Student Problem Set 6

Instructions: Please solve these problems in today’s problem session at 4:30 (and continue later, as needed) in groups and present the solutions on the black/whiteboards in the front of the room. You are encouraged to ask questions and to ask for help.

1. Prove that the irreducible representations of a finite abelian group have dimension 1.

2. Use the previous exercise and the magic formula $|G| = \sum d_R^2$ to prove that an abelian group $G$ has $|G|$ irreducible representations.

3. We want to compute the irreducible representations of $A_4$.
   (a) Find a normal subgroup $H$ of $A_4$ such that the quotient $A_4/H$ is cyclic of order 3.
   (b) Construct 3 representations of degree 1 of $A_4$ from the ones of $A_4/H$.
   (c) Use the magic formula to have a guess for the degree of a missing irreducible representation of $A_4$.
   (d) Construct an irreducible representation of degree 3 of $A_4$ from the isometries of a regular polytope.
   (e) Conclude

4. Prove that $\langle \chi_V, \chi_V \rangle = 1$ if and only if $V$ is irreducible.

5. For $V = P_k$, which denotes the subspace of $C(H_n)$ generated by the $\chi_y$ with $|y| = k$ compute $\langle \chi_V, \chi_V \rangle$ and use previous exercise to derive that $P_k$ is irreducible. Here we are considering the action of $T \rtimes S_n$. Is $P_k$ irreducible for the action of $S_n$?

6. Prove Parseval formula in Frank talk:
   $$\langle f, g \rangle = \sum_{u \in \{0, 1\}^n} \hat{f}(u)\hat{g}(u)$$

7. Prove the formula in Frank talk:
   $$\hat{1}_{A^\perp}(u) = \hat{1}_A(u)\chi_u(y)$$

8. Prove that, for the Krawtchouk polynomials $K_k$ associated to the Hamming cube $\{0, 1\}^n$,
   $$\binom{n}{u} K_k(u) = \binom{n}{k} K_u(k)$$