Chapter 4

Completely regular subsets in the Johnson graphs

The Johnson graph $J(v, \ell)$ is a distance-regular graph constructed on the $\ell$-element subsets of a $v$-set. These graphs provide a framework for the study of combinatorial $t$-designs. Any completely regular subset in the Johnson graph is a $t$-design for some value of $t$. We refer to such a subset as a completely regular $t$-design.

Delsarte [26] investigated the algebraic structure of the Johnson graphs in his thesis. Moreover, he was able to provide an algebraic characterisation for subsets which form combinatorial $t$-designs (Section 4.1). We explore this in the case of completely regular subsets. We exhibit several families of examples of completely regular designs in Section 4.2. These include $t$-designs with block size $t + 1$, Steiner systems with block size $t + 2$, and the Witt designs on twenty-three and twenty-four points. We discuss completely regular zero-designs and one-designs (Theorem 4.2.2). We also construct a family of completely regular two-designs related to projective planes (Proposition 4.2.5).
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In Section 4.3, we obtain bounds on the covering radius and block size of a completely regular design (Proposition 4.3.1 and Theorem 4.3.2). The latter bound is used in Section 4.4 to show that most symmetric designs are not completely regular subsets. We also show that a symmetric design with even minimum distance cannot be completely regular, with one exception (Theorem 4.4.2). In [60], Meyerowitz classifies the completely regular designs of strength zero. We apply this approach to determine all completely regular one-designs with minimum distance at least two in Theorem 4.5.3.

There is a local characterisation for the Johnson graphs: various parts of the theorem are due to Dowling [31], Moon [62], and Blokhuis and Brouwer [10] (see [3, Theorem 9.1.3]). For us, this theorem implies that the Johnson graph admits few quasi-linear partitions (Corollary 4.6.2). We explore some of the remaining possibilities in Section 4.6 (Theorem 4.6.3 and Proposition 4.6.4). A family of graphs closely related to the Johnson graphs $J(2\ell+1, \ell)$ is the family of Odd graphs (Section 4.7). Codes in these graphs were studied by Hammond and Smith [44, 71]. We present their results as well as two new results, one of which (Theorem 4.7.4) states the every perfect code in the Odd graph is a completely regular subset in the Johnson graph.

4.1 The Johnson Graph

Let $\mathcal{V}$ be the set $\{1, 2, \ldots, v\}$ containing the first $v$ positive integers. For an integer $\ell$, $0 \leq \ell \leq v$, we consider the graph $J(v, \ell)$ whose vertices are the $\ell$-element subsets of $\mathcal{V}$. Two vertices $x, y$, are adjacent in $J(v, \ell)$ precisely when their intersection has size $\ell - 1$. It follows that the distance in $J(v, \ell)$ between vertices $x$ and $y$ is equal to $\ell - |x \cap y|$. If $\ell \leq v/2$, then $J(v, \ell)$ has diameter $\ell$. Since $J(v, v - \ell)$ is
isomorphic to \( J(v, \ell) \), we will henceforth restrict ourselves to \( \ell \leq v/2 \). The graph \( J(v, \ell) \) is called a Johnson graph; it is distance-regular. Delsarte [26] was first to analyse the Johnson graph as an association scheme.

Denote by \( \theta_0 > \theta_1 > \cdots > \theta_\ell \) the \( \ell + 1 \) distinct eigenvalues of \( J(v, \ell) \). (This ordering will remain fixed throughout.) Then we have

\[ \theta_j = (\ell - j)(v - \ell - j) - j. \]

The Johnson graph is \( Q \)-polynomial with respect to this ordering of its eigenvalues. The \( j \)-th eigenvalue of \( J(v, \ell) \) has multiplicity

\[ m_j = \binom{v}{j} - \binom{v}{j - 1}. \]

All of this information is derived by Delsarte in [26, Section 4.2].

We are interested in completely regular subsets of the Johnson graph. A collection \( C \) of \( \ell \)-element subsets of \( V \) is called a \( t \)-design, for \( 0 \leq t \leq \ell \), if there exists a constant \( \lambda \) such that every \( t \)-element subset of \( V \) is contained in precisely \( \lambda \) elements of \( C \). We will refer to such a design as a \( t - (v, \ell, \lambda) \) design. We often refer to the elements of \( C \) as blocks. It is not difficult to show that any \( t \)-design is an \( s \)-design for \( 0 \leq s \leq t \). If every \( t \)-set occurs in \( \lambda_t \) blocks, then every \( s \)-set occurs in \( \lambda_s \) blocks where \( \lambda_s \) satisfies

\[ \lambda_s \binom{\ell - s}{t - s} = \lambda_t \binom{v - s}{t - s}. \quad (4.1) \]

The fact that these parameters must be integers is the most effective known nonexistence condition for \( t \)-designs. In fact, the current philosophy is that, for \( v \) large enough, these necessary conditions are also sufficient.

As a collection of \( \ell \)-element subsets of a \( v \)-set, any vertex subset of \( J(v, \ell) \) can be viewed as a \( t \)-design for some appropriate value of \( t \) (possibly zero). We define
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the strength of $C$ to be the largest integer $t$ for which $C$ is a $t$-design. The following theorem of Delsarte has great importance for us:

**Theorem 4.1.1 (Delsarte, [26, Theorem 4.2])** Let $C$ be a subset of the vertices of the Johnson graph $J(v, \ell)$ and let $\chi$ denote the characteristic vector of $C$. Then $C$ is a $t$-design if and only if $\chi$ is orthogonal to the eigenspace corresponding to $\theta_j$ for each $1 \leq j \leq t$. □

Now suppose $C$ is a completely regular subset in $J(v, \ell)$ and suppose $B$ is the quotient matrix corresponding to the distance partition of $J(v, \ell)$ with respect to $C$. Then we know from Corollary 3.2.2 that $Z_j\chi$ is non-zero if and only if $\theta_j$ is an eigenvalue of $B$. Hence, Delsarte's theorem, applied to completely regular subsets, states

**Corollary 4.1.2** Let $C$ be a completely regular subset in the Johnson graph and let $B$ be the quotient matrix of $C$. The strength of $C$ as a combinatorial design is equal to that $t$ for which $\theta_{t+1}$ is the second largest eigenvalue of $B$. □

Accordingly, we shall call a completely regular subset of $J(v, \ell)$ a completely regular $t$-design if its quotient matrix has second largest eigenvalue equal to $\theta_{t+1}$. That is, a completely regular $t$-design is a completely regular subset of strength exactly $t$. It is important to notice that a completely regular $t$-design has dual degree at most $\ell - t$. (Recall the equation on page 70.)

The parameters $\lambda_i$ ($0 \leq i \leq t$) are related to the inner distribution of a completely regular design. If $a = [E_{00}, E_{01}, \ldots, E_{0\ell}]$ denotes the inner distribution of a completely regular $t - (v, \ell, \lambda_i)$ design $C$, then clearly $|C| = \lambda_0 = \sum E_{0i}$. In fact, the inner distribution determines all of the $\lambda_i$. 
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Lemma 4.1.3 If $C$ is a completely regular $t$-$(v, \ell, \lambda_i)$ design with inner distribution $a = [E_{00}, E_{01}, \ldots, E_{0\ell}]$, then, for $0 \leq s \leq t$, we have

$$
\sum_{i=0}^{t} E_{0i}\binom{\ell - i}{s} = \binom{\ell}{s}\lambda_s.
$$

Proof: Fix a block $x$ of the design. Count in two ways the pairs $(y, z)$ where $y$ is an $s$-set contained in $x$ and $z$ is a block containing $y$. The result now follows since there are $E_{0i}$ blocks meeting $x$ in $\ell - i$ points and each of these contains $\binom{\ell - i}{s}$ $s$-subsets of $x$. □

4.2 The known examples of completely regular designs

In this section, I discuss all the designs which I know to be completely regular. In most cases, a family is defined and an argument is given which determines which designs in that family are completely regular.

Suppose $C$ is a completely regular design with covering radius $\rho$. We define the opposing set of design $C$ to be the set $C_\rho$ of $\ell$-sets lying at distance $\rho$ from $C$ where $\rho$ is the covering radius. Since the distance partition of $J(v, \ell)$ with respect to $C$ is the same as the partition of this graph with respect to $C_\rho$, we see that $C_\rho$ is completely regular whenever $C$ is completely regular. This observation is due to Neumaier. Also note that the opposing set of $C$ has strength at least the strength of $C$ (Corollary 3.2.2). By symmetry we get equality: the two designs have the same strength.

Proposition 4.2.1 Let $K$ be a subset of $V$ having size $k \geq \ell$. Then the subset

$$
C := \{x : |x| = \ell, x \subseteq K\}
$$
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is a completely regular 0-design in \( J(v, \ell) \) with covering radius \( \rho = \min(v - k, \ell) \). Similarly, if \( K \) is a subset of \( \mathcal{V} \) with \( |K| \leq \ell \), then

\[
C := \{ x : |x| = \ell, \ K \subseteq x \}
\]

is a completely regular 0-design in \( J(v, \ell) \) with covering radius \( \rho = |K| \).

Proof: Suppose \( K \) has size \( k \geq \ell \) and define \( C := \{ x : |x| = \ell, x \subseteq K \} \). Then the set of all \( \ell \)-sets at distance \( i \) from \( C \) is given by

\[
C_i = \{ y : |y| = \ell, |y \cap K| = \ell - i \}
\]

and this set is nonempty for \( 0 \leq i \leq \min(\ell, v - k) \). A vertex \( y \) in \( C_i \) has \( i(k - \ell + i) \) neighbours in \( C_{i-1} \) and \((\ell - i)(v - k - i)\) neighbours in \( C_{i+1} \). Hence the distance partition of \( J(v, \ell) \) with respect to \( C \) is equitable. The second case is completely analogous. \( \Box \)

Let us look at the subset \( C = \{ x : |x| = \ell, K \subseteq x \} \) for a set \( K \) of size \( k \). Assuming \( \ell \leq v/2 \), this subset has covering radius \( k \). So we expect the quotient matrix \( B \) to have exactly \( k + 1 \) eigenvalues. We now determine these eigenvalues indirectly.

The characteristic vector \( \chi \) of \( C \) is a column of the incidence matrix \( W_{\ell,k} \) of \( \ell \)-subsets of \( \mathcal{V} \) versus \( k \)-subsets of \( \mathcal{V} \). Delsarte proved the following relationship between these incidence matrices and the primitive idempotents of the Johnson scheme \( J(v, \ell) \):

\[
\text{colsp}(W_{\ell,k}) = \text{colsp}(Z_0) \oplus \text{colsp}(Z_1) \oplus \cdots \oplus \text{colsp}(Z_k).
\]

(This is a consequence of Lemma 4.5 in [26].) Hence, for \( j > k \), we must have \( Z_j \chi = 0 \). This guarantees that the eigenvalues of \( B \) are \( \theta_0, \theta_1, \ldots, \theta_k \).
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We next look at a similar class of designs having rather simple structure. Consider the Johnson graph \( J(v, \ell) \) where \( v = qp \) and \( \ell = sp \) with \( p \geq 2 \) and \( q \geq 2s \). Let
\[
\mathcal{X} = \{X_1, X_2, \ldots, X_q\}
\]
be a partition of \( \mathcal{Y} \) into \( q \) "groups", each of size \( p \). The blocks of \( C \) will be the \( \binom{q}{2} \) subsets of the form
\[
\gamma = \bigcup_{i \in I} X_i
\]
where \( I \) is any \( s \)-element subset of \( \{1, 2, \ldots, q\} \). Since \( p \geq 2 \) and \( q > s \), such a design has strength one. It is also clear that \( C \) has minimum distance equal to \( p \). For the following discussion, let us call such a design a group-wise complete design.

Let us now determine for which values of \( p, q \) and \( s \) such a 1-design is completely regular.

**Theorem 4.2.2** Let \( C \) be a group-wise complete design in \( J(v, \ell) \). Then \( C \) is completely regular if and only if one of the following holds:

(i) \( p = \ell \) and \( v = 2\ell \) and \( C \) is an antipodal class (containing two elements);

(ii) \( p = 2 \);

(iii) \( p = 3 \) and \( s = 1 \).

**Proof:** Let \( C \) be a group-wise complete design in \( J(v, \ell) \) with \( v = qp \) and \( \ell = sp \) \((p \geq 2, q \geq 2s)\). Let \( C \) be constructed as described above from partition \( \mathcal{X} \) of \( \mathcal{Y} \) into \( q \) groups of size \( p \). Let \( \gamma = X_1 \cup X_2 \cup \cdots \cup X_s \) be an element of \( C \). If \( q \geq s + 2 \), we take \( X \) and \( X' \) to be distinct groups outside \( \gamma \). Let \( h \) and \( h' \) be points in \( X_1, j, k \) in \( X \) and \( k' \) in \( X' \). Consider the sets \( w := \gamma - h - h' + j + k \) and \( w' := \gamma - h - h' + j + k' \), both at distance two from \( \gamma \). Vertex \( w \) is at distance
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$p - 2$ from the block $z - X_1 + X$ while $w'$ is at distance at least $p - 1$ from every block other than $z$. That is, $w$ and $w'$ have different profiles with respect to $C$. Hence, if $C$ is completely regular, $w$ and $w'$ must be in different cells of the distance partition of $J(u, \ell)$ with respect to $C$. This forces the minimum distance $p$ to be at most three, except in the case where $s = 1$ and $q = 2$. This last case corresponds to an antipodal pair of vertices in $J(2p, p)$ which we already know to be completely regular.

Next, we modify the above argument to deal with the case $p = 3$. Suppose $s \geq 2$ and let $z = X_1 \cup X_2 \cup \cdots \cup X_s$ be a block of the design. Let $h$ and $h'$ be elements of $z$, now in distinct groups. Let $X$ and $X'$ be groups outside $z$ and let $j$, $k$ be elements of $X$ and $k'$ an element of $X'$. The sets $w := z - h - h' + j + k$ and $w' := z - h - h' + j + k'$, are both at distance two from $C$. Yet $w$ is at distance two from three blocks while $w'$ is at distance two from only one. Thus, such a design cannot be completely regular.

The only cases which remain are $p = 3$, $s = 1$ and $p = 2$ (any permissible value of $s$). It is not difficult to verify in both these cases that $C$ is completely regular.

The above class of 1-designs may seem somewhat trivial, but they will arise later, in Section 4.5. In fact, we will show that any completely regular 1-design having minimum distance at least two is of this type. Also, we note that the collection of $\ell$-sets at maximal distance from such a design (i.e., the opposing set) has the form

$$\{w : |w| = \ell, |X_i \cap w| \leq \left\lceil \frac{q}{\ell} \right\rceil \text{ for each } i = 1, 2, \ldots, q\}$$

and is a completely regular design of strength one.

The following two results can be viewed as corollaries of Theorem 2.1.11 on page 28.
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Theorem 4.2.3 Let \( C \) be a \( t \)-design in \( J(v, \ell) \). If \( \ell = t + 1 \), then \( C \) is completely regular.

Proof: By Theorem 4.1.1, such a set has \( \mathbb{Z}_J \chi_C = 0 \) for \( 1 \leq j < \ell \). So it has dual degree \( r = 1 \). Since there are no repeated blocks allowed, we have minimum distance \( \delta \geq 1 \). Now the result follows directly from Theorem 2.1.11. \( \Box \)

The quotient matrix of such a design has the form

\[
\begin{pmatrix}
\ell(\lambda - 1) & \ell(v - \ell - \lambda + 1) \\
\ell \lambda & \ell(v - \ell - \lambda)
\end{pmatrix},
\]

with eigenvalues \( \theta_0 = \ell(v - \ell) \) and \( \theta_\ell = -\ell \), as expected. This result shows that all triple systems \((2 - (v, 3, \lambda) \text{ designs})\) are completely regular.

There exist \( t \)-designs with block size \( t + 2 \) which are not completely regular (almost any non-regular graph will suffice, viewed as a subset of \( J(v, 2) \)). However, we can establish complete regularity for a special case. Recall that a \( t \)-design is called a Steiner system if every \( t \)-set occurs in exactly one block of \( C \); i.e., if \( \lambda_t = 1 \).

Theorem 4.2.4 Let \( C \) be a Steiner system in \( J(v, \ell) \) with \( \ell = t + 2 \). Then \( C \) is completely regular.

Proof: This result, too, follows from Theorems 2.1.11 and 4.1.1. Since \( C \) is an \((\ell - 2)\)-design, it has dual degree at most two. On the other hand, any \((\ell - 2)\)-set occurs in only one block, so the minimum distance \( \delta \) is at least three. Hence, we obtain \( \delta \geq 2r - 1 \) and \( C \) is completely regular. \( \Box \)

With some work, we can determine that the quotient matrix for such a design
is

\[ B = \begin{pmatrix}
0 & \ell(v - \ell) & 0 \\
1 & v - 2 + 2(\ell - 1)(\ell - 2) & (\ell - 1)(v - 3\ell + 3) \\
0 & 2\ell(\ell - 1) & \ell(v - 3\ell + 2)
\end{pmatrix} \]

whenever \( v \geq 3\ell - 2 \). This matrix has eigenvalues \( \theta_0 = \ell(v - \ell) \), \( \theta_{\ell-1} = v - 3\ell + 2 \), and \( \theta_{\ell} = -\ell \). For such a design, any \( \ell \)-set in \( C_1 \) must have \( v - 2 + 2(\ell - 1)(\ell - 2) \) neighbours in \( C_1 \). This implies that \( v \geq 3\ell - 3 \). An exceptional case arises when \( v = 3\ell - 3 \). The subset is still completely regular, but has covering radius one. This is a parameter set for a perfect code in \( J(3\ell - 3, \ell) \). Aside from an antipodal class in \( J(6, 3) \), no examples are known.

Theorem 4.2.4 tells us that all \( 2-(v, 4, 1) \) Steiner systems are completely regular.

We will see below that a projective plane of order \( q \) is not completely regular when \( q \geq 4 \). But we can construct a completely regular design from each of these projective planes by taking the “4-shadow” as follows. We fix a \( 2-(q^2 + q + 1, q + 1, 1) \) design \( P \) and we define a design \( C \) in \( J(q^2 + q + 1, 4) \) by choosing as blocks the four-element sets which are contained in blocks of \( P \).

**Proposition 4.2.5** For \( q \geq 3 \), the 4-shadow of a projective plane of order \( q \) is completely regular.

**Proof:** We will use Theorem 2.1.13. Let \( C \) be such a design and let \( x \) be a block. Then \( x \) is contained in some line \( \ell \) of the geometry. Vertex \( x \) is adjacent to \( 4(q - 3) \) vertices of \( C \), all subsets of \( \ell \). The remaining \( 4q^2 \) neighbours of \( x \) lie in \( C_1 \).

A vertex \( y \) in \( C_1 \) is composed of three points on some line \( \ell \) and one point off \( \ell \). So \( y \) has \( q - 2 \) neighbours in \( C \). Aside from \( \ell \), \( y \) meets only three lines in more than one point. These three meet \( y \) in two points each. So \( y \) is at distance two from
3^{(q-2)} 4-element subsets of \ell and from \binom{q-1}{2} 4-element subsets of each of these other three lines. Since no other line meets \( y \) in more than one point, these are all the vertices of \( C \) at distance two from \( y \). In total, \( y \) is at distance two from \( 3(q-2)^2 \) vertices of \( C \).

The design \( C \) clearly has covering radius two. A vertex \( z \) at distance two from \( C \) consists of four points, no three collinear. There are six lines determined and \( z \) is at distance two from \( \binom{q-1}{2} \) 4-element subsets of each. Thus, \( z \) is at distance two from exactly \( 3(q-1)(q-2) \) vertices of \( C \).

So we have the following partial information concerning the reduced outer distribution matrix of \( C \):

\[
E = \begin{pmatrix}
1 & 4(q-3) & \cdots \\
0 & q-2 & 3(q-2)^2 & \cdots \\
0 & 0 & 3(q-1)(q-2) & \cdots
\end{pmatrix}
\]

Using Theorem 2.1.13, this is sufficient information to conclude that \( C \) is completely regular. \( \Box \)

The 1-shadow and 2-shadow of a 2-design are trivial. The 3-shadow of a projective plane is completely regular by virtue of Theorem 4.2.3. The 5-shadow of a projective plane is never completely regular. Figure 4.1 shows that there are two types of vertices at distance two from such a design.

The Witt designs

Here we analyse the Witt designs and related designs. We look at the unique designs with parameters 4 — (11, 5, 1), 5 — (12, 6, 1), 2 — (21, 5, 1), 3 — (22, 6, 1), 4 — (23, 7, 1), and 5 — (24, 8, 1). Each of these designs is determined up to isomorphism by its parameters. The small Witt design 4 — (11, 5, 1) and its extension 5 — (12, 6, 1) are both completely regular by Theorem 4.2.3. The projective plane of order four
Figure 4.1: Two types of vertices at distance two from the 5-shadow of a projective plane. On the left, we have $D_{a2} = 2$ while, on the right, $D_{y2} = 1$.

$PG(2, 4)$ is not a completely regular subset. This follows from Proposition 4.4.1.

Three designs remain.

**The Witt design $M_{22}$ on 22 points**

The Witt design $M_{22}$ is the first extension of $PG(2, 4)$. We have twenty-two points and seventy-seven blocks. The derived design at any point is isomorphic to $PG(2, 4)$. We view this design as a set of 77 vertices of the Johnson graph $J(22, 6)$.

Any block $b$ of this design is at distance four from sixty blocks and at distance six from sixteen blocks. The $\binom{9}{2} \cdot \binom{16}{2} = 1800$ vertices of $J(22, 6)$ lying at distance two from $b$ are all at distance two from $M_{22}$ itself. If two vertices of $J(v, \ell)$ are at distance four, then there are exactly 36 vertices at distance two from both. Now let $x$ be an arbitrary vertex of $J(22, 6)$ which is at distance two from $M_{22}$. We have that the average number of blocks at distance two from $x$ is $1 + \frac{60 \cdot 36}{1800}$ which is not an integer. We conclude that this number cannot be constant for all such $x$, whence $M_{22}$ cannot be a completely regular subset in $J(22, 6)$.

**The Witt design $M_{24}$ on 24 points**

Since we will use facts about the Witt design on twenty-four points in our
discussion of the $4 - (23, 7, 1)$ design $\mathcal{M}_{23}$, we will discuss this design first.

The large Witt design $\mathcal{M}_{24}$ with 759 blocks is a completely regular subset in $J(24, 8)$. This was observed by Delsarte [26] and follows from Theorem 2.1.11, stating that a subset with dual degree $r$ and minimum distance at least $2r - 1$ is necessarily completely regular. The large Witt design has dual degree two (indeed it is balanced in all representations except those corresponding to $\theta_0$, $\theta_6$, and $\theta_8$). Yet its minimum distance is four.

We now compute the quotient matrix of $J(24, 8)$ corresponding to the distance partition with respect to $\mathcal{M}_{24}$. Delsarte obtains

$$E = \begin{pmatrix}
1 & 0 & 0 & 0 & 280 & 0 & 448 & 0 & 30 \\
0 & 1 & 0 & 35 & 140 & 231 & 252 & 85 & 15 \\
0 & 0 & 4 & 32 & 130 & 256 & 228 & 96 & 13
\end{pmatrix}.$$ 

Using Corollary 2.1.7, we determine

$$B = \begin{pmatrix}
0 & 128 & 0 \\
1 & 22 & 105 \\
0 & 16 & 112
\end{pmatrix}.$$ 

The Witt design $\mathcal{M}_{23}$ on 23 points

The Witt design $\mathcal{M}_{23}$ is a one-point extension of $\mathcal{M}_{22}$. The derived design at any point is isomorphic to $\mathcal{M}_{22}$ and the derivation at any two points leaves a copy of $PG(2, 4)$.

**Proposition 4.2.6** The Witt design $\mathcal{M}_{23}$ on twenty-three points is completely regular.
Figure 4.2: The Witt design on 23 points is viewed as the set of blocks of $\mathcal{M}_{24}$ containing a fixed point 1.

**Proof:** We will use a slight variation of Theorem 2.1.13. This design has minimum distance four. So we have the following partial information regarding the reduced outer distribution matrix.

$$E = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & ? & ? & \cdots \\ 0 & 0 & 0 & ? & \cdots \end{pmatrix}$$

We will prove that $E_{22}$ and $E_{23}$ are well-defined. Then, rather than determining $E_{33}$ we will determine $\alpha_3$ and $\gamma_3$ directly (cf. proof of Theorem 2.1.13).

We view $\mathcal{M}_{23}$ as the derived design of $\mathcal{M}_{24}$ at some point, "1", say. The vertices of $J(23, 7)$ are identified with the vertices of $J(24, 8)$ containing 1. The large Witt design has covering radius two; let $\sigma = \{M_0, M_1, M_2\}$ be the distance partition of $J(24, 8)$ with respect to this subset. Write $M_0 = C_0 \cup N_0$ where $C_0$ consists of the blocks containing 1 and $N_0$ is the remainder. We wish to determine the distance partition of $J(23, 7)$ with respect to $C_0$. The vertices at distance one from $C_0$ are easily determined since $M_0$ has minimum distance four.
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From above, any 8-set at distance two from \( M_0 \) is at distance two from four blocks of the design. However, each pair of these blocks meets in (at most) four points. So if \( x = \{1, 2, 3, 4, 5, 6, 7, 8\} \) is at distance two from the large design, then, for some partition of \( x \), \( \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \) say, the four blocks each meet \( x \) in three cells of this partition and miss the fourth. So precisely three of these blocks contain 1. This proves that \( C_2 \) contains \( \{x \in M_2 : 1 \in x\} \). We also have \( D_{x2} = 3 \) for each \( x \) of this type. The only vertices of \( J(23, 7) \) which remain unaccounted for are the vertices of \( M_1 \) which contain 1 but are not adjacent to any vertex in \( C_0 \). Denote the set of such vertices by \( R \). Such a vertex is adjacent to a unique vertex in \( N_0 \). Since any two blocks of the large design are at distance four or more, a vertex of \( R \) is at distance at least three from each vertex of \( C_0 \). Since \( M_{24} \) is completely regular, each vertex of \( R \) is at distance three from some vertex of \( C_0 \). So \( R = C_3 \). This also implies that \( C_2 = \{x \in M_2 : 1 \in x\} \) since all other vertices are accounted for. Consequently \( E_{22} = 3 \) is constant for all \( x \in C_2 \).

We know that each vertex in \( C_3 \) is adjacent to exactly one vertex of \( N_0 \) (blocks not containing 1) and at distance at least three from all others. For each block of \( N_0 \), there are eight vertices in \( C_3 \) (remove any element and insert 1) and these induce a complete graph \( K_8 \). To sum up, this implies that the induced graph on \( C_3 \) is isomorphic to \( 506 \cdot K_8 \); so we have \( \alpha_3 = 7 \). Therefore, \( \gamma_3 = 105 \) and we no longer need to prove that \( D_{x3} \) is constant for \( x \in C_3 \).

It remains only to prove that \( D_{x3} \) is constant for \( x \in C_2 \). From the reduced outer distribution matrix for \( M_{24} \), we see that any vertex in \( M_2 \) is at distance three from thirty-two blocks. Let \( x = \{1, 2, 3, 4, 5, 6, 7, 8\} \) be a vertex in \( C_2 \) and let \( b_1, b_2, b_3, b_4 \) be the four blocks at distance two from \( x \). Let \( \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\} \) be the partition of \( x \) determined by the intersection of \( x \) with these four blocks. A block \( b \) at distance three from \( x \) must contain five of these points; but no more than four
in any of the blocks $b_1, \ldots, b_4$. So $b$ contains one cell of this partition and meets the other three in one point each. But there are thirty-two such blocks $b$ and there are only thirty-two such five-element subsets of $x$. So the blocks at distance three from $x$ meet $x$ in all thirty-two possible ways. In particular, any point of $x$ lies in exactly $\frac{32 \times 8}{8} = 20$ of these blocks. This proves that $E_{23} = 20$ is well-defined. □

The quotient matrix of $M_{23}$ in $J(23, 7)$ is as follows:

$$
B = \begin{pmatrix}
0 & 112 & 0 & 0 \\
1 & 21 & 90 & 0 \\
0 & 12 & 98 & 2 \\
0 & 0 & 105 & 7 \\
\end{pmatrix},
$$

with eigenvalues 112, 17, 4, and $-7$.

### 4.3 Bounds on the covering radius and block size

The covering radius of a design is an interesting parameter which seems to have received very little attention in the literature. In our discussion (restricted to completely regular designs), the covering radius will play an important role. For now, we simply mention the following elementary result.

**Proposition 4.3.1** The covering radius of a $t-(v, \ell, \lambda)$ design is at most $\ell - t$.

**Proof:** Every vertex $x$ of $J(v, \ell)$ contains a $t$-set and each $t$-set lies in at least one block $y$. So $\text{dist}(x, C) \leq \text{dist}(x, y) \leq \ell - t$. □

It may be that $\lambda_1$ is so large that the covering radius is less than $\ell - t$. Also note that this fact follows from two inequalities involving the dual degree which we have
already discussed. The first is \( r \leq \ell - t \) which was explained after the statement of Corollary 4.1.2. The second is \( \rho \leq r \) which was proved at the end of Section 2.1.2.

We now apply Delsarte’s definition of complete regularity to obtain an upper bound on the block size of a completely regular design.

**Theorem 4.3.2** If \( C \) is a completely regular design with block size \( \ell \) and largest block intersection \( m > 0 \), then \( \ell \leq 3m + 1 \).

**Proof:** Let \( b \) and \( b' \) be distinct blocks of \( C \) meeting in \( m \) points. Set \( s := \lfloor \frac{\ell - m}{2} \rfloor \). Assume \( s \geq m \). Let \( S_1 \) be an \( s \)-subset of \( b \setminus b' \), and let \( S_2 \) be an \( s \)-subset of \( b \) containing \( b \cap b' \). Let \( S_3 \) be an \( s \)-subset of \( b' \setminus b \). Each of the sets

\[ r_1 := (b \setminus S_1) \cup S_3 \]

and

\[ r_2 := (b \setminus S_2) \cup S_3 \]

is at distance \( s \) from \( b \). Since \( C \) has minimum distance \( \ell - m \), sets \( r_1 \) and \( r_2 \) are both at distance \( s \) from \( C \) itself. The set \( r_1 \) is at distance \( \ell - m - s \) from \( b' \) since \( S_3 \subseteq b' \) and \( S_1 \cap b' = \emptyset \). Since \( C \) is completely regular and both \( r_1 \) and \( r_2 \) are at distance \( s \) from \( C \), there must be a block \( b'' \in C \) at distance \( \ell - m - s \) from \( r_2 \). But \( b'' \) can have at most \( m \) points in common with \( b \cap r_2 \) since \( m \) is the maximum cardinality of any block intersection. Since \( m \geq 1 \), \( b'' = b' \) is impossible. So \( b'' \) can have at most \( m \) points in common with \( b' \cap r_2 \). Since \( r_2 \subseteq b \cup b' \), these inequalities imply \( \text{dist}(b'', r_2) \geq \ell - 2m \). The resulting inequality

\[ \ell - m - s \geq \ell - 2m \]

forces \( s \leq m \), finishing the proof. \( \square \)
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One way to interpret this theorem is to say that the minimum distance $\delta$ of a completely regular design is bounded above:

$$\delta \leq \frac{2\ell + 1}{3}.$$

If $C$ is a Steiner system $t - (v, \ell, 1)$, then no $t$-set occurs in two blocks. So the maximum block intersection is at most $t - 1$. Thus, for Steiner systems, Theorem 4.3.2 gives the bound

$$\ell \leq 3t - 2.$$

4.4 Symmetric Designs

A 2-design is called symmetric if the number of blocks equals the number of points. The class of symmetric designs includes all projective geometries. Symmetric designs are discussed at length in Lander [56]. If $C$ is a symmetric $2 - (v, \ell, \lambda)$ design, then any two distinct blocks meet in exactly $\lambda$ points. So, by Theorem 4.3.2, $C$ cannot be completely regular unless

$$\ell \leq 3\lambda + 1.$$

**Corollary 4.4.1** Let $C$ be a projective plane of order $q$ in $J(q^2 + q + 1, q + 1)$. If $q \geq 4$, then $C$ cannot be completely regular. $\square$

It follows from Theorem 4.2.3 that the Fano plane $PG(2, 2)$ is completely regular in $J(7, 3)$ and from Theorem 4.2.4 that the projective plane of order three is completely regular in $J(13, 4)$. For biplanes, the bound in Theorem 4.3.2 implies $\ell \leq 7$. But, in fact, there are no non-trivial completely regular biplanes.
The $2 - (11, 5, 2)$ biplane can be viewed as a subset of the Odd graph $O_6$ and, in the Odd graph, it is a completely regular subset. It has covering radius three in this graph and so its dual degree is three in the Johnson graph $J(11, 5)$ (cf. Section 4.7). But in the Johnson graph, this design has covering radius two. Since $\rho < r$, we clearly cannot have complete regularity.

The $(16, 6, 2)$ biplanes are all handled by the following theorem.

**Theorem 4.4.2** Suppose $C$ is a completely regular symmetric $2 - (v, \ell, \lambda)$ design. Then $\ell - \lambda$ is odd unless $C$ is the Fano plane.

**Proof:** Assume the minimum distance $\ell - \lambda$ is even. We define $h := \frac{\ell - \lambda}{2}$. Let $D$ be the outer distribution matrix of $C$. By considering the values $D_{xy}$ for vertices at distance $h$ from $C$, we will prove that

$$\binom{v - \ell}{h} \binom{\ell}{h} \leq (v - 1) \binom{\ell - \lambda}{h}^2$$

(4.2)

holds whenever $C$ is completely regular.

Let $b \in C$ be a fixed block of the design. The average number of blocks containing any $h$-set disjoint from $b$ is $(v - 1)(\binom{\ell - \lambda}{h})/\binom{v - \ell}{h}$. Let $H$ be an $h$-element subset of $V \setminus b$ which is contained in as few blocks as possible. Let $W$ be the collection of all $\ell$-sets containing $H$ and at distance $h$ from $b$. Then $W \subseteq C_h$ and since the minimum distance of $C$ is $2h$, every vertex in $W$ is at distance $h$ from at least one block of the design different from $b$. Any block containing $H$ is at distance $h$ from $\binom{\ell - \lambda}{h}$ elements of $W$. So

$$|W| \leq \binom{\ell - \lambda}{h} (# \text{ blocks containing } H) \leq \frac{(v - 1) (\binom{\ell - \lambda}{h})^2}{\binom{v - \ell}{h}}.$$ 

The set $W$ contains $\binom{\ell}{h}$ elements. This yields the bound given in Inequality (4.2).
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Since we only work with \( v \) and \( \ell \) satisfying \( v \geq 2\ell \), we have \( \ell \geq 2\lambda + 1 \). We aim to show that the bound (4.2) fails whenever \( \ell \geq 64 \). For such \( \ell \), we have

\[
\left( \frac{\ell}{4} - 1 \right) \log_2 2 > 2 \log_2 \ell.
\]

Since \( (\ell - 1)/\lambda \geq 2 \) and \( \ell - \lambda - 2 \geq \frac{\ell}{2} - 2 \), this yields

\[
\left( \frac{\ell - 1}{\lambda} \right)^{\frac{\ell - 2}{2}} > \ell,
\]

\[
\left( \frac{\ell - 1}{\lambda} \right)^{h} > \frac{\ell(\ell - 1)}{\lambda},
\]

\[
\left( \frac{\ell - 1}{\lambda} \right)^{h} > v - 1,
\]

using the identity \( \lambda(v - 1) = \ell(\ell - 1) \) for symmetric designs. Now compare the binomial coefficients \( \binom{v-\ell}{h} \) and \( \binom{\ell-\lambda}{h} \). The ratio of the first to the second is

\[
\frac{(v-\ell)(v-\ell-1)\cdots(v-\ell-h+1)}{(\ell-\lambda)(\ell-\lambda-1)\cdots(\ell-\lambda-h+1)} > \left( \frac{\ell - 1}{\lambda} \right)^{h}
\]

since \( (v-\ell)/(\ell-\lambda) = (\ell - 1)/\lambda \) is the largest of the individual terms. Yet we have \( (\frac{\ell-1}{\lambda})^{h} > v - 1 \). This implies that

\[
\binom{v-\ell}{h} > (v-1)\binom{\ell-\lambda}{h}.
\]

So the bound fails whenever \( \ell \geq 64 \). The remaining cases (\( \ell \leq 63 \)) are verified by computer. The only parameters satisfying Inequality (4.2) are those of \( PG(2,2) \).

\( \square \)

By a projective geometry \( PG(n,q) \), we refer to the design whose points are the one-dimensional subspaces of \( GF(q)^{n+1} \) and whose blocks are the hyperplanes (\( n \)-dimensional subspaces) of this vector space. A block \( h \) of the design is said to include the point \( p \) if \( p \leq h \) as a subspace. Theorem 4.3.2 implies that any
design with the same parameters as the projective geometry $PG(n, q)$ cannot be completely regular if $n \geq 2$ and $q \geq 4$. For $q$ even, Theorem 4.4.2 shows that only $PG(2, 2)$ is completely regular. We dismiss the case $n = 1$ as trivial and proceed to resolve the question for the remaining projective geometries (i.e., those with $q = 3$).

**Proposition 4.4.3** For $n \geq 3$ and for all $q$, the projective geometry $PG(n, q)$ is not completely regular.

**Proof:** We have $v = \frac{(q^{n+1} - 1)}{(q - 1)}$ points, $\ell = \frac{(q^n - 1)}{(q - 1)}$ points on each block, and any pair of distinct blocks meet in $\lambda = \frac{(q^{n-1} - 1)}{(q - 1)}$ points. Let $h$ be a hyperplane and let $x$ and $y$ be any two distinct points on $h$. For each pair of distinct points $p, q$ outside $h$, define

$$S_{pq} := h - x - y + p + q.$$  

Each $S_{pq}$ is at distance two from $h$ in $J(v, \ell)$ and, since $PG(n, q)$ has minimum distance $\ell - \lambda = q^{n-1} \geq 4$, this is also the distance from $S_{pq}$ to the design. We compare the value of

$$a(p, q) := |\{h' : h' \text{ a hyperplane, } \operatorname{dist}(h', S_{pq}) = \ell - \lambda - 2\}|$$

over all the choices of $\{p, q\}$. Any block $h'$ at distance $\ell - \lambda - 2$ from $S_{pq}$ must meet $h - x - y$ in $\lambda$ points and must contain both $p$ and $q$. Hence the line through $p$ and $q$ meets $h$ in a point outside $\{x, y\}$. Clearly, this will happen for some choices of $p$ and $q$, yielding $a(p, q) > 0$, and will not happen for other pairs $\{p, q\}$, yielding $a(p, q) = 0$. Since $a(p, q)$ is not constant for $\ell$-sets at distance two from the design, the outer distribution matrix of the design has more than $\rho + 1$ distinct rows. □
4.5 Completely regular designs of strength zero and one

Let $C$ be a completely regular $t$-design in $J(v, \ell)$. To learn about $C$, we represent $C$ in a way in which its strength and complete regularity are evident. Since the number of elements of $C$ containing a $(t+1)$-set varies, we represent $C$ as a function on $(t+1)$-sets. Let $W_{t+1, \ell}$ be the incidence matrix of $(t+1)$-subsets versus $\ell$-subsets of $V$. For each vertex $x$ of $J(v, \ell)$ we map $x$ to the $x$-th column of $W_{t+1, \ell}$ and we map $C$ to the sum of the columns corresponding to its elements. In this way, $C$ becomes a non-constant function on sets of size $t+1$. Moreover, as we will see, the inner product of the $x$-th column of $W_{t+1, \ell}$ and the image of $C$ depends only on the distance from $x$ to $C$.

A similar approach was used by Meyerowitz to prove:

Theorem 4.5.1 (Meyerowitz [60]) Suppose $C$ is a completely regular zero-design in $J(v, \ell)$. Then either $C$ is of the form

$$\{z: |z| = \ell, \ z \subseteq K\}$$

for some subset $K \subseteq V$ or $C$ is of the form

$$\{z: |z| = \ell, \ z \supseteq K\}$$

for some subset $K \subseteq V$. □

Let us now apply this technique to completely regular 1-designs. For $x$ a vertex of $J(v, \ell)$, we have $e_x$ the elementary vector having a one in position $x$. We define,
$u(x) := W_{2,\ell}e_x$, which is the $x$-th column of $W_{2,\ell}$. For a subset $C$, we map $C$ to $u(C) := W_{2,\ell} \chi_C$. It is not difficult to see that

$$W_{2,\ell}^T W_{2,\ell} = \sum_{i=0}^{\ell} \binom{\ell - i}{2} A_i$$

(see [26, Equation 4.22]). So the inner product $\langle u(x), u(y) \rangle$ is given by $\binom{|x \cap y|}{2}$.

**Lemma 4.5.2** Given a completely regular 1-design $C$ in $J(v, \ell)$ and the function $u$ as defined above, the inner product $\langle u(x), u(C) \rangle$ is a constant $\omega_h$ which only depends on the distance $h$ from $x$ to $C$. Moreover, $\omega_0 > \omega_1 > \cdots > \omega_\rho$ where $\rho$ is the covering radius of $C$.

**Proof:** Let $\pi = \{C_0, C_1, \ldots, C_\rho\}$ be the distance partition of $J(v, \ell)$ with respect to $C$. Suppose $C$ has reduced outer distribution matrix $E$ (that is, the outer distribution matrix $D$ satisfies $D = \Pi E$). Then, for $z \in C_h$,

$$\langle u(z), u(C) \rangle = \sum_{i=0}^{\ell} E_{hi} \binom{\ell - i}{2}$$

since there are precisely $E_{hi}$ vertices of $C$ at distance $i$ from $z$ for each $i$. Denote this constant by $\omega_h$.

Now $\omega_h$ is the $z$-entry in the vector $W_{2,\ell}^T W_{2,\ell} \chi_C$ for each $z \in C_h$. Delsarte shows that

$$W_{2,\ell}^T W_{2,\ell} = \binom{\ell}{2} \binom{v-2}{\ell-2} Z_0 + \binom{\ell-1}{\ell-2} \binom{v-3}{\ell-2} Z_1 + \binom{v-4}{\ell-2} Z_2$$

(see [26, p47]). This gives

$$\omega_h = \binom{\ell}{2} \binom{v-2}{\ell-2} \sum_{v \in C} (Z_0)_{zu} + \binom{\ell-1}{\ell-2} \binom{v-3}{\ell-2} \sum_{v \in C} (Z_1)_{zu} + \binom{v-4}{\ell-2} \sum_{v \in C} (Z_2)_{zu}.$$

Now $Z_0$ is a multiple of the all-ones matrix and, since $C$ is a 1-design, $Z_1 \chi_C = 0$. So it is sufficient to prove that the sequence $x_0, x_1, \ldots, x_\rho$ given by

$$x_h = \sum_{v \in C} (Z_2)_{zu}$$