# There are finitely many $Q$-polynomial association schemes with given first multiplicity at least three

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*Dedicated to Professor Eiichi Bannai on the occasion of his 60th birthday.*

**Abstract**

In this paper, we will prove a result which is formally dual to the long-standing conjecture of Bannai and Ito which claims that there are only finitely many distance-regular graphs of valency $k$ for each $k > 2$. That is, we prove that, for any fixed $m_1 > 2$, there are only finitely many cometric association schemes $(X, \mathcal{R})$ with the property that the first idempotent in a $Q$-polynomial ordering has rank $m_1$. As a key preliminary result, we show that the splitting field of any such association scheme is at most a degree two extension of the rationals.

All of the tools involved in the proof are fairly elementary yet have wide applicability. To indicate this, a more general theorem is proved here with the result alluded to in the title appearing as a corollary to this theorem.

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# 1 Introduction

We begin with a review of the basic definitions and our choice of notation. The reader is referred to [3] or [5] for much more background.

A *(symmetric) association scheme* $(X, \mathcal{R})$ consists of a finite set $X$ of size $v$ and a set $\mathcal{R}$ of binary relations on $X$ satisfying

(i) $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a partition of $X \times X$;

(ii) $R_0$ is the identity relation;

(iii) $R_i^\top = R_i$ for each $i$;

(iv) there exist integers $p_{ij}^k$ such that $|\{c \in X : (a,c) \in R_i \text{ and } (c,b) \in R_j\}| = p_{ij}^k$ whenever $(a,b) \in R_k$, for each $i, j, k \in \{0, \ldots, d\}$.

As usual, we define $A_i$ to be the matrix with rows and columns indexed by $X$ with $(a,b)$-entry equal to one if $(a,b) \in R_i$ and zero otherwise. In this way, we obtain a collection of $v \times v$ symmetric 01-matrices $\mathbf{A} = \{A_0, A_1, \ldots, A_d\}$ such that:

(i) $A_0$ is the identity matrix,

(ii) $\sum_{i=0}^d A_i = J$ where $J$ is the all 1's matrix,

(iii) The set $\mathbf{A}$ forms the basis of a commutative matrix algebra $\mathcal{A}$ called the *Bose-Mesner algebra*.

Since no two matrices in $\mathbf{A}$ have a nonzero entry in the same location, the Bose-Mesner algebra is also closed under entrywise (or Schur) multiplication, denoted $\circ$.

Elementary linear algebra tells us that the matrices $\mathbf{A}$ can be simultaneously diagonalized; there are $d+1$ maximal common eigenspaces for $\mathbf{A}$ known as the *eigenspaces* of the scheme, and it can be shown that the primitive idempotents $E_0, E_1, \ldots, E_d$ representing orthogonal projection onto these eigenspaces form another basis of $\mathcal{A}$. If we let $P_{ji}$ denote the eigenvalue of $A_i$ on the $j^{\text{th}}$ eigenspace of the scheme, i.e.,

$$A_i E_j = P_{ji} E_j,$$

then the $(d+1) \times (d+1)$ matrix $P$ containing $P_{ji}$ as its entry in the $j^{\text{th}}$ row, $i^{\text{th}}$ column is called the *first eigenmatrix* of the association scheme.

The *second eigenmatrix* $Q$ of the scheme is defined as $Q = vP^{-1}$ but also satisfies a second "orthogonality relation". If $v_i$ denotes the valency of the relation $R_i$ (i.e., the common row sum of the matrix $A_i$) and $m_j$ denotes the dimension of the $j^{\text{th}}$ eigenspace (i.e., the rank of $E_j$), then we have, for all $i$ and $j$,

$$v_i Q_{ij} = m_j P_{ji} \tag{1.1}$$

(Equation (3), [5, p46]). Furthermore, we have $E_j = \frac{1}{v} \sum_i Q_{ij} A_i$ so that the entry in row $a$, column $b$ of $E_j$ is $Q_{ij}/v$ whenever $(a,b) \in R_i$.

An association scheme is *metric* (or $P$-polynomial) if there is an ordering $R_0, R_1, \ldots, R_d$ on the relations so that, for each $i$, $A_i$ may be expressed as a matrix polynomial of degree exactly $i$ in $A_1$. Such an ordering is called a *$P$-polynomial ordering*. Delsarte [8] showed that metric association schemes, with specified $P$-polynomial ordering, are in one-to-one correspondence with distance-regular graphs (see [5, Prop. 2.7.1] or [3, Prop. III.1.1]). By analogy, an association scheme is said to be *cometric* (or $Q$-polynomial) if there is an ordering $E_0, E_1, \ldots, E_d$ on the primitive idempotents so that, for each $j$, $E_j$ may be expressed as a polynomial of degree exactly $j$ applied entrywise to the values in $E_1$. Such an ordering is called a *$Q$-polynomial ordering*.

In their 1984 monograph, Bannai and Ito conjectured [3, Conjecture 1, p237] that there are only finitely many distance-regular graphs with any given valency $v_1 > 2$ (the polygons are all distance-regular with $v_1 = 2$). There has been much recent activity on this problem (see, e.g., [1] and the references therein), and to date the result is known to hold at least for $v_1 \leq 10$ [2]. In fact, it has just recently been announced that the full conjecture has been proven by Bang, Koolen and Moulton.

Now consider the dual of this conjecture, which is quite natural but does not appear in [3]: for fixed $m_1 > 2$, there are only finitely many cometric association schemes with $\operatorname{rank} E_1 = m_1$. This is the result we prove.

We finish this section with a summary of several earlier results where parameters of association schemes are bounded by a multiplicity. In [4], Bannai and Bannai classified the primitive symmetric association schemes with an eigenvalue of multiplicity three: only the complete graph $K_4$ arises. In that paper (see the remark in Section 3 on p72), they point out, for primitive schemes, the valency of a certain relation is bounded above by a function of the multiplicity; this result is quite similar to our Lemma 4.3. In 1988, Godsil proved that there are only finitely many distance-regular graphs with an eigenvalue of any given multiplicity $m > 2$. His proof also involves a bound $k \leq (m-1)(m+2)/2$ on the valency using spherical 2-distance sets, as well as a diameter bound $d \leq 3m - 4$ (see, e.g., [5, Theorem 5.3.2]). Another bound on the valency, which is often stronger, is the bound of Terwilliger [17]: if $G$ is a distance-regular graph with valency $k$ and girth $g \geq 4$ and $\theta \neq \pm k$ is an eigenvalue of $G$ having multiplicity $m$, then $m \geq k(k-1)^{\lfloor g/4 \rfloor - 1}$ if $g \equiv 0, 1 \pmod 4$ and $m \geq 2(k-1)^{\lfloor g/4 \rfloor}$ if $g \equiv 2, 3 \pmod 4$.

## 2 The splitting field

The *splitting field* of an association scheme is the smallest extension of the rationals containing all of the entries of $P$. This and related fields were studied by Munemasa [13]. Our first theorem concerns actions of automorphisms of the splitting field on the $E_i$, and is true for any association scheme.

**Theorem 2.1.** *Let $(X, \mathcal{R})$ be an association scheme and let $\mathcal{A}$ be its Bose-Mesner algebra. Let $\mathbb{F}$ be the splitting field of this scheme and $G = Gal\,(\mathbb{F}/\mathbb{Q})$. Let $G$ act on the matrices of the Bose-Mesner algebra $\mathcal{A}$ entrywise. Then this induces a faithful action of $G$ on the primitive idempotents $E_0, \ldots, E_d$.*

*Proof.* Let $(X, \mathcal{R})$ be a $d$-class association scheme, and $E_0, E_1, \ldots, E_d$ the primitive idempotents of the scheme. Let $\mathbb{F}$ be the splitting field of the scheme, generated by the entries $P_{ji}$ of the matrix $P$, and suppose $[\mathbb{F} : \mathbb{Q}] = n$. Then $\mathbb{F}$ is the splitting field of the product of the minimal polynomials of the $A_i$, and is, therefore, a Galois extension of the rational field $\mathbb{Q}$. Then the Galois group $G = Gal\,(\mathbb{F}/\mathbb{Q})$ is of order $n$, and application of field automorphisms yields an action of $G$ on $\mathcal{A}$. Note that the matrix $Q$ is a rational multiple of the inverse of $P$, hence $\mathbb{F}$ is also the field generated by the entries of $Q$.

We now consider the action of $G$ on the matrices of $\mathcal{A}$, where elements of $G$ are applied entrywise. Note that since all of the matrices $A_i$ are fixed by $G$, this induces an action of $G$ on the primitive idempotents $E_0, \ldots, E_d$. Let $\sigma \in G$ where $\sigma$ is not the identity. Since $\sigma$ fixes all of the matrices $A_i$ we must have that $\sigma$ induces a permutation on the matrices $E_j$. Since $\mathbb{F}$ is generated by the entries of $Q$, there must be some entry $Q_{ij}$ which is not fixed by $\sigma$. Then, using the identity $A_i \circ E_j = \frac{Q_{ij}}{v} A_i$, we see that $E_j$ has at least 1 entry equal to $Q_{ij}/v$, implying that $E_j$ is not fixed by $\sigma$. Therefore, no non-identity member of $G$ fixes all of the $E_j$, so the action of $G$ on $\{E_0, \ldots, E_d\}$ is faithful. $\square$

The following result has been obtained independently by Cerzo and Suzuki [6].

**Theorem 2.2.** *The splitting field of any cometric association scheme with $m_1 > 2$ is at most a degree two extension of the rationals.*

*Proof.* Let $(X, \mathcal{R})$ be a $Q$-polynomial association scheme of diameter $d$, and let $E_0, E_1, \ldots, E_d$ be a $Q$-polynomial ordering of the primitive idempotents. Let $\mathbb{F}$ be the splitting field of the scheme, generated by the entries $Q_{ij}$ of the matrix $Q$, and suppose $[\mathbb{F} : \mathbb{Q}] = n$.

Note that if $\sigma$ is in $G$, then $E_0^\sigma, E_1^\sigma, \ldots, E_d^\sigma$ is also a $Q$-polynomial ordering. Since $|G| = n$, and the action of $G$ is faithful on the $E_j$, there must then be at least $n$ different $Q$-polynomial orderings of the $E_j$. By a result of Suzuki [16] there can be at most two $Q$-polynomial orderings of the $E_j$, so $|G| \leq 2$, therefore, $[\mathbb{F} : \mathbb{Q}] \leq 2$. $\square$

# 3 Spherical codes

In our proof of the main theorem, we will use bounds on the size of a spherical code. These are studied in [9], but in fact we need only the most elementary observation. For a subset $T \subset [-1, 1)$ of the possible inner products among unit vectors, a *spherical $T$-code* in $\mathbb{R}^m$ is a subset $Y$ of the unit sphere $S^{m-1}$ having the property that $x \cdot y \in T$ for any distinct $x, y \in Y$ where $\cdot$ denotes the ordinary dot product. If $T$ is bounded away from 1 — the case of interest is $T = [-1, \eta]$ for some fixed $\eta < 1$ — then $Y$ must be finite. We can obtain an upper bound, which in this paper we denote by $U(m, \eta)$, on the size of such a set $Y$ just using a sphere-packing argument. Suppose that $\eta = \cos\alpha$. The unit sphere has fixed finite volume $\|S^{m-1}\|$ and the points of $Y$ can be surrounded by pairwise disjoint spherical caps of angular radius $\alpha/2$ and thus these caps all have some fixed positive volume $V$. We then find $|Y| \leq \|S^{m-1}\|/V$.

An interesting special case is that of spherical $A$-codes where $A = [-1, \frac{1}{2}]$. The optimal size of such a spherical code in $\mathbb{R}^m$ is called the *kissing number* $\tau_m$ (see [7, p21]). Kissing numbers are well-studied and there has been some exciting recent activity in this area. See [14] for a survey of recent work. At this point, we need no more than the result that the kissing number in dimension $m$ is bounded by a function of $m$. So we appeal to a result (see

[7, p23]) of Kabatianski and Levenshtein from 1978 which established the bound

$$\tau_m \leq 2^{0.401m(1+o(m))}.$$

# 4 Main theorem

**Theorem 4.1.** *Let $m \geq 2$ and $n \geq 1$. Then there are, up to isomorphism, only finitely many symmetric association schemes $(X, \mathcal{R})$ with the following properties*

*(i) for some $1 \leq j \leq d$, $Q_{0j} = m > Q_{ij}$ for all $i = 1, \ldots, d$, and*

*(ii) each eigenvalue $P_{ji}$ of each adjacency matrix $A_i$ has minimal polynomial (over the rationals) of degree at most $n$.*

Before proving this result, we show how it specializes to the case of $Q$-polynomial schemes. If $E_1$ is the first idempotent in a $Q$-polynomial ordering for the cometric association scheme $(X, \mathcal{R})$, then we know that all entries $Q_{i1}$ are distinct. Moreover, by Theorem 2.1, the splitting field of the scheme $(X, \mathcal{R})$ is at most a degree two extension of $\mathbb{Q}$, so it certainly holds that each $P_{ji}$ has minimal polynomial of degree at most two. Thus we immediately get our main result:

**Corollary 4.2 (Compare Conjecture 1, [3, p237]).** *For any fixed $m_1 > 2$, there are only finitely many cometric association schemes $(X, \mathcal{R})$ with some $Q$-polynomial ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents satisfying* $\operatorname{rank} E_1 = m_1$. □

The proof of Theorem 4.1 is broken into three steps. We examine the geometry of the $j^{\text{th}}$ eigenspace (where $j$ is as stipulated in the statement of the theorem) and we focus on the relation $R_1$ selected so that $m > Q_{1j} \geq Q_{ij}$ for all $i > 1$. First, we prove a lemma which implies that this valency $v_1$ is bounded above by some function of $m$. Second, we prove that there are only finitely many possible eigenvalues for $A_1$ in such an association scheme with $\operatorname{rank} E_j = m$. The proof then proceeds to derive a contradiction using these tools.

We remark that the next lemma applies to arbitrary association schemes and may therefore be of independent interest.

**Lemma 4.3 (Cf. [4, Remark, Section 3]).** *Let $(X, \mathcal{R})$ be an association scheme and let $E_j$ be a primitive idempotent with rank $m_j$. Suppose $m_j > Q_{1j} \geq Q_{ij}$ for all $i > 1$. Then $v_1 \leq K$ for some $K$ depending only on $m_j$.*

*Proof.* Fix $a \in X$ and consider the configuration

$$Y' = \left\{ \bar{b} : (a, b) \in R_1 \right\}$$

where $\bar{b}$ denotes the $b^{\text{th}}$ column of $E_j$. In $\mathbb{R}^{m_j}$, the Euclidean distance from $\bar{a}$ is

$$d(\bar{a}, \bar{b}) = d_1 := \sqrt{2(m_j - Q_{1j})/v}$$

for each such $b$. Since this is the smallest distance between any two distinct columns of $E_j$, we have $d(\bar{b}, \bar{c}) \geq d_1$ for any distinct $b, c \in R_1(a)$. Now (after a translation and renormalization) $Y'$ forms a spherical code in $\mathbb{R}^{m_j - 1}$ with center $\frac{Q_{1j}}{m_j}\bar{a}$ since each vector in $Y'$ lies in the hyperplane $\{\bar{x} : \bar{x} \cdot \bar{a} = Q_{1j}/v\}$ inside $\operatorname{colsp} E_j$, the column space of $E_j$.

We next show that the minimum angle formed by distinct vectors in this spherical code is at least $60°$. Denote the point $\bar{a}$ by $A$ and the center of the sphere $\frac{Q_{1j}}{m_j}\bar{a}$ by $O$. Now if $B$ and $C$ are distinct points from $Y'$, then $\angle BAC \geq 60°$ since $d(B, C) \geq d(A, B) = d(A, C)$. So, since $\angle AOB = \angle AOC = 90°$, we must have $\angle BOC > 60°$.

Now we have enough to show that $v_1$ is bounded by a function of $m_1$. Our hypothesis that $m_j > Q_{1j} \geq Q_{ij}$ for all $j > 1$ guarantees that the columns of $E_j$ are all distinct. So we have $v_1 = |Y'|$. But, from the observation in the previous paragraph, $Y'$ can be scaled to a spherical $[-1, \frac{1}{2}]$-code in $\mathbb{R}^{m_j - 1}$. Thus the size of $Y'$ is bounded by the kissing number $\tau_{m_j - 1}$ in dimension $m_j - 1$. It follows that there is a positive integer $K$ depending only on $m_j$ such that $v_1 \leq K$ where $R_1$ is chosen as above. $\square$

Now we need to bound the second largest eigenvalue of $A_1$ away from the valency.

**Theorem 4.4.** *Let $K > 0$ and $S$ the set of all monic polynomials with degree $n$ over the integers, all of whose roots lie in $[-K, K]$. Then $|S|$ is finite.*

*Proof.* Let $f \in S$ have degree $n$, and write $f = x^n + \sum_{i=0}^{n-1} f_i x^i$. Let $s$ be the maximum absolute value of all the roots of $f$. Then the coefficient $f_i$ satisfies

$$-\binom{n}{i} s^{n-i} \leq f_i \leq \binom{n}{i} s^{n-i}.$$

In particular, we have

$$-\binom{n}{i} K^{n-i} \leq f_i \leq \binom{n}{i} K^{n-i}.$$

Since this bound depends only on the degree of $f$ and choice of $i$, there are only finitely many possible values for each integer coefficient $f_i$, so the set $S$ must be finite. $\square$

**Corollary 4.5.** *Let $K > 0$ and let $n$ be a positive integer. Then there are only finitely many algebraic integers $a$ satisfying*

*(i) the minimal polynomial of $a$ over the rationals has degree at most $n$, and*

*(ii) $a$ and all of its algebraic conjugates lie in the interval $[-K, K]$.* $\square$

Now we are ready to complete our proof.

*Proof.* [Theorem 4.1] Suppose, by way of contradiction, that for some fixed $m \geq 2$, there are infinitely many non-isomorphic association schemes with $\operatorname{rank} E_j = m$ (for some $j$, but without loss of generality we may always take $j = 1$). Assume further that for each of these schemes, all $P_{ji}$ have minimal polynomials over $\mathbb{Q}$ of degree $n$ or less. Let $\mathcal{F}$ denote this family of association schemes and henceforth for each scheme in this family, order the relations $R_0, R_1, \ldots, R_d$ so that

$$m = Q_{01} > Q_{11} \geq Q_{21} \geq \cdots \geq Q_{d1}.$$

Let $K$ be the bound of Lemma 4.3, where $K$ depends only on $m$. Let $\mathcal{B}$ denote the ring of algebraic integers and, for $b \in \mathcal{B}$, write $m_b(t)$ for the minimal polynomial of $b$ over the rationals. Now consider the set

$$S = \left\{ b \in \mathcal{B} \setminus \mathbb{Z} \;\middle|\; \deg m_b \leq n,\ -K \leq \xi \leq K \text{ whenever } m_b(\xi) = 0 \right\}$$

and let

$$r = \max\left( \frac{K-1}{K}, \max_{s \in S} \frac{s}{\lceil s \rceil} \right).$$

Then $r$ is well-defined by Corollary 4.5 and $r < 1$. Choose $\delta$ so that $1 > 1 - \delta > r$.

Let $M = U(m, 1-\delta)$ be the bound on spherical codes from Section 3. Since our family $\mathcal{F}$ is assumed to be infinite, we can find a member of $\mathcal{F}$ with more than $M$ vertices. For this association scheme, we must have

$$m_1 > Q_{11} > (1-\delta)m_1.$$

(In fact, one can prove something a bit stronger; see Proposition 5.1.) Since $P_{11} = v_1 Q_{11}/m_1$ (from (1.1)), this gives

$$v_1 > P_{11} > (1-\delta)v_1$$

or $1 > \frac{P_{11}}{v_1} > r$. But this is impossible since either $P_{11} \in \mathbb{Z}$ (in which case $\frac{P_{11}}{v_1} \leq \frac{K-1}{K} \leq r$) or $P_{11}$ belongs to the set $S$: it is an algebraic integer in some extension $\mathbb{Q}[\zeta]$ of degree at most $n$ and any conjugate $\xi$ of $P_{11}$ is an eigenvalue of $A_1$ so satisfies

$$-K \leq -v_1 < \xi < v_1 \leq K.$$

So we have arrived at the desired contradiction. $\square$

**Example 4.6.** *For any fixed $m \geq 2$, there are only finitely many primitive association schemes with some idempotent having rank $m$ and all eigenvalues either integral or belonging to quadratic extensions of the rationals.*

# 5 Further results and open questions

The study of cometric association schemes seems wide open. Very few examples are known which do not come from the theory of distance-regular graphs. In addition to the paper [16] used in Section 2, Suzuki also proved in [15] that an imprimitive cometric association scheme with more than six classes must be either $Q$-bipartite or $Q$-antipodal. Together with M. Muzychuk, the authors studied imprimitive cometric schemes in [12]. In that paper, we list all the cometric schemes known to us which are neither metric nor duals of metric association schemes.

While we have proven that there are only finitely many cometric schemes for a given multiplicity $m_1$, we have no useful concrete bound on the number of such schemes. We do not even have a list of all cometric association schemes satisfying $\operatorname{rank} E_1 = 3$ although we expect all such examples are known. Bannai and Bannai [4] proved that the only primitive example is the tetrahedron.

The search for more examples and the analysis of their structure both depend on the development of more tools for working with them. We finish by proving the following result.

**Proposition 5.1.** *Let $m$ be a fixed integer, at least two. Let $\mathcal{F}$ be the family of symmetric association schemes with some primitive idempotent $E_j$ having rank $m$ and having all columns distinct. Then either $\mathcal{F}$ is finite or*

*(i) for each $\delta > 0$ and each positive integer $M$, there is a member of $\mathcal{F}$ in which the graph*

$$G = (X, \bigcup_{\frac{Q_{ij}}{m} > 1-\delta} R_i)$$

*has a clique of size $M$ or larger, this $j$ being any integer which satisfies* $\operatorname{rank} E_j = m$, *and*

*(ii) for each $\delta > 0$ and each positive integer $N$, there is a member of $\mathcal{F}$ for which*

$$|\{i : m > Q_{ij} > (1-\delta)m\}| > N.$$

*Proof.* Let $U(m, 1-\delta)$ be the maximum size of a spherical $T$-code $X$ in $\mathbb{R}^m$ with $T = [-1, 1-\delta]$. Now take $M' = \max(M, U(m, 1-\delta) + 1)$. For each member $(X, \mathcal{R})$ of $\mathcal{F}$, the partition $(G, \bar{G})$ is a two-edge-coloring of the complete graph on $X$. For $|X| > R(M', M')$ (the two-color Ramsey number), either $G$ or $G'$ must admit a clique of size $M'$. But such a clique in $\bar{G}$ would map to a spherical $[-1, 1-\delta]$-code of size $M'$ in $\mathbb{R}^m$, which is impossible. So there must be such a clique in $G$. This proves the first claim. For the second claim, we simply observe that each graph $R_i$ has clique number at most $m+1$ since mutually adjacent vertices are mapped to a regular simplex in $\mathbb{R}^m$ by $b \mapsto \bar{b}$ and the Ramsey number $R(m+2, m+2, \ldots, m+2)$ (with $N$ operands) is known to be finite. $\square$


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