Completely Regular Subsets

by

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Abstract

We explore completely regular subsets in distance-regular graphs. We make an attempt to develop the general combinatorial and algebraic theory of these objects and also to delve deeply into several more specific existence questions.

We work within the framework of the theory of association schemes. Thus we look for results which are consistent with current graph-theoretic, algebraic, and representation-theoretic approaches in the study of distance-regular graphs themselves. In the course of this treatise, we also obtain several results regarding equitable partitions and quotients of general association schemes.

In the latter part of the thesis, focus is placed on the search for completely regular subsets in the Johnson graphs. This family of distance-regular graphs provides a setting for the algebraic analysis of combinatorial design theory. The problem of classifying completely regular subsets in the Johnson graphs is posed and several cases are addressed. In particular, we classify completely regular one-designs having minimum distance at least two. The existence question for perfect codes in the Johnson graphs is discussed; a theorem establishing the complete regularity of the derived design of a perfect code is proved and its implications for several cases are explored.

Related to the Johnson graph are the Grassman graph and the bilinear forms graph. We generalise a result of Roos, finding large anticode in these graphs which enable us to provide new proofs of a result due to Chihara which rules out perfect codes in these graphs. We also spend some time exploring very special types of subsets in general distance-regular graphs, for example completely regular subsets of size two and three.
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To my parents
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Chapter 1

Introduction

In this chapter, I first provide a brief history of the theory of association schemes. After this, I give an overview of the thesis, outlining the main results. The latter part of the chapter is a terse summary of background material. It will be referred to in later chapters.

1.1 History

Association schemes were introduced by Bose and Nair [12] in 1939 in connection with partially balanced incomplete block designs. The actual term “association scheme” was not used in that paper; it was first used by Bose and Shimamoto [13] in 1952. The fundamental linear-algebraic theory emerged in the 1959 paper of Bose and Mesner [14]. Much of this algebraic theory had already been developed in the analysis of centraliser rings and Schur rings in the study of finite permutation groups (see [1]). In 1967, D. G. Higman [49] introduced the very general concept of a coherent configuration; this generalises both the association scheme and the
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P. Delsarte, in his 1973 thesis [26], was motivated by problems in coding theory and seemingly analogous problems in combinatorial design theory. His work advances the theory of association schemes, explores subsets in association schemes and focuses on the Hamming and Johnson schemes in particular. He is able to make precise the "duality" between codes and designs in an association scheme. Delsarte generalised Lloyd's theorem from coding theory to the study of perfect codes in distance-regular graphs. About this time (1973), Biggs [6] proved the same result by different means.

1.2 Overview

In Chapter 2 of this thesis, we discuss equitable partitions of association schemes. After making some general observations (Proposition 2.1.4 is new here), we focus on equitable partitions determined by a subset. In particular, we investigate simple subsets in association schemes and completely regular subsets in distance-regular graphs. We argue that, in a sense, a completely regular subset behaves much like an association scheme. Next, we focus on examples. We determine several classes of completely regular subsets, all of which turn out to be cliques. (Theorems 2.3.3, 2.3.4, 2.3.6 and Proposition 2.3.5). The last section of Chapter 2 deals with quotients of association schemes and distance-regular graphs. Our main theorem is Theorem 2.4.2 which describes a rather general condition under which an association scheme has a quotient. This theorem encompasses several earlier results characterising various types of quotients. Finally, we study the relationship
between completely regular subsets in a graph and those in a quotient of that graph (Theorem 2.4.6).

To each eigenspace of an association scheme is associated a mapping from the vertices to a polytope in a Euclidean space. These polytopes have provided several researchers [73, 63, 40, 72] with an elegant geometric approach to certain questions about distance-regular graphs. These "representations" also provide a nice setting for the study of equitable partitions and completely regular subsets. In Chapter 3, we study the relationship between representations and equitable partitions (Theorem 3.2.1 and Proposition 3.3.1). These generalise Godsil's results on representations of completely regular subsets. We are also able to find the representation of a quotient graph in that of its cover (Theorem 3.5.1). Well-known bounds in spherical geometry then allow us to prove Proposition 3.4.1, Theorem 3.4.3, and Corollary 3.5.2. These provide bounds on the parameters of our objects in terms of the multiplicity of an eigenvalue of the graph.

The second half of the thesis explores completely regular subsets in the Johnson graphs and related graphs. These graphs provide an algebraic setting for problems in combinatorial design theory. For this reason, the Johnson graphs serve as a wonderful proving ground for the theory. Theorem 5.2.1 in Chapter 5 states that the derived design of a perfect code in the Johnson graph is a completely regular subset; this main result is another motivation for the study of completely regular subsets in these graphs. Thus Chapter 4 is devoted to general existence questions of this type. Examples of completely regular subsets come from $t$-designs with block size $t+1$ and Steiner systems with block size $t+2$. (We also discuss the Witt designs in Section 4.2.) In Section 4.3, it is shown that a completely regular 2-design cannot have large minimum distance (Theorem 4.3.2). For example, this shows that most projective geometries are not completely regular. Meyerowitz [60] has
already classified the completely regular subsets of strength zero. In Theorems 4.2.2 and 4.5.3, we determine the completely regular subsets of strength one having minimum distance at least two.

A theorem of Dowling et al. provides a local characterisation of the Johnson graphs. We use this theorem to rule out many quotients of Johnson graphs (Corollary 4.6.2). Some of the remaining cases are discussed in Section 4.6. Finally, we investigate completely regular subsets in the Odd graph in Section 4.7. In particular, we show in Theorem 4.7.4 that a perfect code in the Odd graph forms a completely regular subset in the corresponding Johnson graph.

Chapter 5 investigates perfect codes in the Johnson graph and the application of Delsarte's anticode condition to perfect codes in related graphs. Our main theorem is Theorem 5.2.1 described above. We investigate the implications of this theorem for perfect codes of radius one and two. Finally, in Section 5.4, we use a technique of Roos to construct large antico
codes in the Grassman graphs and the bilinear forms graphs. With these antico
codes, Delsarte's anticode condition allows us to prove that there are no perfect codes in these graphs (Theorem 5.4.3 and Theorem 5.4.4). In both cases, Chihara [20] has already shown that there are no perfect codes; only the proofs are new.

1.3 Distance-Regular Graphs

A simple graph G is a pair G = (V(G), E(G)) where V(G) is a finite set of elements called vertices, having size n say, and E(G) is a collection of distinct two-element subsets of V(G) called edges. For efficiency we will omit the set brackets and will denote edge e = \{x, y\} simply by e = xy. Distinct vertices which occur together in an edge are adjacent and are called neighbours of one another. The notation x \sim y
means “$x$ is adjacent to $y$.” The distance between vertices $x$ and $y$ is the length of a shortest path joining them. A graph can be identified with its zero-one adjacency matrix $A$. We refer to Bondy and Murty [11] for basic facts about graphs.

A graph $G$ of diameter $d$ is distance-regular if, for each pair of vertices $x$ and $y$, and each $i$ and $j$ ($0 \leq i, j \leq d$), the number of vertices at distance $i$ from $x$ and $j$ from $y$ is a constant only depending on $i$, $j$ and $k := \text{dist}(x,y)$ and not on the choice of $x$ and $y$ themselves. If we define the sphere of radius $i$ about vertex $x$ to be

$$G_i(x) := \{z \in V(G) : \text{dist}(x,z) = i\},$$

then we can say that $G$ is distance-regular if and only if there exist constants $p_{ij}(k)$ ($0 \leq i, j, k \leq d$) such that whenever $\text{dist}(x,y) = k$,

$$|G_i(x) \cap G_j(y)| = p_{ij}(k).$$

Define the $i$-th distance matrix of $G$ to be the $n \times n$ matrix $A_i$ with $(x,y)$-entry equal to

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \text{dist}(x,y) = i; \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, we have $A_0 = I$ and $A_1 = A$, the adjacency matrix of $G$. Moreover, the matrices $A_i$ ($0 \leq i \leq d$) sum to the all-ones matrix $J$. The reader may verify that $G$ is distance-regular if and only if the real vector space spanned by $\{A_0, A_1, \ldots, A_d\}$ is closed under matrix multiplication. In fact, we have

$$A_iA_j = \sum_{k=0}^{d} p_{ij}(k)A_k.$$ 

We denote this commutative algebra by $A$.

Let $G$ be a graph and let $x$ be a vertex of $G$. The distance partition of $G$ with respect to $x$ is the partition \{$G_0(x),G_1(x),\ldots,G_d(x)$\}. A vertex $y$ in $G_i(x)$
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can only have neighbours in $G_{i-1}(x)$, $G_{i}(x)$, and $G_{i+1}(x)$. When $G$ is distance-
regular, the number of neighbours of each type is constant. That is, $y$ has precisely $c_i := p_{(i-1)1}(i)$ neighbours in $G_{i-1}(x)$, $a_i := p_{i1}(i)$ neighbours in $G_i(x)$, and $b_i := p_{(i+1)1}(i)$ neighbours in $G_{i+1}(x)$. These constants $c_i (1 \leq i \leq d)$, $a_i (0 \leq i \leq d)$, and $b_i (0 \leq i \leq d - 1)$ are thus independent of the choice of $x$ and $y$. It can be shown that distance-regularity of $G$ is equivalent to the existence of the constants $a_i$, $b_i$, and $c_i$, independent of the choice of $x$ and $y$ [4, Proposition 20.8]. We define the intersection array of $G$ to be the array

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}.$$  

Using the $a_i$, $b_i$, and $c_i$, we can simplify the above product of distance matrices in the case where $j = 1$. We get

$$A_1A_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$$  \hspace{1cm} (1.1)

for $1 \leq i \leq d - 1$. If we define $c_0 = b_d = 0$ and $c_{d+1} = 0$, then the identity holds also for $i = 0$ and $i = d$. Using this recurrence, we define polynomials $v_0(x), v_1(x), \ldots, v_d(x)$ as follows: we set $v_0(x) = 1$, $v_1(x) = x$. The remaining polynomials are obtained from the recurrence

$$v_{i+1}(x) = \frac{1}{c_{i+1}}((x - a_i)v_i(x) - b_{i-1}v_{i-1}(x)).$$

It is clear that $v_0(A_1) = A_0$ and $v_1(A_1) = A_1$. The recurrence in Equation (1.1) for the distance matrices verifies that $v_i(A_1) = A_i$. Moreover, the polynomial $v_i(x)$ has degree $i$ for $0 \leq i \leq d$. If the distinct eigenvalues of $A_1$ are $\theta_0 > \theta_1 > \ldots > \theta_d$, then the eigenvalues of $A_i$ are $v_i(\theta_0), v_i(\theta_1), \ldots, v_i(\theta_d)$. As we will see, this is one characterising property of $P$-polynomial (or "metric") association schemes.

Throughout this thesis, we will make occasional reference to results contained in the book of Brouwer, Cohen, and Neumaier [3].
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1.4 Association Schemes

A distance-regular graph is an example of an association scheme. Much of the study of distance-regular graphs is properly set within the theory of association schemes; hence, the present thesis makes frequent reference to this material. We now present the basic concepts of the theory of association schemes.

Let \( X = \{1, 2, \ldots, n\} \) and let \( A_0, A_1, \ldots, A_d \) be zero-one symmetric matrices of size \( n \times n \), each with rows and columns indexed by \( X \). Set \( A := \{A_0, A_1, \ldots, A_d\} \). We say \( (X, A) \) is an association scheme if the following properties hold:

I. \( A_0 = I \) where \( I \) is the identity matrix of size \( n \);

II. \( \sum_{i=0}^{d} A_i = J \) where \( J \) is the all-ones matrix of size \( n \times n \);

III. for each \( h \) and each \( i \) (\( 0 \leq h, i \leq d \)), the matrix \( A_h A_i \) can be written as a linear combination of elements of \( A \).

(The above definition corresponds to what is usually called a symmetric association scheme. In this thesis, all schemes will be symmetric.) In particular, if we take \( h = i \) in (III) and consider the diagonal of \( A_i A_i \), we infer that each matrix \( A_i \) has constant row sums. Denote by \( k_i \) the row sum of \( A_i \); the integers \( k_0, k_1, \ldots, k_d \) are called the valencies of the association scheme.

To each matrix \( A_i \) (\( 1 \leq i \leq d \)) we may associate a simple graph \( G_i \) on vertex set \( X \). Since the matrix \( A_i \) and the graph \( G_i \) record the same information, we will use the two interchangeably. We now present an alternative, more intuitive definition of a (symmetric) association scheme. The graphs \( G_i \) (\( 1 \leq i \leq d \)) constitute an edge-colouring of the complete graph \( K_n \). If edge \( xz \) is assigned colour \( i \), we say \( z \) is \( i \)-related to \( x \). An edge-colouring of the complete graph on \( X \) is an association
scheme if and only if, for any \( x, y \in X \) and any colours \( h \) and \( i \), the number of vertices \( z \) which are \( h \)-related to \( x \) and \( i \)-related to \( y \) is independent of the choice of \( x \) and \( y \) and only depends on \( h, i \), and the colour \( k \) of edge \( xy \) (including some colour "0" for the case \( x = y \)).

If we denote by \( \mathcal{A} \) the real vector space spanned by the matrices \( A_0, A_1, \ldots, A_d \), then the definition implies that \( \mathcal{A} \) is a commutative algebra of symmetric matrices. We call this the Bose-Mesner algebra of the association scheme. Since \( \mathcal{A} \) is a semisimple algebra, there is a unique basis \( \{ Z_0, Z_1, \ldots, Z_d \} \) for \( \mathcal{A} \) satisfying

\[
Z_j Z_k = \delta_{jk} Z_j
\]

and

\[
\sum_{j=0}^{d} Z_j = I.
\]

That is, we have a basis of orthogonal idempotents with respect to ordinary matrix multiplication. Since each \( \text{colsp}(Z_j) \) is an eigenspace of each matrix in \( \mathcal{A} \) and the all-ones matrix is in \( \mathcal{A} \), one of these idempotents is a multiple of the all-ones matrix. Without loss of generality, we write \( Z_0 = \frac{1}{n} J \).

In our work with association schemes, we also consider the Schur product of matrices. For \( n \times n \) matrices \( M \) and \( N \) define \( M \circ N \) to be the matrix with \((i, j)\)-entry equal to \( M_{ij} N_{ij} \). Thus the Schur product is simply componentwise multiplication of matrices. It is easy to see that the set \( \mathcal{A} = \{ A_0, A_1, \ldots, A_d \} \) constitutes a basis of orthogonal idempotents for \( \mathcal{A} \) with respect to Schur multiplication. That is,

\[
A_h \circ A_i = \delta_{hi} A_h.
\]

In particular, the space \( \mathcal{A} \) is also closed under Schur multiplication.

The following theorem provides us with an algebraic criterion for determining
when we have an association scheme. See, e.g., Higman [52] or Brouwer et al. [3, Theorem 2.6.1].

**Theorem 1.4.1** Let \( A \) be a vector space of symmetric matrices. Then \( A \) is the Bose-Mesner algebra of an association scheme if and only if \( A \) is closed under both ordinary and Schur multiplication and contains both \( I \) and \( J \). □

### 1.5 Eigenmatrices

Each \( A_i \) can be written uniquely as a linear combination of the matrices \( Z_j \) \((0 \leq j \leq d)\), and conversely. For \( 0 \leq i, j \leq d \), define constants \( p_i(j) \) by the equations

\[
A_i = \sum_{j=0}^{d} p_i(j) Z_j, \tag{1.4}
\]

and define constants \( q_j(i) \) by the equations

\[
Z_j = \frac{1}{n} \sum_{i=0}^{d} q_j(i) A_i. \tag{1.5}
\]

Since the \( Z_j \) are mutually orthogonal idempotent matrices, we see that the values \( p_i(0), p_i(1), \ldots, p_i(d) \) are the eigenvalues of \( A_i \) and Equation (1.4) is related to the spectral decomposition of the matrix \( A_i \). Hence we call the constants \( p_i(j) \) the eigenvalues of the association scheme. There is no obvious combinatorial interpretation for the constants \( q_j(i) \) (which are sometimes called the "dual eigenvalues"). However, they will arise in Chapter 3 when we discuss representations. As suggested by the notation, we often view \( p_i(j) \) as a function of \( j \) for each \( i \). In fact, we can find \( d + 1 \) polynomials \( v_0(x), v_1(x), \ldots, v_d(x) \) each of degree at most \( d \), and constants \( \theta_0, \theta_1, \ldots, \theta_d \) such that the numbers \( p_i(j) := v_i(\theta_j) \) satisfy the equations above. Similarly, we can view \( q_j(i) \) as an evaluation of a polynomial \( w_j(x) \) at \( x = \omega_i \).
for some appropriately chosen constants $\omega_i$ ($0 \leq i \leq d$). As we have seen, in the case of a distance-regular graph, we can choose the polynomials $v_0(x), \ldots, v_d(x)$ so that $v_i(x)$ has degree $i$ for $0 \leq i \leq d$.

Equations (1.4) and (1.5) allow us to compute the following products:

$$A_i Z_j = p_i(j)Z_j,$$

(1.6)

and

$$Z_j \circ A_i = \frac{1}{n} q_j(i)A_i.$$  

(1.7)

We now define the eigenmatrices of the association scheme $(X, A)$. Let $P$ be the $(d + 1) \times (d + 1)$ matrix with $(j, i)$-entry equal to $p_i(j)$. We call $P$ the first eigenmatrix of the scheme. Let $Q$ be the $(d + 1) \times (d + 1)$ matrix with $(i, j)$-entry equal to $q_j(i)$. We call $Q$ the second eigenmatrix of the association scheme. Since $\sum A_i = J = nZ_0$, we have $q_0(i) = 1$ for $0 \leq i \leq d$. Symmetrically, the convention $A_0 = I$ and the property $\sum Z_j = I$ imply that $p_0(j) = 1$ for $0 \leq j \leq d$. Moreover, each graph $G_i$ is regular of valency $k_i$ and hence has the all-ones vector as an eigenvector with the valency $k_i$ as its eigenvalue. This means that $A_i$ has eigenvalue $p_i(0) = k_i$ with corresponding idempotent $Z_0$.

Look again at an idempotent $Z_j$. This matrix has all eigenvalues equal to zero or one. Furthermore, since $Z_j$ is a linear combination of the matrices $A_i$, it has constant diagonal. The rank of $Z_j$ is equal to the sum of its eigenvalues which, in turn, is equal to its trace. So the diagonal entries of $Z_j$ are equal to $\frac{1}{n}$ times its rank. Accordingly, if we denote by $m_j$ the rank of $Z_j$, we have $q_j(0) = m_j$. The integers $m_0 = 1, m_1, \ldots, m_d$ are called the multiplicities of the association scheme $(X, A)$. 


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Polynomial schemes

Let \((X, \mathcal{A})\) be an association scheme and let \(A_0, A_1, \ldots, A_d\) be a specified ordering of its adjacency matrices. The scheme \((X, \mathcal{A})\) is defined to be \(P\)-polynomial (or "metric") with respect to this ordering if there exist polynomials \(v_0(x), v_1(x), \ldots, v_d(x)\) and constants \(\{\theta_j : 0 \leq j \leq d\}\) such that: (i) \(v_i(x)\) has degree at most \(i\) for each \(i\); and, (ii) for all \(i\) and \(j\), \(p_i(j) = v_i(\theta_j)\). Analogously, the association scheme \((X, \mathcal{A})\) is defined to be \(Q\)-polynomial (or "cometric") with respect to a specified ordering \(Z_0, Z_1, \ldots, Z_d\) of its idempotents if there exist polynomials \(w_0(x), w_1(x), \ldots, w_d(x)\) and constants \(\{\omega_i : 0 \leq i \leq d\}\) such that: (i) \(w_j(x)\) has degree at most \(j\) for each \(j\); and, (ii) for all \(i\) and \(j\), \(q_j(i) = w_j(\omega_i)\). In fact, \(P\)-polynomial association schemes are just distance-regular graphs. There is no combinatorial interpretation known for the \(Q\)-polynomial property, although it is very important in the study of codes and designs. In particular, the Hamming scheme and the Johnson scheme are both \(P\)-polynomial and \(Q\)-polynomial.

Again, every distance-regular graph yields an association scheme as follows. Suppose \(G\) is a distance-regular graph of diameter \(d\) and suppose \(A_i\) is the \(i\)-th distance matrix of \(G\). Set \(\mathcal{A} = \{A_0, A_1, \ldots, A_d\}\). Then, from the definitions, we see that \((V(G), \mathcal{A})\) is an association scheme. Moreover, Equation (1.1) shows that this scheme is \(P\)-polynomial.

1.6 Orthogonality relations

Let \((X, \mathcal{A})\) be an association scheme with eigenmatrices \(P\) and \(Q\). Composing Equations (1.4) and (1.5), we obtain the equation

\[ PQ = nI \]
where $n$ is the number of vertices of the association scheme.

Recall the valencies $k_0, k_1, \ldots, k_d$ and the multiplicities $m_0, m_1, \ldots, m_d$ introduced above. Define $K$ to be the diagonal matrix with $i$-th diagonal entry equal to $k_i$ ($0 \leq i \leq d$) and define $M$ to be the diagonal matrix with $j$-th diagonal entry equal to $m_j$ ($0 \leq j \leq d$).

There is a second "orthogonality" relation between the two eigenmatrices:

$$MP = QTK.$$  \hfill (1.9)

This is obtained (entrywise) by comparing the sum of the entries of $A_i \circ Z_j$ with the trace of $A_i^T Z_j$.

### 1.7 Intersection numbers and Krein parameters

The definition of an association scheme asserts that the product $A_h A_i$ can be written as a linear combination of the elements $\{A_0, A_1, \ldots, A_d\}$. We define, for all $h, i$, and $k$, ($0 \leq h, i, k \leq d$), the intersection numbers $p_{hi}(k)$ of the association scheme $(X, A)$ by the equations

$$A_h A_i = \sum_{k=0}^{d} p_{hi}(k) A_k.$$  \hfill (1.10)

The coefficients $p_{hi}(k)$ are clearly non-negative integers. They have already been introduced in Section 1.3 in the case of distance-regular graphs. For a fixed $i$, we collect the intersection numbers $p_{hi}(k)$ in a matrix: define $L_i$ to be the $(d+1) \times (d+1)$ matrix with $(k, h)$-entry equal to $p_{hi}(k)$. We call $L_i$ the $i$-th intersection matrix.

The vector space $A$ is also closed under Schur multiplication. We write the Schur product $Z_j \circ Z_k$ as a linear combination of the matrices in the basis of orthogonal
idempotents:

\[ Z_j \circ Z_k = \frac{1}{n} \sum_{\ell=0}^{d} q_{jk}(\ell)Z_{\ell}. \]  

(1.11)

The coefficients \( q_{jk}(\ell) \) (0 \( \leq j, k, \ell \leq d \)) are called the Krein parameters of the association scheme. These numbers are not necessarily integers. However an important result due to Scott [69] (or, see [3, Theorem 2.3.2]) asserts that they are always non-negative. The inequalities \( q_{jk}(\ell) \geq 0 \) are known as the Krein conditions.