . Let C be a completely regular code in Γ which is Q -polynomial with respect to the ordering $\eta_0, \eta_1, \dots, \eta_\rho$ of $\text{Spec}(C)$. Then the intersection numbers $\alpha_i, \beta_i, \gamma_i$ ($0 \leq i \leq \rho$) are completely determined (as is the covering radius ρ , from $\beta_\rho = 0$) by the eigenvalues η_1 and η_2 of C together with the parameters $\lambda_0, \lambda_1, \tau_1$ and τ_2 as defined in Proposition 3.5.*

Proof: Since C is Q -polynomial, we have

$$u^{(2)}(\eta_1) = \lambda_0 u(\eta_0) + \lambda_1 u(\eta_1) + \lambda_2 u(\eta_2) \quad (22)$$

and

$$u^{(3)}(\eta_1) = \tau_0 u(\eta_0) + \tau_1 u(\eta_1) + \tau_2 u(\eta_2) + \tau_3 u(\eta_3). \quad (23)$$

Looking at the zero entry on both sides of each equation, we find $\lambda_0 + \lambda_1 + \lambda_2 = 1$ and $\tau_0 + \tau_1 + \tau_2 + \tau_3 = 1$. Now C is Leonard by Theorem 3.10, so there exist scalars $\sigma_1, \sigma_2, \sigma_3$ for which

$$u(\eta_1) \circ u(\eta_2) = \sigma_1 u(\eta_1) + \sigma_2 u(\eta_2) + \sigma_3 u(\eta_3). \quad (24)$$

Moreover, we have $\sigma_1 + \sigma_2 + \sigma_3 = 1$. Next, we may use this and Equation (22) to obtain an alternative expression for $u^{(3)}(\eta_1)$:

$$u^{(3)}(\eta_1) = \lambda_0 \lambda_1 u(\eta_0) + (\lambda_0 + \lambda_1^2 + \lambda_2 \sigma_1) u(\eta_1) + \lambda_2 (\lambda_1 + \sigma_2) u(\eta_2) + \lambda_2 \sigma_3 u(\eta_3).$$

Comparing coefficients against those in Equation (23), we find

$$\begin{aligned} \lambda_0 \lambda_1 &= \tau_0 \\ \lambda_0 + \lambda_1^2 + \lambda_2 \sigma_1 &= \tau_1 \\ \lambda_2 (\lambda_1 + \sigma_2) &= \tau_2 \\ \lambda_2 \sigma_3 &= \tau_3 \end{aligned}$$

so that $\lambda_2, \tau_0, \tau_3$ are determined by knowledge of $\lambda_0, \lambda_1, \tau_1$ and τ_2 . Now all we need are the eigenvalues needed in Proposition 3.5. But we know $\eta_0 = k$,

the valency of Γ , we are given η_1 and η_2 by hypothesis and we may then solve for η_3 by looking at the $i = 1$ entry on both sides of (23):

$$\tau_0 + \tau_1 \frac{\eta_1 - \alpha_0}{k - \alpha_0} + \tau_2 \frac{\eta_2 - \alpha_0}{k - \alpha_0} + \tau_3 \frac{\eta_3 - \alpha_0}{k - \alpha_0} = \left(\frac{\eta_1 - \alpha_0}{k - \alpha_0} \right)^3$$

where we have used the evaluation (9) $u_1(\theta) = (\theta - \alpha_0)/(k - \alpha_0)$. Now the result follows from Proposition 3.5. \blacksquare

Conjecture 3.12 *Every completely regular code in a Q -polynomial distance-regular graph with sufficiently large covering radius is a Leonard completely regular code.*

We finish this section with a description of an interesting family of codes in the n -cubes.

Example 3.13 *In any $\binom{m}{2}$ -cube for integer $m \geq 3$, there exist Leonard completely regular codes which are not of Krawtchouk type. Following [9], for natural numbers $m \geq 3$ and $2 \leq l < m$, define E_l^m as the set of all binary vectors of length m and weight l . Denote by $H^{(m,l)}$ the binary matrix of size $m \times \binom{m}{l}$, whose columns are exactly all vectors from E_l^m . Rifa and Zinoviev consider the binary linear code $C^{(m,l)}$ whose parity check matrix is the matrix $H^{(m,l)}$; they show that the code $C^{(m,2)}$ is completely regular and its coset graph is the halved m -cube. As the halved m -cube is Q -polynomial, it follows that $C^{(m,2)}$ is Leonard, but it is of dual Hahn, not Krawtchouk, type.*

4 Harmonic completely regular codes

In a companion paper [6], we explore a well-structured class of Leonard completely regular codes in the Hamming graphs. These *arithmetic* completely regular codes are defined as those whose eigenvalues are in arithmetic progression: $\text{Spec}(C) = \{k, k - t, k - 2t, \dots\}$. These codes have a rich structure and are intimately tied to Hamming quotients of Hamming graphs. In [6], we study products of completely regular codes and completely classify the possible quotients of a Hamming graph that can arise from the coset partition of a linear arithmetic completely regular code. For families of distance-regular

graphs other than the Hamming graphs, we need to look at a slightly weaker definition to probe the same sort of rich structure.

We next introduce the class of *harmonic* completely regular codes and we will see that this class lies strictly between the arithmetic completely regular codes and the Leonard completely regular codes.

Definition 4.1 *Let Γ be a Q -polynomial distance-regular graph with respect to the ordering $\theta_0, \theta_1, \dots, \theta_D$ of its eigenvalues and C be a completely regular code of Γ . We call the code C harmonic if $\text{Spec}(C) = \{\theta_{ti} \mid i = 0, \dots, \rho\}$ for some positive integer t .*

Let Γ be a Q -polynomial with respect to the ordering $\{\theta_0, \theta_1, \dots, \theta_D\}$ of its eigenvalues and let $C \subseteq V\Gamma$ be a code. Then *strength* of C , $t(C)$ is defined as the $\min\{i \geq 1 \mid \theta_i \in \text{Spec}^*(C)\} - 1$.

Example 4.2 *The following are examples of harmonic completely regular codes:*

- (1) *the repetition code in a hypercube;*
- (2) *cartesian products of a completely regular code of a Hamming graph $C \times \dots \times C$ where C is covering radius 1;*
- (3) *in the Grassmann Graph $J_q(n, t)$, whose vertices are all t -dimensional subspaces of a some n -dimensional vector space V over $GF(q)$, we find the following two families:*

- *C consists of all t -dimensional subspaces of a given $(n - s)$ -dimensional subspace of V , where $0 < s < n - t$;*
- *C consists of all t -dimensional subspaces of V containing a fixed s -dimensional subspace U of V , where $0 < s \leq t < n$.*

(We note that the Johnson graph $J(n, t)$ contains examples analogous to these.)

- (4) *any completely regular code of strength 0 in a Q -polynomial distance-regular graph.*

Lemma 4.3 *Let Γ be a Q -polynomial distance-regular graph with respect to the ordering $\theta_0, \theta_1, \dots, \theta_D$ of its eigenvalues. Then any harmonic completely regular code is a Leonard completely regular code.*

Proof: Since Γ is Q -polynomial, there exist numbers $\omega_{h,j}$ such that $E_t \mathbf{x}_0 \circ E_{jt} \mathbf{x}_0 = \sum_{h=0}^p \omega_{h,j} E_{ht} \mathbf{x}_0$ and the following holds:

$$\omega_{h,j} \begin{cases} = 0 & \text{if } |ht - jt| > t \\ \neq 0 & \text{if } |ht - jt| \leq t \end{cases} .$$

So,

$$\omega_{h,j} \begin{cases} = 0 & \text{if } |h - j| > 1 \\ \neq 0 & \text{if } |h - j| \leq 1 \end{cases} .$$

Hence the matrix representing $\mathcal{A}^*(\theta_t)$ is irreducible tridiagonal with respect to \mathcal{B} . ■

Finally, we remark that the codes given in Example 3.13 are Leonard but not harmonic.

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References

- [1] E. BANNAI AND T. ITO, *Algebraic Combinatorics I: Association Schemes*, Benjamin-Cummings Lecture Note Ser. 58, Benjamin/Cummings, London, 1984.

- [2] A.E. BROUWER, A.M. COHEN, AND A. NEUMAIER, *Distance-Regular Graphs*, Springer, Heidelberg, 1989.
- [3] P. J. CAMERON, J. M. GOETHALS, AND J. J. SEIDEL, *The Krein condition, spherical designs, Norton algebras and permutation groups*. Proc. Kon. Nederl. Akad. Wetensch. (Indag. Math.) 40 no. 2, (1978), pp. 196-206.
- [4] P. DELSARTE, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. No. 10 (1973), vi+97.
- [5] C. D. GODSIL, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.
- [6] J. H. KOOLEN, W. S. LEE AND W. J. MARTIN, Arithmetic completely regular codes, Preprint, October 2009.
- [7] W.J. MARTIN, *Symmetric designs, sets with two intersection numbers, and Krein parameters of incidence graphs*. J. Combin. Math. and Combin. Comput., 38 (2001), pp. 185-196.
- [8] A. NEUMAIER, *Completely regular codes, in: A collection of contributions in honour of Jack van Lint*, Discrete Math., 106/107 (1992), pp. 353-360.
- [9] J. RIFA, V.A. ZINOVIEV, *On a class of binary linear completely regular codes with arbitrary covering radius*, Preprint (2008).
- [10] P. TERWILLIGER, *Two linear transformations each tridigonal with respect to an eigenbasis of the other*, Linear Algebra and its Applications, 330 (2001), pp. 149-203.
- [11] P. TERWILLIGER, *Two linear transformations each tridigonal with respect to an eigenbasis of the other: comments on the split decomposition*, Journal of Computational and Applied Mathematics, 178 (2005), pp. 437-452.
- [12] P. TERWILLIGER, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array*, Des. Codes Cryptogr., 34 (2005), no. 2-3, pp. 307-332.