Almost Orthogonal Vectors in Euclidean Space

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Chapter 1

Introduction

This Major Qualifying Project is motivated by two problems. The first is to consider how many lines can be fit through the origin in d-dimensional space such that the angle between any two lines is close to 90 degrees. If the lines are exactly 90 degrees apart, then the coordinate axes are the optimal configuration and the answer is the dimension of the space. However, if a tolerance away from 90 degrees is allowed, then the answer to this question is not obvious. For example in 57-dimensional space, allowing the angle between any two lines to be between 89 and 90 degrees does not yield any improvement and you still only get 57 lines. However, in 58-dimensional space with the same tolerance you get 59 lines.

There are interesting examples in three-dimensional space that are easier to understand. In $\mathbb{R}^3$, the optimal configuration for four vectors through the origin with maximum angle is given by the regular simplex. For five vectors, remove a diagonal from the icosahedron to find the optimal configuration. The most interesting part of this example is the amount of separation of the lines does not change if the missing diagonal is included. It is also interesting to note the case of 7 lines in $\mathbb{R}^3$. In this case, the number of lines is as close to the upper bound we have found as possible for that tolerance. These examples give the flavor for the problem.

The other problem this project considers is how to find large matrices very close to the identity with very low rank. By close to the identity we mean all diagonal entries are 1 and all off diagonal entries are close to 0. This is close to the identity in the ℓ_∞ metric. These two problems may seem very different, but they are in fact the same when the matrices are restricted to symmetric positive semidefinite matrices with all diagonal entries 1 and off diagonal entries close to zero. The following theorem will show the connection.

Theorem 1 *Let $E_{n\times n}$ be symmetric and positive semidefinite, then E is of rank d if and only if there are n vectors, $x_1, x_2, ..., x_n$ in $\mathbb{R}^d$ such that $E_{ij} = \langle x_i, x_j \rangle$.*

Proof: We will use the following facts in the proof:

1. Let $A_{n\times n}$ be a real symmetric matrix, then $\theta_{min} = \min_{u\in\mathbb{R}^n} \frac{u^T Au}{u^T u}$ with $u \neq 0$. These are known as Rayleigh quotients and this implies that A is positive semidefinite if and only if for all u in $\mathbb{R}^n$, $u^T Au \geq 0$.

2. All principal sub-matrices of A are positive semidefinite. This is easy to prove, suppose

B is a principal sub-matrix and the rows and columns of A are reordered such that the matrix

$$A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix}.$$

So $u^T B u = [u^T|0] A [\frac{u}{0}]$ where the u vectors are appended with zeros. Since A is positive semidefinite, $x^T A x \geq 0$ for all x and in particular the u appended with zeros. This implies $u^T B u \geq 0$.

In the direction $E_{ij} = \langle x_i, x_j \rangle$ for n vectors in $\mathbb{R}^d$ implies E is a symmetric positive semidefinite matrix of rank d is easiest. First define $U = [x_1|x_2|...|x_n]$ with each x_i in $\mathbb{R}^d$. Then $E = U^T U$ is $n \times n$ and symmetric, and $E_{ij} = \langle x_i, x_j \rangle$. To prove E is positive semidefinite, fact 1 will be used. First we observe $x^T E x = x^T U^T U x$. Then, using properties of the transpose we have $x^T U^T U x = (Ux)^T (Ux)$. We then know that $(Ux)^T (Ux) = ||Ux||^2$. From this we know $||Ux||^2 \geq 0$ so $x^T E x \geq 0$.

Rather than give a full proof, we will outline the argument in the reverse direction. Suppose the E is a symmetric positive semidefinite matrix of rank d. Then construct the first vector, x_1 such that the first component is α, where $\alpha = \sqrt{E_{11}}$. Using fact 2, we know that E_{11} must be positive because it is a principal sub-matrix of a positive semidefinite matrix and since the eigenvalue of $[E_{11}]$ is E_{11}, it must be nonnegative. From here construct the rest of the vectors by solving the linear system $E_{ij} = \langle x_i, x_j \rangle$. An example is given: Let

$$E = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}.$$

Then

$$U = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix}.$$

From this you can see $E = U^T U$.

Definition 1 *A set of unit vectors X in $\mathbb{R}^d$ is called ϵ-almost orthogonal if for any two distinct vectors x_1, x_2 in X, $|\langle x_1, x_2 \rangle| \leq \epsilon$.*

The goal of this project is to establish upper and lower bounds for the cardinality of the set X as a function of the dimension of the space, d and the tolerance ϵ. The upper bounds will be established by using a family of polynomials called Gegenbauer polynomials from spherical harmonics. The lower bounds will be constructed from known examples found from either distance regular graphs, known theorems, a table of examples from Neil Sloane at AT&T, and propagation techniques developed during this project. Selections from a table of lower bounds we constructed are in Chapter 5. From this, growth rates can be established for tolerances that are functions of the dimension of the space. This project considers Euclidean space. We did not consider the complex case or other Grassmanian packings.

A large portion of Chapter 2 is an exposition of “Spherical Codes and Designs” by Delsarte, Goethals and Seidel which discusses how to establish upper bounds[1]. Proofs of theorems and lemmas that are within my ability will be provided and examples of others will be explored. This will all be discussed in the chapter on linear programming bounds.

During this project, growth rates for the cardinality of a set of ϵ-almost orthogonal vectors were investigated. Let c be a constant. In our approach we consider ϵ to be a function of the dimension of the space. We also assume that as $d \to \infty$, $\epsilon(d) \to 0$. If $\epsilon = \frac{c}{d}$, then $|X| = \Theta(d)$. If $\epsilon = \frac{c}{\sqrt{d}}$, then $|X| \geq \Omega(d^2)$. By the probabilistic method, for $\epsilon = \frac{c}{\log d}$, evidence suggests that $|X| = \Omega(d^k)$, for any k. Note that $f(n) = O(g(n))$ means that there exists $C > 0$ such that for sufficiently large n, $f(n) < Cg(n)$. Also, $f(n) = \Omega(g(n))$ means that $g(n) = O(f(n))$, or in other words there exits $C > 0$ for sufficiently large n such that $f(n) > Cg(n)$. Lastly, $f(n) = \Theta(g(n))$ means that there exists $C_1 > 0$ and $C_2 > 0$ for sufficiently large n such that $C_1 g(n) \leq f(n) \leq C_2 g(n)$.

It is then conjectured that for $\epsilon = c$, $|X| = O(a^d)$. This conjecture has not yet been proved. The Johnson graphs, a particular type of distance regular graph, should provide exponential growth, however the details of the proof are incomplete. There is data consistent with the other growth rates as will be explained in the body of the report.

Chapter 2

Bounds from Linear Programming

In the paper "Spherical Codes and Designs" by Delsarte, Goethals, and Seidel, a theory about the combinatorics of finite sets of points on spheres is developed. In particular, they devise bounds on the size of a set X of unit vectors given the number of possible inner products among its elements or given a candidate set $A \subset [-1, 1)$ of potential values for the inner products among distinct elements of X. Their approach uses ideas from spherical harmonics. Central to the linear programming bound, which we will examine in some detail, is a family of orthogonal polynomials called "Gegenbauer polynomials".

2.1 Gegenbauer Polynomials

The family of polynomials we will consider are the Gegenbauer (or ultraspherical) polynomials $\{Q_k(x) : k \in \mathbb{N}\}$ defined for a fixed dimension, d. This is a family of orthogonal polynomials with respect to the measure $(1 - x^2)^{\frac{d-3}{2}}$.

$$\int_{-1}^{1} Q_k(x) Q_l(x) (1 - x^2)^{\frac{d-3}{2}} dx = a_d Q_k(1) \delta_{k,l},$$

where a_d is some positive constant and $\delta_{k,l}$ is the Kronecker symbol. The following definitions and theorem will explain the significance of these polynomials.

Definition 2 *Let $A(X) = \{\langle x, y \rangle : x \neq y \in X\}$. Also suppose A is a particular subset of $[-1, 1)$ containing $A(X)$, then X is called a spherical A-code.*

From this definition we find a relationship between the Gegenbauer coefficients and the cardinality of X that corresponds with a particular A-code.

Definition 3 *Let $A \subset [-1, 1)$ and $F : [-1, 1] \to \mathbb{R}$. F is said to be compatible with A if $F(\alpha) \leq 0$ for all α in A.*

In order to find a relationship between the Gegenbauer coefficients and the cardinality of X, we need a function F such that it is a nonnegative linear combination of Gegenbauer polynomials and compatible with A. In our case, $A = (-\epsilon, \epsilon)$.

Definition 4 *The Gegenbauer Polynomial $Q_k(x)$ of degree k is defined by*

$$\lambda_{k+1}Q_{k+1}(x) = xQ_k(x) - (1-\lambda_{k-1})Q_{k-1}(x),$$

where

$$\lambda_k = k/(d+2k-2), \qquad Q_0(x) = 1, \qquad Q_1(x) = dx.$$

The following are the first few Gegenbauer polynomials.

$$\begin{aligned}
2Q_2(x) &= (d+2)(dx^2-1), \\
6Q_3(x) &= d(d+4)\left((d+2)x^3 - 3x\right), \\
24Q_4(x) &= d(d+6)\left((d+2)(d+4)x^4 - 6(d+2)x^2 + 3\right), \\
120Q_5(x) &= d(d+2)(d+8)\left((d+4)(d+6)x^5 - 10(d+4)x^3 + 15x\right).
\end{aligned}$$

These Gegenbauer polynomials are related to the usual Gegenbauer polynomials, $C_k^m(x)$ for $d \geq 3$ by

$$Q_k(x) = \frac{d+2k-2}{d-2} C_k^{(d-2)/2}(x).$$

The regular Gegenbauer polynomials are defined as

$$C_k^m(x) = \frac{\Gamma(m+1/2)\Gamma(k+2m)}{\Gamma(2m)\Gamma(k+m+\frac{1}{2})} P_k^{(m-\frac{1}{2},m-\frac{1}{2})}(x),$$

where $P_k^{(a,b)}$ is the usual Jacobi polynomial defined as

$$P_k^{(a,b)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-a}(1+x)^{-b}\frac{d^k}{dx^k}[(1-x)^{a+k}(1+x)^{b+k}].$$

For the case where $d = 2$ and $k \geq 1$, the Gegenbauer polynomials are related to the Chebyshev polynomials of the first kind $T_k(x)$ by

$$Q_k(x) = kC_k^0(x) = 2T_k(x).$$

It is convenient to observe that the Gegenbauer polynomials and Chebyshev polynomials can be recursively defined as well.

$$\begin{gathered}
(k+1)C_{k+1}^m(x) = 2(k+m)xC_k^m(x) - (k+2m-1)C_{k-1}^m(x), \\
C_0^m(x) = 1, \qquad C_1^m(x) = 2mx
\end{gathered}$$

$$\begin{gathered}
T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \\
T_0(x) = 1, \qquad T_1(x) = x.
\end{gathered}$$

From the definitions above it is easy to see that both of the first two Chebyshev polynomials satisfy the relations relating our Gegenbauer polynomials to the usual Gegenbauer polynomials and Chebyshev polynomials for $k = 0$ and $k = 1$. To show that it is true in general is easy. If we assume the relationship between our Gegenbauer polynomials and the usual ones holds up to k and using our recursive definition we have

$$\frac{k+1}{d+2k} Q_{k+1}(x) = \frac{(d+2k-2)}{d-2} x C_k^{\frac{d-2}{2}} - \frac{d+k-3}{d+2k-4} \frac{d+2k-4}{d-2} C_{k-1}^{\frac{d-2}{2}}.$$

This implies that

$$(k+1)Q_{k+1}(x) = \frac{d+2k}{d-2} \left[(d+2k-2) x C_k^{\frac{d-2}{2}} - (d+k-3) C_{k-1}^{\frac{d-2}{2}} \right].$$

And from the recurrence relation for the usual Gegenbauer polynomials, we see that

$$(k+1)Q_{k+1}(x) = \frac{d+2k}{d-2} (k+1) C_{k+1}^{\frac{d-2}{2}}.$$

By canceling the $k+1$ on both sides of the equation we see that the proof is complete. For the Chebyshev polynomials, again knowing that it works for $k = 0$ and $k = 1$, and using the recurrence relations we get

$$Q_{k+1}(x) = 2(2x T_k(x) - \frac{1}{2} 2 T_{k-1}(x)).$$

From the recurrence relation for the Chebyshev polynomials we see that

$$Q_{k+1}(x) = 2T_{k+1}(x)$$

when $d = 2$.

Definition 5

$$R_k(x) = \sum_{i=0}^{k} Q_i(x),$$

$$C_k(x) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} Q_{k-2i}(x).$$

Apart from a constant, $R_k(x)$ is the usual Jacobi polynomial and $C_k(x)$ is the usual Gegenbauer polynomial $C_k^{\frac{d}{2}}(x)$. From the above definitions, the following theorem is easily proved.

Theorem 2

$$Q_k(1) = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1},$$

$$R_k(1) = \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1},$$

$$C_k(1) = \binom{d+k-1}{d-1}, \qquad k \geq 1.$$

Proof: By induction we need to check if

$$Q_k(1) = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1}$$

for $k=0$ and $k=1$. If this is true, then

$$Q_0(1) = \binom{d-1}{d-1} - \binom{d-3}{d-1}.$$

Since $\binom{d-3}{d-1} = 0$ and $\binom{d-2}{d-1} = 0$, then from the above, $Q_0(1) = 1$ and $Q_1(1) = d$, both of which are true by definition. Since we have the recursive definition, then

$$\lambda_{k+1} Q_{k+1}(1) = 1Q_k(1) - (1-\lambda_{k-1})Q_{k-1}(1).$$

If we assume the above formula is true up to k, then

$$\lambda_{k+1} Q_{k+1}(1) = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1} - \frac{d+k-3}{d+2k-4}\left[\binom{d+k-2}{d-1} - \binom{d+k-4}{d-1}\right].$$

Dividing both sides by λ_{k+1} and simplifying gives us

$$Q_{k+1}(1) = \frac{(d+2k)(d+k-2)!}{(d-2)!(k+1)!}.$$

But

$$\binom{d+k}{d-1} - \binom{d+k-2}{d-1} = \frac{(d+2k)(d+k-2)!}{(d-2)!(k+1)!},$$

so the formula for $Q_k(1)$ is proved. The formula for $R_k(1)$, is true for $k=0$. Then assume

$$R_k(1) = \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1}$$

holds up to k. By definition,

$$R_{k+1}(1) = \sum_{i=0}^{k+1} Q_i(1),$$

which implies that

$$R_{k+1}(1) = R_k(1) + Q_{k+1}(1).$$

Using our induction hypothesis and what we already proved about $Q_k(1)$ we get

$$R_{k+1}(1) = \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1} + \binom{d+k}{d-1} - \binom{d+k-2}{d-1}.$$

From this we get the result that

$$R_{k+1}(1) = \binom{d+k}{d-1} + \binom{d+k-1}{d-1},$$

so the formula for $R_k(1)$ has been proved. For the formula for $C_k(1)$, we need to consider two cases because of how it is defined. Clearly the formula is true for $k = 0$ and $k = 1$. Because of the definition we will show the formula is true for $C_{2k}(1)$ and $C_{2k+1}(1)$. We know it is true for $k = 0$, so assume the formula works up to $2k$. Also, by definition,

$$C_{2k+2}(1) = \sum_{i=0}^{k+1} Q_{2k+2-2i}(1).$$

Using the induction hypothesis and what we proved about $Q_k(1)$, we have

$$C_{2k+2}(1) = C_{2k}(1) + Q_{2k+2}(1).$$

But this implies

$$C_{2k+2}(1) = \binom{d+2k-1}{d-1} + \binom{d+2k+1}{d-1} - \binom{d+2k-1}{d-1}.$$

So

$$C_{2k+2}(1) = \binom{d+2k+1}{d-1},$$

which is what we wanted. Since we know the formula works for $C_1(1)$ and using the above ideas, we know

$$C_{2k+3}(1) = C_{2k+1}(1) + Q_{2k+3}(1).$$

Again, from the induction hypothesis and what was already proved,

$$C_{2k+3}(1) = \binom{d+2k}{d-1} + \binom{d+2k+2}{d-1} - \binom{d+2k}{d-1},$$

which gives us

$$C_{2k+3}(1) = \binom{d+2k+2}{d-1},$$

which is what we wanted and thus completes the proof of the theorem. □

We will also define $F(x) = \sum_{k=0}^{\infty} f_x Q_k(x)$ to be the Gegenbauer expansion of any polynomial $F(x)$.

The following lemmas will be useful and can be proved using a formula for the usual Gegenbauer polynomials from [2]. First we'll define some notation.

$$(a)_k = (a)(a+1)...(a+k-1),$$

or

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The following formula can be proved by induction.

$$C_i^m(x)C_j^m(x) = \sum_{k=0}^{\min(i,j)} \frac{i+j+m-2k}{i+j+m-k} \frac{(m)_k(m)_{i-k}m_{j-k}(2m)_{i+j-k}(i+j-2k)!}{k!(i-k)!(j-k)!(m)_{i+j-k}(2m)_{i+j-2k}} C_{i+j-2k}^m(x).$$

It is easy to see that this formula holds if at least one of i or j is zero. Indeed,

$$C_0^m(x)C_j^m(x) = \frac{j+m}{j+m}\frac{(m)_j(2m)_j j!}{j!(m)_j(2m)_j}C_j^m(x)$$

and from this it is clear that

$$C_0^m(x)C_j^m(x) = C_j^m(x),$$

which is what we expect. Let's now look at when both i and j are one.

$$C_1^m(x)C_1^m(x) = \frac{2+m}{2+m}\frac{m^2(2m)(2m+1)2!}{2!m(m+1)(2m)(2m+1)}C_2^m(x) + \frac{m}{m+1}\frac{m(2m)}{m}C_0^m(x).$$

Keeping in mind that

$$C_2^m(x) = -m + 2m(1+m)x^2,$$

we see that

$$C_1^m(x)C_1^m(x) = 4m^2x^2,$$

which is what we expected. Assuming that the formula holds if i is one and up to j, we can show it will hold for $j+1$. Since

$$C_1^m(x)C_{j+1}^m(x) = C_1^m(x)\left[\frac{1}{j+1}(2(j+m)xC_j^m(x) - (j+2m-1)C_{j-2}^m(x))\right],$$

and observe that $x = \frac{1}{2m}C_1^m(x)$, the linearization formula must hold since it works for the terms of the recurrence relation. Also note that since by holding i fixed to either zero or one, the formula works for all j. Then

$$C_{i+1}^m(x)C_j^m(x) = \frac{1}{i+1}\left(\frac{i+m}{m}C_1^m(x)C_i^m(x) - (i+2m-1)C_{i-1}^m(x)\right)C_j^m(x).$$

Again, since the formula works for all j up to i, it works for $i+1$. To make this formula work for our Gegenbauer polynomials, we will use

$$C_k^{\frac{d-2}{2}}(x) = \frac{d-2}{d+2k-2}Q_k(x).$$

Now that we know this formula works, the following lemmas will be easy to prove.

Lemma 1 *Let* $Q_i(x)Q_j(x) = \sum_{k=0}^{i+j} q_0(i,j)Q_k(x)$. *Then*

$$q_0(i,j) = Q_i(1)\delta_{i,j} \qquad \text{and} \qquad q_k(i,j) \geq 0$$

for all i, j, k, *with* $q_k(i,j) > 0$ *if and only if* $|i-j| \leq k \leq i+j$ *and* $k \equiv i+j \pmod 2$.

It is easy to see from the linearization formula that you can write the products this way with nonnegative coefficients. To prove what q_0 is simply requires looking at the formula. It is the case when $i = j$ and the term with $k = i$. So $q_0(i,j) = 0$ if $i \neq j$ and looking at the regular Gegenbauer coefficient for C_0^m we have

$$q_0 = \frac{m}{m+i} \frac{\Gamma(m+i)\Gamma(2m+i)\Gamma(m)}{i!\Gamma(m+i)\Gamma(m)\Gamma(2m)}.$$

After making the substitution $m = \frac{d-2}{2}$ we obtain

$$q_0 = \frac{(d-2)\Gamma(d-2+i)}{(d+2i-2)\Gamma(d-2)i!}.$$

In order to get the coefficient for our Gegenbauer polynomial we then need to multiply by $\frac{(d+2i-2)^2}{(d-2)^2}$, so

$$q_0(i,i) = \frac{(d+2i-2)(d-2+i)!}{(d-2)!i!}.$$

From above we know that

$$q_0(i,i) = \binom{d+i-1}{d-1} - \binom{d+i-3}{d-2},$$

which is $Q_i(1)$, so the lemma is proved. □

The next lemma will be even easier to prove. We will use Gegenbauer expansions $F(x) = \sum_{k=0}^{\infty} f_k Q_k(x)$ and $G(x) = \sum_{k=0}^{\infty} g_k Q_k(x)$.

Lemma 2 *Let $G(x) = Q_l(x)F(x)/Q_l(1)$ for some $l \in \mathbb{N}$. Then*

$$g_0 = f_l,$$

and if f_k is nonnegative for all k then g_k is nonnegative for all k, where g_k and f_k are the Gegenbauer coefficients for the expansion of the corresponding polynomials.

From our linearization of products formula we get that $g_0 = Q_l(1)f_l$ from the Gegenbauer expansion of $F(x)$. In the lemma we divide by $Q_l(1)$ so $g_0 = f_l$. It is also easy to see from the linearization formula that any g_k is a nonnegative linear combination of the coefficients of $F(x)$ so if all of the coefficients of the expansion for $F(x)$ are nonnegative, then the coefficients for the expansion of $G(x)$ is also nonnegative. This is a consequence of Lemma 1.

The next lemma requires a bit of induction, but is simple.

Lemma 3 *Let $G(x) = x^l F(x)$ for some $l \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, g_k is a convex linear combination — with strictly positive coefficients — of the f_{k+l-2i} for $i = 0, 1, ..., \min(l, \lfloor \frac{1}{2}(k+l) \rfloor)$.*

Proof: First let us consider $l = 1$. It should be clear that g_k should be a linear combination of f_{k-1} and f_{k+1}. Also remember that $xC_k^m(x) = \frac{1}{2m}C_1^m(x)C_k^m(x)$. From this we can see that if h_k is the coefficient for the usual Gegenbauer polynomials then

$$h_k = \frac{1}{2m}\left[f_{k-1}\frac{m(m)_{k-1}(2m)_k k!}{(k-1)!(m)_k(2m)_k} + f_{k+1}\frac{k+m}{k+1+m}\frac{m(m)_k(2m)_{k+1}k!}{k!(m)_{k+1}(2m)_k}\right].$$

Simplifying and converting to our Gegenbauer polynomial coefficients we get

$$g_k = \lambda_k f_{k-1} + (1-\lambda_k) f_{k+1}.$$

This is a convex linear combination so it works for $l = 1$ since $\lambda_k + 1 - \lambda_k = 1$. When $l = 2$, apply the same process to the new coefficients e. Then

$$e_k = \lambda_k g_{k-1} + (1-\lambda_k) g_{k+1},$$

but that is

$$e_k = \lambda_k \left(\lambda_{k-1} f_{k-2} + (1-\lambda_{k-1}) f_k\right) + (1-\lambda_k)\left(\lambda_{k+1} f_k + (1-\lambda_{k+1}) f_{k+2}\right).$$

It is easy to see that this process continues to produce convex linear combinations as described before.

After all of this it is interesting to consider the orthogonality again. Using Maple I found that

$$\int_{-1}^{1} (1-x^2)^{\frac{d-3}{2}} dx = \frac{\Gamma(\frac{d-1}{2})\sqrt{\pi}}{\Gamma(\frac{d}{2})}$$

which is a_d. It is also easy to show that $Q_1(x)$ is orthogonal to $Q_0(x)$ by simply showing that

$$\int_{-1}^{1} Q_1(x)(1-x^2)^{\frac{d-3}{2}} dx = 0.$$

Since $Q_1(x) = d \cdot x$, not to be confused with the differential, and noting that the integrand below is odd, we know that

$$\int_{-1}^{1} dx(1-x^2)^{\frac{d-3}{2}} dx = 0.$$

Using Maple to show that $Q_2(x)$ is orthogonal to $Q_0(x)$, we can then use induction to show that for all $k \neq 0$, $Q_k(x)$ is orthogonal to $Q_0(x)$. Assuming this is valid up to i, then

$$\int_{-1}^{1} Q_{i+1}(x)(1-x^2)^{\frac{d-3}{2}} dx = \frac{1}{\lambda_{i+1}} \int_{-1}^{1} (xQ_i(x) - (1-\lambda_{i-1})Q_{i-1}(x))(1-x^2)^{\frac{d-3}{2}} dx.$$

We already know by hypothesis that

$$\int_{-1}^{1} Q_{i-1}(x)(1-x^2)^{\frac{d-3}{2}} dx = 0.$$

Also, note by using the linearization formula we get a term with a $Q_{i+1}(x)$ and $Q_{i-1}(x)$, so solving for the integral with $Q_{i+1}(x)$, we get an expression that equals 0 so

$$\int_{-1}^{1} Q_{i+1}(x)(1-x^2)^{\frac{d-3}{2}} dx = 0.$$

Now we know that for all $k > 0$, $Q_k(x)$ is orthogonal to $Q_0(x)$. Also note from the linearization formula the product of any two Gegenbauer polynomials is a linear combination of Gegenbauer polynomials and for $Q_k(x)Q_k(x)$ we have a term that is $Q_k(1)Q_0(x)$. From this and what we just proved we have that

$$\int_{-1}^{1} Q_i(x)Q_j(x)(1-x^2)^{\frac{d-3}{2}} dx = Q_i(1) a_d \delta_{i,j},$$

which is what we said it would be when we first talked about orthogonality.

2.1.1 Harmonic Polynomials

We will consider Ω_d to be the unit sphere in Euclidean space $\mathbb{R}^d$ with measure ω_d and inner product $\langle , \rangle$. Let $Hom(k) = Hom_d(k)$ for $k \geq 0$ denote the linear space of all functions $V : \Omega_d \to \mathbb{R}$ which are represented by polynomials $V(x) = V(x_1, ..., x_d)$ homogeneous of total degree k. The *Laplace operator* is a differential operator which maps $Hom(k)$ onto $Hom(k-2)$ via

$$\Delta V = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \cdots + \frac{\partial^2 V}{\partial x_d^2}.$$

A polynomial V is said to be *harmonic* if $\Delta V = 0$. Let $Harm(k)$ denote the subspace $Hom(k)$ consisting of functions represented by harmonic polynomials of degree k. $Harm(k)$ is invariant under the orthogonal group $O(d)$ of $\mathbb{R}^d$.

To prove this let us consider a monomial from $Hom(k)$,

$$U = x_1^{a_1} \cdots x_d^{a_d} \qquad \text{with} \sum_{i=1}^{d} a_i = k.$$

Then

$$\Delta U = \sum_{i=1}^{d} a_i(a_1 - 1) x_1^{a_1} \cdots x_i^{a_i - 2} \cdots x_d^{a_d}.$$

Now applying some orthogonal transformation we get

$$L_M(\Delta U) = \sum_{i=1}^{d} a_i(a_i - 1) y_1^{a_1} ... y_i^{a_i - 2} ... y_d^{a_d},$$

where $M = [m_{ik}]$ and

$$y_i = \sum_{i=1}^{d} m_{ik} x_k.$$

If the orthogonal transformation is applied first, then

$$L_M(U) = y_1^{a_1} ... y_d^{a_d}.$$

Keeping in mind the product rule and chain rule when we take the Laplacian of the above we see that

$$\Delta L_M(U) = \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} m_{ik} m_{jk} a_i a_j \frac{y_i^{a_i - 1} y_j^{a_j - 1}}{y_i^{a_i} y_j^{a_j}} y_1^{a_1} ... y_d^{a_d} + \sum_{i=1}^{d} a_i(a_i - 1) y_1^{a_1} ... y_i^{a_i} ... y_d^{a_d} (\sum_{k=1}^{d} m_{ik}^2).$$

From this we can see that

$$\Delta L_M(U) = \sum_{i=1}^{d} \sum_{j=1}^{d} a_i a_j y_1^{a_1} ... y_i^{a_i - 1} ... y_j^{a_j - 1} ... y_d^{a_d} \sum_{k=1}^{d} (m_{ik} m_{jk}) + \sum_{i=1}^{d} a_i(a_i - 1) y_1^{a_1} ... y_i^{a_i} ... y_d^{a_d} \sum_{k=1}^{d} (m_{ik}^2).$$

Since M is orthogonal, we know that $\sum_{k=1}^{d} m_{ik} m_{jk} = 0$ and $\sum_{k=1}^{d} m_{ik}^2 = 1$, so we have shown the invariance. By linearity, we have $L_M(\Delta f) = \Delta L_M(f)$ for any polynomial. So since L_M is invertible, $\Delta f = 0$ if and only if $\Delta L_M(f) = 0$.

There is a direct sum decomposition of $Hom(k)$ which is stated in the following theorem.

Theorem 3

$$Hom(k) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} Harm(k-2i),$$

$$Hom(k) \oplus Hom(k-1) = \sum_{i=0}^{k} Harm(i).$$

Clearly if the first line is true then the second line must be as well. It is also important to remember that these linear spaces are of functions that are representable by polynomials on Ω_d. Some of the theorems in this section will not be proved, however some examples will be given to illustrate this theorem.

First a list of the span of the first few spaces with $d = 2$ will be given.

$$\begin{aligned} Hom(0) &= span\{1\} & Harm(0) &= span\{1\}, \\ Hom(1) &= span\{x, y\} & Harm(1) &= span\{x, y\}, \\ Hom(2) &= span\{x^2, xy, y^2\} & Harm(2) &= span\{xy, x^2 - y^2\}, \\ Hom(3) &= span\{x^3, x^2y, xy^2, y^3\} & Harm(3) &= span\{x^3 - 3xy^2, y^3 - 3x^2y\}. \end{aligned}$$

The first interesting example is with $Hom(2)$. By the theorem,

$$Hom(2) = Harm(2) + Harm(0) = span\{xy, x^2 - y^2, 1\}.$$

But remember that these are functions represented on the unit sphere, in this case Ω_2. In this case, it is just the unit circle, so

$$Harm(2) + Harm(0) = span\{xy, x^2 - y^2, x^2 + y^2\}.$$

Adding the second two elements yields $2x^2$ and subtracting the second element from the third yields $2y^2$, so

$$Harm(2) + Harm(0) = span\{xy, x^2, y^2\},$$

which is $Hom(2)$. The next interesting example is

$$Hom(3) = Harm(3) + Harm(1).$$

Note that

$$Harm(3) + Harm(1) = span\{x^3 - 3xy^2, y^3 - 3x^2y, x, y\}.$$

Again remembering that we can multiply x and y by one, but in the sphere that is $x^2 + y^2$, we have

$$Harm(3) + Harm(1) = span\{x^3 - 3xy^2, y^3 - 3x^2y, x^3 + xy^2, x^2y + y^3\}.$$

From this we get

$$Harm(3) + Harm(1) = span\{x^3, x^2y, xy^2, y^3\},$$

which is again what we wanted. From this it should be clear that

$$Hom(3) \oplus Hom(2) = \sum_{i=0}^{3} Harm(i),$$

since it is an orthogonal direct sum, the elements from the decomposition of each will be independent of one another.

The next theorem relates the dimension of the spaces described above to the polynomials $Q_k(x)$, $C_k(x)$, and $R_k(x)$ defined above.

Theorem 4

$$dimHom(k) = C_k(1), \qquad dimHarm(k) = Q_k(1),$$
$$dimHom(k) \oplus Hom(k-1) = R_k(1).$$

All that is needed is to prove that

$$dimHarm(k) = Q_k(x),$$

because the rest follows from the previous theorem and the definitions of those polynomials. Again, I will only give examples. It should be clear that it is true for $k = 0$ and $k = 1$. The rest will be given explicitly,

$$dimHom(2) = 3 \qquad C_2(1) = 3,$$
$$dimHarm(2) = 2 \qquad Q_2(1) = 2,$$
$$dimHom(2) \oplus Hom(1) = 5 \qquad R_2(1) = 5,$$
$$dimHom(3) = 4 \qquad C_3(1) = 4,$$
$$dimHarm(3) = 2 \qquad Q_3(1) = 2,$$
$$dimHom(3) \oplus Hom(2) = 7 \qquad R_3(1) = 7.$$

The following addition formula will also just be stated with no examples. It relates the Gegenbauer polynomial $Q_k(x)$, and any orthogonal basis $W_{k,i} : i = 1, 2, ...Q_k(1)$ of $Harm(k)$, with $normW_{k,i} = \omega_d^{\frac{1}{2}}$.[1]

Theorem 5

$$\sum_{i=1}^{Q_k(1)} W_{k,i}(x)W_{k,i}(y) = Q_k(\langle x, y\rangle)$$

such that $x, y \in \Omega_d$.

Using the above theorem and the following definitions will allow a proof of the next theorem and corollaries.

Definition 6 *Let X be a finite non-empty set of Ω_d of size n. For any orthogonal basis $W_{k,i}$ of $Harm(k)$, with norm$W_{k,i} = \omega_d^{\frac{1}{2}}$, and for any fixed numbering of these, the $n \times Q_k(1)$ matrix*

$$H_k = [W_{k,i}(x)], \qquad x \in X, i \in \{1, 2, ..., Q_k(1)\},$$

is called the kth characteristic matrix of X. Thus, H_0 is the all-one vector of size n.

Definition 7 *For any $X \in \Omega_d$ of size n, and for any $\alpha \in \mathbb{R}, -1 \leq \alpha \leq 1$, the $n \times n$ distance matrix D_α is defined by its elements $D_\alpha(x, y) = 1$ for $\langle x, y \rangle = \alpha$, and $D_\alpha(x, y) = 0$ otherwise, for $x, y \in X$. The sum of the elements of D_α is denoted by d_α.*

Using these definitions and the previous theorem, the following theorem is easy to prove.

Theorem 6 *Let $X \in \Omega_d$, and let A' be a finite set containing all inner products of the vectors of X. Then*

$$H_k H_k^T = \sum_{\alpha \in A'} Q_k(\alpha) D_\alpha,$$

where the $Q_k(x)$ are the Gegenbauer polynomials, H_k the characteristic matrices, and D_α the distance matrices.

Proof: Using the addition formula from above and the definition of the characteristic matrices, we get

$$H_k H_k^T = [Q_k(\langle x, y \rangle)]_{x,y \in X}.$$

Next, apply the definition of the distance matrices. From these two definitions, the theorem is clear. □

Let's consider an example. Let

$$X = \{(1, 0), (0, 1), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}.$$

From this X, we get

$$A' = \{0, 1, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\}.$$

Then

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$D_{\frac{1}{\sqrt{2}}} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$D_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$D_{\frac{-1}{\sqrt{2}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since H_0 is the all one vector of size 4 in this case, $H_0 H_0^T$ is the 4×4 all ones matrix. Also, since $Q_0(x) = 1$ and the sum of the distance matrices shown above is the all ones matrix, that case has been shown to work. Since the paper [1] doesn't define ω_d, we are not sure how to scale the polynomials so they have norm $\omega^{\frac{1}{2}}$. However, we will see what the matrix looks like without scaling. Since the polynomials x and y from $Harm(1) = span\{x, y\}$ and both are orthogonal to each other over the sphere, the rows of H_1 will be the vectors of X. So,

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

From this we have

$$H_1 H_1^T = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 1 \end{pmatrix}.$$

Remembering that $Q_1(x) = d \cdot x$, and $d = 2$ then we have the following

$$Q_1(0) = 0, \qquad Q_1(1) = 2, \qquad Q_1(\frac{1}{\sqrt{2}}) = \sqrt{2}, \qquad Q_1(\frac{-1}{\sqrt{2}}) = -\sqrt{2}.$$

So

$$\sum_{\alpha \in A'} Q_1(\alpha) D_\alpha = \begin{pmatrix} 2 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 2 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 & 0 \\ \sqrt{2} & -\sqrt{2} & 0 & 2 \end{pmatrix}.$$

These two matrices differ by a constant. This is due to the fact that we don't know how to scale the basis vectors of $Harm(1)$. The following corollary is fairly simple.

Corollary 1

$$\| H_k^T H_0 \|^2 = \sum_{\alpha \in A} Q_k(\alpha) d_\alpha.$$

Proof: Since the matrix norm in this case is the square root of the sum of the square of the elements, then $\| H_k^T H_0 \|^2$ is the sum of the elements of $H_k H_k^T$. Also, taking the sum of the right side of the equation form the previous theorem, it is clear that that equals $\sum_{\alpha \in A'} Q_k(\alpha) d_\alpha$. Thus the proof is complete. □

The last corollary is also a clear consequence of the corollary just proved.

Corollary 2 *For any polynomial $F(x)$, with Gegenbauer coefficients $f_0, f_1, \ldots$, the following holds:*

$$f_0 n^2 + \sum_{k=1}^{\infty} f_k \parallel H_k^T H_0 \parallel^2 = \sum_{\alpha \in A'} F(\alpha) d_\alpha.$$

Proof: Use the fact that $F(x) = \sum_{k=0}^{\infty} f_k Q_k(x)$. Then apply the last theorem to get

$$\sum_{k=0}^{\infty} f_k H_k H_k^T = \sum_{\alpha \in A'} F(\alpha) D_\alpha.$$

Then take the sum of the elements of the matrices on both sides as with the previous corollary.

2.2 Bounds

From all of this work we have the following theorem.

Theorem 7 (Delsarte, Goethals and Seidel [1]) *Let $F(x)$, with Gegenbauer coefficients $f_0 > 0$ and $f_k \geq 0$, for all k, be compatible with the set A. Then the cardinality of any A-code X satisfies*

$$|X| \leq F(1)/f_0.$$

Equality holds if and only if, for all $x \neq y$ in X, and for all $k \geq 1$,

$$F(\langle x, y \rangle) = 0, \qquad f_k H_k^T H_0 = 0.$$

Proof: Since $d_1 = n$, if we divide both sides of the equation from the second corollary by $n f_0$, we have

$$n + \frac{1}{n f_0} \sum_{k=1}^{\infty} f_k \parallel H_k H_k^T \parallel^2 = \frac{1}{f_0 n} \sum_{\alpha \in A'} F(\alpha).$$

Since $F(\alpha) \leq 0$ for all $\alpha \in A$, and $A' = A \cup \{1\}$, then

$$n = \frac{F(1)}{f_0} + \frac{1}{n f_0} \left[\sum_{\alpha \in A} F(\alpha) d_\alpha - \sum_{k=1}^{\infty} f_k \parallel H_k^T H_0 \parallel^2 \right] \leq \frac{F(1)}{f_0}.$$

This implies that

$$n \leq \frac{F(1)}{f_0}.$$

From this it is clear that we have equality if and only if

$$F(\langle x, y \rangle) = 0, \quad \text{for all } x \neq y \text{ in } X, \text{ and} \quad f_k H_k^T H_0 = 0 \quad \text{for all } k.$$

Thus the theorem in proved.

Let us consider some examples. Let $\beta \in [-1, 0]$, let A be any subset of the interval $[-1, \beta]$. The polynomial $F(x) = x - \beta$ is compatible with A, and that $f_0 = -\beta > 0$, $f_1 = \frac{1}{d} > 0$ so we can apply the theorem. From this we see that

$$n \leq 1 - \frac{1}{\beta}.$$

It is then interesting to note that the bound is achieved if the A-code is an r-dimensional simplex with $\beta = \frac{-1}{r}$.

For the next example let

$$-1 \leq \alpha \leq \beta < 1, \qquad \alpha + \beta \leq 0, \qquad \alpha\beta > \frac{-1}{d},$$

and let A be any subset of $[\alpha, \beta]$. The polynomial $F(x) = (x - \alpha)(x - \beta)$ has the following Gegenbauer coefficients

$$f_0 = \alpha\beta + \frac{1}{d}, \qquad f_1 = \frac{-(\alpha + \beta)}{d}, \qquad f_2 = \frac{2}{d(d+2)},$$

which are nonnegative so the theorem applies and gives us

$$n \leq \frac{d(1-\alpha)(1-\beta)}{(1 + d\alpha\beta)}$$

for any such A code X. If $\alpha = -\epsilon$ and $\beta = \epsilon$ then we get bounds for the cardinality of sets of almost orthogonal vectors. More on this will be discussed later.

For the next example let $0 \leq \beta < d^{\frac{-1}{2}}$, Let A be any subset of $[-1, \beta]$. Then define

$$\alpha = \frac{-(1+\beta)}{(1 + d\beta)},$$

so $-1 \leq \alpha < 0$. The polynomial

$$F(x) = (x - \alpha)^2(x - \beta)$$

is compatible with A with nonnegative Gegenbauer coefficients so the theorem applies and we obtain

$$n \leq \frac{d(1-\beta)(2 + (d+1)\beta)}{(1 - d\beta^2)}$$

for any A-code X.

For the last example let $-1 \leq \alpha \leq \beta \leq \gamma < 1$, and let $A \subset [-1, \alpha] \cup [\beta, \gamma]$. The polynomial $F(x) = (x-\alpha)(x-\beta)(x-\gamma)$ is compatible with A. It has nonnegative Gegenbauer coefficients, with $f_0 > 0$, if

$$\alpha + \beta + \gamma \leq 0, \qquad \alpha\beta + \beta\gamma + \gamma\alpha \geq \frac{-3}{(d+2)},$$
$$\alpha + \beta + \gamma < -d\alpha\beta\gamma.$$

The theorem gives us

$$n \leq \frac{-d(1-\alpha)(1-\beta)(1-\gamma)}{(\alpha+\beta+\gamma+d\alpha\beta\gamma)}.$$

It should be noted that these bounds hold whenever X is an $\{\alpha, \beta\}$-code or $\{\alpha, \beta, \gamma\}$-code.

Before we look at some bounds in particular for the cardinality of sets of almost orthogonal vectors, we shall look at one more theorem that produces bounds based only on the cardinality of A and not on the elements themselves.

Theorem 8 *For a given $s = |A| < \infty$, the cardinality n of any A-code X satisfies $n \leq R_s(1)$.*

Proof: For A define the annihilator polynomial

$$F(x) = \prod_{\alpha \in A} \frac{(x-\alpha)}{(1-\alpha)}.$$

For any $y \in X$ define the function $F_y : \Omega_d \to \mathbb{R}$ by

$$F_y(x) = F(\langle x, y \rangle), \qquad x \in \Omega_d.$$

Thus F_y belongs to the linear space $Hom(s) \oplus Hom(s-1)$, which has dimension $R_s(1)$. By definition we get

$$F_y(x) = \delta_{x,y}, \text{for all} x \in X,$$

so that the functions F_y are linearly independent. Therefore $n = |X|$ cannot exceed the dimension of the linear space. This proves the theorem.

There are a two examples that will be examined. The first has $s = 1$, which clearly indicates that $n \leq d+1$, with equality if and only if X is a regular d-simplex, as with the first example of the previous theorem.

The next example is with $s = 2$ which results in $n \leq \frac{d}{2}(d+3)$. Interestingly there are examples of sets of equiangular lines with cardinalities

$$6 \text{ in } \mathbb{R}^3, \qquad 28 \text{ in } \mathbb{R}^7, \qquad 276 \text{ in } \mathbb{R}^{23};$$

no other examples are known where equality holds in the bound.

2.2.1 The degree 2 polynomial

The second example of the main theorem gives us the second degree bound. The same bound can be constructed by considering nonnegative linear combination of $Q_0(x)$ and $Q_2(x)$. This construction will be illustrated and this method will be used to construct other bounds. Without loss of generality we take $f_0 = 1$; and write

$$F(x) = 1 + f_2(d+2)(dx^2-1)/2.$$

There are no $Q_1(x)$ because we want a second degree polynomial to be symmetric about the origin with zeros at $-\epsilon$ and ϵ.

In order to find f_2, all that is needed is to solve,

$$F(\epsilon) = 0,$$

for f_2. From this polynomial we get

$$|X| \leq d(1-\epsilon^2)/(1-d\epsilon^2).$$

This bound was established by Delsarte, Goethals, and Seidel, but it is only good for $\epsilon < \sqrt{1/d}$. We can do better than this however with the degree four polynomial.

A sample table of the dimension and the bound for $\epsilon = 2/d$ is provided below.

d	5	6	7	8	9	10	11	12	13	14	15	16
$f(d, \frac{2}{d})$	21	16	15	15	15.4	16	16.7	17.5	18.3	19.2	20.1	21

2.2.2 The degree 4 polynomial

The degree four polynomial improves the bounds for relatively large epsilon. The polynomial is then

$$F(x) = 1 + f_2(d+2)(dx^2-1) + f_4 d(d+6)((d+2)(d+4)x^4 - 6(d+2)x^2 + 3).$$

In order for F to be non-positive in a symmetric interval we just need $F(0) \leq 0$, but for simplicity we choose $F(0) = 0$. By solving for f_2 we get the best f_2, so $f_2 = \frac{(8+d(d+6)f_4)}{(4(d+2))}$. Then, again solving $F(\epsilon) = 0$ for f_4, we get,

$$f_4 = \frac{24}{3d^2 + 30d - d^3\epsilon^2 - 12d^2\epsilon^2 - 44d\epsilon^2 - 48\epsilon^2 + 72}.$$

With the Gegenbauer coefficients defined this way, and since we have $f_0 = 1$, the cardinality of $X \leq F(1)$ and

$$F(1) = \frac{d(d+2)(1-\epsilon^2))}{3-(d+2)\epsilon^2}.$$

This bound is good for $\epsilon \leq \sqrt{(3/(d+2))}$.

The new bound also becomes relevant for $\epsilon = \sqrt{\frac{1}{d}}$ and the bound for that ϵ, $|X| \leq \frac{d(d+2)}{2}$

d	5	6	7	8	9	10	11	12	13	14	15	16
$f(d, \epsilon)$	17.5	24	31.5	40	49.5	60	71.5	84	97.5	112	127.5	144

2.2.3 Degree 6 and Higher

For higher degree polynomials such as the degree six polynomial slightly different methods are needed. Since the degree of each term is even, replacing x with $\sqrt{x}$, the problem becomes much simpler. So, for the sixth degree polynomial, first define

$$F(x) = 1 + f_2 Q_2(x) + f_4 Q_4(x) + f_6 Q_6(x).$$

Then making the substitution for $\sqrt{x}$, define $G(x) = F(\sqrt{x})$. The first steps will be similar. First, solve $G(0) = 0$ for f_2, $G(\epsilon^2) = 0$ for f_4 and $G(-\epsilon^2) = 0$ for f_6. The order this is done does not matter, but by doing this we construct a polynomial so that when we replace x with x^2 we will have our polynomial $F(x)$ which will be compatible with our desired A. This will produce the Gegenbauer expansion of $F(x) = x^2(x^2 - \epsilon^2)(x^2 + \epsilon^2)$, which will be compatible with $A = (-\epsilon, \epsilon)$. Because the roots are complex and the net effect is to have a positive factor, it does not change the sign anywhere from the fourth degree polynomial.

The resulting polynomial is

$$F(x) = \frac{dx^2(\epsilon^4 d^2 - x^4 d^2 + 6\epsilon^4 d - 6x^4 d - 8\epsilon^4 + 8x^4)}{\epsilon^4 d^2 + 6\epsilon^4 d - 15 + 8\epsilon^4}.$$

This means that

$$|X| \leq \frac{d(\epsilon^4 d^2 - d^2 + 6\epsilon^4 d - 6d - 8\epsilon^4 + 8)}{\epsilon^4 d^2 + 6\epsilon^4 d - 15 + 8\epsilon^4}.$$

In particular, if $\epsilon = \sqrt{\frac{3}{d+2}}$ then

$$|X| \leq \frac{d(d+4)(d+5)}{6}.$$

This can be illustrated in Figure 2.1, which shows the degree six polynomial with $d = 50$ and $\epsilon = .25$.

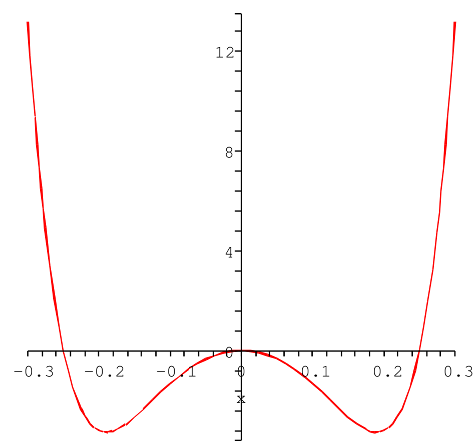

Figure 2.1: This plot shows $F(x) \leq 0$ for $x \in [-\epsilon, \epsilon]$ for the six degree polynomial with $d = 50$ and $\epsilon = .25$

This bound holds for $\epsilon < \sqrt[4]{\frac{15}{(d+2)(d+4)}}$. The following is a table of sample values for the bounds, for $\epsilon = \sqrt{\frac{3}{d+2}}$.

d	5	6	7	8	9	10	11	12	13	14	15	16
$f(d,\epsilon)$	75	110	154	208	273	350	440	544	663	798	950	1120

To obtain higher degree polynomials a similar process can be followed. Let

$$F(x) = x^2(x^2 - \epsilon^2) \prod_{i=2}^{\lfloor \frac{n}{2} \rfloor - 2} (x^2 + \epsilon^i).$$

Then find the Gegenbauer expansion to find upper bounds for larger ϵ.

However, we know from both of the theorems that we can only come close to attaining the bounds that are interesting for the degree six polynomial. The degree six bound at the epsilon where the degree four polynomial blows up to infinity is

$$n \leq \frac{d(d+5)(d+4)}{6}.$$

2.3 Jacobi Polynomials

This exploration of Jacobi Polynomials was motivated by trying to simplify the problem of finding the Gegenbauer coefficients. Though the Jacobi polynomials are not used directly, it was motivation to develop the techniques used for the higher degree polynomials. Below we will see the relationship between the Jacobi polynomials and the Gegenbauer polynomials.

These are not the usual Jacobi polynomials, but those used in [4], scaled similarly to the Gegenbauer polynomials we use above.

In the case of $\mathbb{R}^n$, with $n \geq 2$, and for $\epsilon \in \{0, 1\}$ we will define the Jacobi polynomials $J_{k,\epsilon}(x)$. We take $0/0 = 0$ and $0^0 = 1$.

Definition 8 *For $\epsilon \in \{0,1\}$ and integer $k \geq 0$, the polynomials $J_{k,\epsilon}(x)$ are defined by the recurrence relations*

$$\nu_{k+1} J_{k+1,0}(x) = x J_{k,1}(x) - (1 - \nu_k) J_{k,0}(x),$$
$$\mu_{k+1} J_{k+1,1}(x) = J_{k+1,0}(x) - (1 - \mu_k) J_{k,1}(x),$$

with the initial values $J_{-1,\epsilon}(x) = 0, J_{0,0}(x) = 1$. In the case of $\mathbb{R}^d$ the coefficients are

$$\nu_k = \frac{2k}{d + 4k - 2}, \qquad \mu_k = \frac{2k+1}{d+4k}.$$

2.3.1 Jacobi and Gegenbauer polynomials

There is a relationship between the Jacobi and Gegenbauer polynomials that allows simplification in our search for polynomials compatible with $[-\epsilon, \epsilon]$. In this section we will refer to the Gegenbauer polynomials by $Q_k(x)$ and their coefficients by λ_k as we have throughout.

Proposition 1 $x^\epsilon J_{n,\epsilon}(x^2) = Q_{2n+\epsilon}(x)$

Proof:

By induction we see

$$x^0 J_{0,0}(x^2) = J_{0,0}(x^2) = 1, \quad x^1 J_{0,1}(x^2) = xJ_{0,1}(x^2) = d,$$

which shows that

$$J_{0,0}(x^2) = Q_0(x)$$

and

$$xJ_{0,1}(x^2) = Q_1(x).$$

We then assume $x^\epsilon J_{k,\epsilon}(x^2) = Q_{2k+1}(x)$.

$$\nu_{k+1} J_{k+1,0}(x^2) = x^2 J_{k,1}(x^2) - (1 - \nu_k) J_{k,0}(x^2)$$

By our induction hypothesis,

$$x^2 J_{k+1,1}(x^2) - (1 - \nu_k) J_{k,0} = x(Q_{2k+1}(x)) - (1 - \nu_k) Q_{2k}(x).$$

Looking back at the definition of the Gegenbauer polynomials and the constants, it can be seen that $\nu_k = \lambda_{2k}$. Making the substitutions above we see we that

$$xQ_{2k+1}(x) - (1 - \lambda_{2k}) Q_{2k}(x) = \lambda_{2k+2} Q_{2k+2}(x),$$

so

$$J_{k+1,0}(x^2) = Q_{2k+2}(x),$$

which is the first half of what we wanted. For the second half,

$$\mu_{k+1} J_{k+1,1}(x^2) = J_{k+1,0}(x^2) - (1 - \mu_k) J_{k,1}(x^2),$$

which implies again from the induction hypothesis that

$$J_{k+1,0}(x^2) - (1 - \mu_k) J_{k,1}(x^2) = Q_{2k+2}(x) - (1 - \lambda_{2k+1}) \frac{Q_{2k+1}(x)}{x}.$$

And, multiplying both sides of the equation by x and applying the definition of the Gegenbauer polynomials we see

$$x\mu_{k+1} J_{k+1,1}(x^2) = \lambda_{2k+3} J_{2k+3}(x).$$

and since $\mu_{2k+1} = \lambda_{2k+3}$ as can be seen from the definitions,

$$xJ_{k+1,1}(x^2) = Q_{2k+3}(x).$$

and thus the proof is complete.

Chapter 3

Propagation Techniques

In order to construct a table of lower bounds we apply propagation techniques to the examples we have. These techniques include direct sum of spaces, tensor product of spaces, projection down one dimension, and we also make sure the table is monotone in ϵ and d.

3.1 Direct Sum of Spaces

Suppose we have two sets of ϵ-almost orthogonal vectors, X_1 and X_2 such that X_1 is in $\mathbb{R}^{d_1}$ and X_2 is in $\mathbb{R}^{d_2}$. Consider the direct sum of the two spaces. The first d_1 components for the vectors from X_1 are all zeros and the last d_2 components of the vectors in X_2 are zeros and the rest of the components for both are the same as the original vectors. It is clear that the vectors originally from X_1 are orthogonal to the vectors originally in X_2.

Let $L : \mathbb{N} \times [0, 1] \mapsto \mathbb{N}$ denote our lower bound function; i.e., $L(d, \epsilon)$ is the cardinality of the largest set X of ϵ-almost orthogonal vectors in $\mathbb{R}^d$ known to us. With this direct sum of spaces we have$L(d_1 + d_2, \epsilon) \geq L(d_1, \epsilon) + L(d_2, \epsilon)$.

3.2 Tensor Products of Spaces

Consider again, two sets of ϵ-almost orthogonal vectors, X_1, X_2 in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. Now consider the Gram matrix representation for each as A and B respectively. If $n_1 = |X_1|$ and $n_2 = |X_2|$, then the rank of A and B are d_1 and d_2 respectively and A is $n_1 \times n_1$ and B is $n_2 \times n_2$. As a reminder of how the tensor product works, in this case, if we take $A \otimes B$, then this is a block matrix

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & ... & a_{1,n_1}B \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n_1,1}B & a_{n_1,2}B & ... & a_{n_1,n_1}B \end{pmatrix}.$$

If both A and B are Gram matrices, then from this definition it should be clear that there will be $n_1 n_2$ vectors and $rank(A \otimes B) = rank(A)rank(B)$. Also note that we will again have ones on the diagonal and the off-diagonal entries will be close to zero. If we define $\epsilon_1 = \max|a_{ij}|$ and $\epsilon_2 = \max|b_{ij}|$ for $i \neq j$, then after the tensor product we will have $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. From this it can be shown that

$$L(d_1 d_2, \epsilon) \geq L(d_1, \epsilon) L(d_2, \epsilon).$$

3.3 Projecting Down

This propagation technique was motivated by asking what happens if we remove one vector in d-dimensional space, but also move down to $(d-1)$-dimensional space. We can do this by supposing that one of the vectors is at the north pole of Ω_{d-1}, then by deleting that vector. Then the rest of the vectors must be close to the equator, so we project those vectors onto the equator. Now we have a set of vectors that are almost orthogonal on Ω_{d-2}. The case of moving from Ω_2 to Ω_1 is illustrated below.

An explanation of this construction follows. We can assume one vector in the set of almost orthogonal vectors X is of the form $x = [1, 0, ...0]$, since we can rotate the set such that one is of this form. Since all of the vectors are almost orthogonal, let $y, v \in X$ be any other vectors. Then the inner product

$$-\epsilon \leq \langle x, y \rangle \leq \epsilon, \quad -\epsilon \leq \langle x, v \rangle \leq \epsilon,$$

which implies that $-\epsilon \leq y_1 \leq \epsilon$ and $-\epsilon \leq v_1 \leq \epsilon$. Therefore,

$$-(\epsilon^2 + \epsilon) \leq \langle (0, u_2, u_3, ..., u_n), (0, v_2, v_3, ..., v_n) \rangle \leq (\epsilon^2 + \epsilon).$$

And therefore if $L(d, \epsilon)$ is our function of the dimension d and tolerance ϵ, then

$$L(d-1, \epsilon^2 + \epsilon) \geq L(d, \epsilon) - 1.$$

This is illustrated in the following figure.

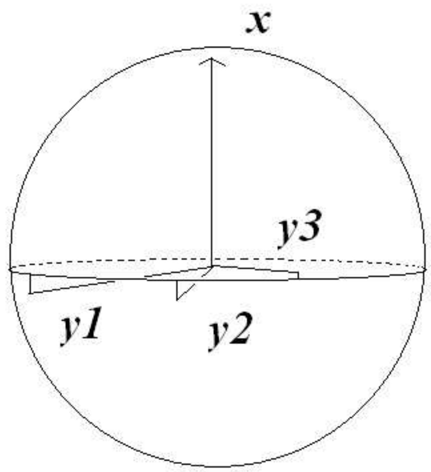

Figure 3.1: Projecting the vectors y_1, y_2, y_3 onto the equator and deleting x gives us almost orthogonal vectors with tolerance $\epsilon^2 + \epsilon$

The next question to ask is, can we iterate this process some cd times for some $c, 0 < c < 1$? This is because we would like to know if this projection technique can be repeated to smooth out the table of lower bounds. If it was possible to smooth out the table without making the ϵ too large, then suppose we had exponential growth. Then we can move down one dimension and delete only one vector and we would only need a little more tolerance. This however would contradict the exponential growth because we will have only subtracted a polynomial number of vectors without increasing ϵ considerably. So we hope that this problem will not arise.

To test this process, it is sufficient to show that $g(x) = x^2 + x$ composed with itself evaluated at $\frac{1}{d^t}$ any multiple of d times is less than one. This fails for any $c, 0 < c < 1$ if $t < 1$. We can show this by looking at the inverse compositions and thus defining the sequence,

$$a_0 = 1, \qquad a_{n+1} = \frac{-1 + \sqrt{1 + 4a_n}}{2}.$$

A key fact will be that for $n > 0$, $\frac{1}{a_n} > n - \ln(n)$. This can be proved by induction. It is easy to see that $\frac{1}{a_1} > 1 - \ln(1) = 1$. Assume up to k, $\frac{1}{a_k} > k - \ln(k)$. Then $\frac{1}{k - \ln(k)} > a_k$ which implies that $\frac{-1+\sqrt{1+\frac{4}{k-\ln(k)}}}{2} > a_{k+1}$. Now all that is needed is to verify $k+1-\ln(k+1) \geq \frac{-1+\sqrt{1+\frac{4}{k-\ln(k)}}}{2}$. Assuming it is, then

$$2k + 3 - 2\ln(k+1) \geq \sqrt{1 + \frac{4}{k - \ln(k)}},$$

squaring both sides results in

$$4k^2 + 12k - 4k\ln(k+1) + 9 - 6\ln(k+1) + \ln^2(k+1) \geq 1 + \frac{4}{k - \ln(k)}.$$

Subtracting 1 from both sides and dividing by 4 we get

$$k^2 + 3k - k\ln(k+1) + 2 - \frac{3\ln(k+1)}{2} + \frac{\ln^2(k+1)}{4} \geq \frac{1}{k - \ln(k)},$$

since $k - \ln(k) > 0$ for all $k > 1$, if we multiply both sides of the inequality by $k - \ln(k)$, the sign stays the same; $k^2 + 3k - k\ln(k+1) - \frac{3(\ln(k+1))}{2} > 0$. This is easily shown since it is true for $k = 1$ even though this is not a continuous function. Take the derivative of the continuous counterpart, $x^2 + 3x - x\ln(x+1) - \frac{3\ \ln(x+1)}{2}$, is, $2x + 3 - \ln(x+1) - \frac{x}{x+1} - \frac{3}{2(x+1)}$. Also note that $2x > \ln(x+1)$, since the same derivative test applies at $x = 1$. Then the inequality still holds when taking the derivative of both sides: $2 > \frac{1}{x+1}$. For the second portion we first simplify to get $3 > \frac{2x+3}{2(x+1)}$, then at $x = 1$ the inequality holds. Also note that if you take the limit on the right side it approaches one and it does so monotonically.

From this we suppose that we have an $\epsilon = \frac{1}{d^t}$ that is greater than a_k. Since we have this good upper bound it is enough to say that $\frac{1}{k-\ln(k)} > \frac{1}{d^t}$. This means that we can project at most k times and have $d^t > k - \ln(k)$ for some k and keeping $\epsilon < 1$. Also note that for sufficiently large d and for $0 < c < 1$, $cd > d^t$. So this process can only be repeated a limited number of times preserving exponential growth.

Chapter 4

Sources of Lower bounds

The source of the lower bounds comes from a few of theorems, distance regular graphs, and examples from an already existing table. The sources of the examples will be explained in the following sections.

4.1 Mutually Unbiased Bases in Complex Space

Mutually unbiased bases in complex space appear in quantum mechanics. It was determined that information could not be retrieved in one basis from a measurement of a state prepared in another basis. The definition and conditions for finding extremal sets of mutually unbiased bases were described in [5].

Definition 9 *Two orthonormal bases B and B' of the vector space $\mathbb{C}^d$ are called mutually unbiased if and only if $|\langle b, b' \rangle|^2 = \frac{1}{d}$ for all $b \in B$ and $b' \in B'$. In this case, $\langle , \rangle$ is the Hermitian inner product.*

It is known that the cardinality of any collection of mutually unbiased bases in $\mathbb{C}^d$ is at most $d+1$. It is known that if d is a prime power, then extremal sets containing $d+1$ mutually unbiased bases are known to exist.

From the theorems on the linear programming bound, we can see that when we can attain extremal sets of mutually unbiased bases we attain the bound from our degree four polynomial. From the definition of mutually unbiased bases we know that in the set of vectors we have

$$A = \{\frac{-1}{\sqrt{d}}, 0, \frac{1}{\sqrt{d}}\}.$$

We know that this configuration is optimal and we see that for each α in A, $F(\alpha) = 0$, where $F(x)$ is the fourth degree polynomial described above, so this is consistent with our theorems.

4.2 Table of Examples

Neil Sloane from AT&T constructed a table of examples of n-dimensional Grassmanian packings of m-dimensional spaces with the distance between these subspaces listed. We adapted this table to our situation by using only the one dimensional subspaces and converted the distance into our ϵ. Sloane's metric was equivalent to $\sin^2(\theta)$, so to convert I used the fact that we can get the cosine by computing $\sqrt{1-\sin^2(\theta)}$. This table can be found at `http://www.research.att.com/~njas/grass/grassTab.html`.

4.3 Distance Regular Graphs

Distance regular graphs are associated with many geometrical structures that can be helpful in finding almost orthogonal sets of vectors.

Definition 10 *A graph G is called distance regular if there exist integers b_i, c_i $(i \geq 0)$ such that for any two vertices, $x, y \in G$, at distance $i = d(x, y)$, there are c_i neighbors of y in $G_{i-1}(x)$ and b_i neighbors of y in $G_{i+1}(x)$.*

The array

$$i(G) = \{b_0, b_1, ..., b_{d-1}, c_1, c_2, ..., c_d\}$$

where $m = diam(G)$ is called the intersection array of G. From the intersection array we can construct matrices P and Q which are called the eigenmatrices of G. From this we construct a set of almost orthogonal vectors. Code was written to take in an intersection array, find the matrices and report the dimension of a space, an ϵ, and a number of vectors.

This works by using the spectral decomposition of the adjacency matrix. The adjacency matrix $A = [a_{ij}]$ is defined as $a_{ij} = 1$ if vertices i and j are adjacent and $a_{ij} = 0$ otherwise. Since this is a real symmetric matrix, by the spectral theorem it has real eigenvalues. If for distinct eigenvalues, $\theta_0, \theta_1, ..., \theta_k$, we construct matrices U_i which are $n \times m$ where m is the multiplicity of eigenvalue θ_i, with the columns being the eigenvectors associated with that eigenvalue. Now let $E = UU^T$. E forms an orthogonal projection matrix and projects onto the column space of U. To prove this, let $W = colsp(U)$. Then let $w \in W$. This means that for some x, $w = Ux$. Next, observe that $Ew = UU^TUx$. Since each U is orthogonal, $U^TU = I$, so $Ew = Ux = w$, so E fixes w. Now suppose $z \perp W$. Then $U^Tz = 0$, so $Ez = UU^Tz = 0$. This proves that E is an orthogonal projection matrix onto the column space of U.

Also note that A is diagonalizable. We can assume that the diagonalized matrix D, with the eigenvalues of A on the diagonal and zeros off the diagonal, the eigenvalues are grouped together so that identical eigenvalues are adjacent and the P matrix equal to the columns being each U_i. This is equivalent to a matrix $P' = [\theta_0 U_0 ... \theta_k U_k]$ so

$$A = P'P^T.$$

And since we know that for any two matrices $A_{m\times n}$ and $B_{n\times p}$, $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ we can say that

$$A = \sum_{i=0}^{k} \theta_i U_i U_i^T.$$

Because of the way A is diagonalized and from the definitions we have, the following two facts follow,

$$I = \sum_{i=0}^{k} E_i, \qquad A = \sum_{i=0}^{k} \theta_i E_i.$$

Now if we define

$$\langle A \rangle = \{f(A) : \text{f a polynomial}\},$$

where f is a polynomial, then $\{E_0, ...E_k\}$ forms a basis of $\langle A \rangle$. The sum of all of the E_i is the identity and any power of A is just a power of the sum above. These matrices thus form a basis of $\langle A \rangle$ since it was shown that there are linear combinations that produce I and A, so by taking powers and multiplying by the E matrices, you can get every element of $\langle A \rangle$.

The following will be stated without proof and the reader can look for more at [3]. Now in the case of distance regular graphs, the ith-distance matrix A_i with entry (a, b) equal to one if $d(a, b) = i$ and zero otherwise. This A_i can be expressed as a polynomial of A, so it belongs to $< A >$. If we call that polynomial $p_i(x)$, then $p_i(A) = A_i$. Now for a graph of diameter k, we have a family of orthogonal polynomials $p_0(x), p_1(x), ..., p_k(x)$. Now the (a, b) entry of E_j is equal to $\frac{p_i(\theta_j)m_j}{n_j}$ where $m_j = rank(E_j)$ and $n_i = rowsum(A_i)$. The n_i is also the number of vertices that are at a distance i from a given vertex. We now know that E_j has $k + 1$ distinct entries and so we search for graphs G and eigenspaces j such that the off-diagonal entries of E_j are close to zero.

The code we have accomplishes this by creating a tridiagonal matrix $B_{m+1 \times m+1}$ with the b_i's on the upper diagonal, the c_i's on the lower diagonal and for $i > 1$, $a_i = B_{ii} = B_{1,2} - B_{i,i+1} - B_{i,i-1}$ and $B_{1,1} = 0$. From this matrix we get the eigenvalues and construct a new matrix $P_{m+1 \times m+1}$ with each entry in the zeroth column is a one and the first column contains the eigenvalues in descending order going down the column. Then construct the following polynomials,

$$p_{i+1}(x) = \frac{1}{c_{i+1}} \left[(x - a_i)p_i(x) - b_{i-1}p_{i-1}(x) \right]$$

with,

$$p_0(x) = 1 \qquad p_1(x) = x.$$

Then, $P_{j,i} = p_i(P_{j,1}) = p_i(\theta_j)$. Then let $N = \sum_{i=1}^{m+1} P_{1,i}$ and let $Q = NP^{-1}$. From this we get m different positive semidefinite matrices. From each we can find the dimension of the space of a set of vectors from $Q_{1,j+1}$, the cardinality of the set of vectors, N, and the tolerance which is $\epsilon = \max_{i=2..m+1} \frac{Q_{i,j+1}}{Q_{1,j+1}}$.

4.3.1 Johnson graphs

Johnson graphs are a particular distance regular graph denoted by $J(v, k)$. The vertex set is $\binom{v}{k}$, the collection of k-subsets of v. Two vertices x, y are adjacent if $|x \cap y| = 1$. In this case, the number of vectors $N = \binom{v}{k}$, the dimension $d = \binom{v}{j} - \binom{v}{j-1}$ where j goes from one to k. For $j = k$, we have $\epsilon = \frac{1}{\binom{v-k}{k+1}}$.

The parameters for the Johnson graph are

$$p_i(\theta_j) = \sum_{\ell=0}^{i} (-1)^\ell \binom{j}{\ell} \binom{k-j}{i-\ell} \binom{v-k-j}{i-\ell},$$

$$m_j = \binom{v}{j} - \binom{v}{j-1}, \qquad n_i = \binom{k}{i} \binom{v-k}{i}.$$

The most interesting examples seem to be $j = k$ and $j = k-1$, which give low-rank matrices close to I.

From this, a number of examples were constructed from examples in the book "Distance-Regular Graphs" by Brouwer, Cohen, and Neumaier, as well as the Johnson graph.

4.4 A Table of Lower Bounds

Once these examples were assembled, a table of lower bounds was constructed. This table is 100×251 where the rows are the dimension of a space and the columns are tolerances with $0 \leq \epsilon \leq .250$ with each column an increment of .001. For $\epsilon < \frac{1}{d}$, the table entries are initialized to d, and the rest are then initialized to $d+1$. Then, from our examples, we replace any of the entries with an example if it is greater than what is already there. Also, for $d = 2p^t$ where p is prime, at the column entry corresponding to $\epsilon = \sqrt{\frac{1}{d}}$, the entry in the table is replaced by $\frac{d(d+2)}{2}$. This is because of the mutually unbiased bases described above. We also entered in any of the examples from Sloane's table that were better than the ones already in it. Once this is complete, the propagation techniques are applied. First the projection technique is applied, then in a combined check of the direct sum and tensor product, as well as insuring that the table is monotone in d and ϵ. Both operations are only needed once each. This is because for any examples in which the sum of spaces is useful, if the projection of the component spaces is in still in the table and produces new examples, then the sum of those spaces will be in the table. Thus, once projecting and finding the sum once, all possible improvements are found.

Chapter 5

Results from the Table

The table constructed of lower bounds contains some interesting components. Below is a table of data where the the top row is the dimension of a space and the bottom row is ϵ. The point at which the table goes to $d+2$ is the first point at which the table becomes interesting.

d	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
ϵ	.240	.200	.183	.163	.143	.135	.124	.112	.107	.099	.091	.091	.091	.091	.091

d	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
ϵ	.091	.090	.090	.083	.083	.077	.077	.072	.072	.067	.067	.063	.063	.059	.059

d	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
ϵ	.056	.056	.053	.053	.050	.050	.048	.048	.046	.046	.044	.044	.042	.042	.040

d	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65
ϵ	.040	.039	.039	.038	.038	.036	.036	.035	.035	.034	.034	.033	.033	.032	.032

d	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
ϵ	.031	.031	.030	.030	.029	.029	.028	.028	.028	.028	.027	.027	.026	.026	.025

d	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95
ϵ	.025	.025	.025	.024	.024	.024	.024	.023	.023	.023	.023	.022	.022	.022	.022

d	96	97	98	99	100
ϵ	.021	.021	.021	.021	.020

The following table shows two interesting features in the complete table. The first is examples of quadratic growth in the number of vectors, such as in $\mathbb{R}^{64}$ and $\epsilon = .125$ and in $\mathbb{R}^{62}$ and $\epsilon = .128$. There are large jumps and the table creates a sort of staircase pattern. It is also interesting to observe in $\mathbb{R}^{63}$ and $\epsilon = .141$. At that point, the projection technique shows up. The 2111 vectors come from the 2112 vectors in $\mathbb{R}^{64}$ projected down one dimension.

ϵ	.125	.126	.127	.128	.129	.130	.131	.132	.133	.134	.135	.136	.137	.138
45	90	90	90	90	90	90	90	90	90	90	90	90	90	90
46	91	91	91	91	91	91	91	91	91	91	91	91	91	91
47	92	96	96	96	96	96	96	96	96	96	96	96	96	96
48	97	97	97	97	97	97	97	97	97	97	97	97	97	97
49	100	100	100	100	100	100	100	100	100	100	100	100	100	100
50	101	101	101	101	101	101	101	101	101	101	101	101	101	101
51	102	102	102	102	102	102	102	102	102	102	102	102	102	102
52	103	103	103	103	103	103	103	107	107	107	107	107	107	107
53	108	108	108	108	108	108	108	108	108	108	108	108	108	108
54	109	109	109	109	109	109	111	111	111	111	111	111	1512	1512
55	112	112	112	112	112	112	112	112	112	112	112	112	1513	1513
56	113	113	113	113	113	113	113	113	113	113	113	113	1514	1514
57	114	114	114	114	114	114	114	114	114	114	114	114	1515	1515
58	119	119	119	119	119	119	119	1740	1740	1740	1740	1740	1740	1740
59	120	120	120	120	120	120	120	1741	1741	1741	1741	1741	1741	1741
60	123	123	123	123	123	123	123	1742	1742	1742	1742	1742	1742	1742
61	124	124	124	124	124	124	124	1743	1743	1743	1743	1743	1743	1743
62	125	125	125	1984	1984	1984	1984	1984	1984	1984	1984	1984	1984	1984
63	126	126	126	1985	1985	1985	1985	1985	1985	1985	1985	1985	1985	1985
64	2112	2112	2112	2112	2112	2112	2112	2112	2112	2112	2112	2112	2112	2112
65	2113	2113	2113	2113	2113	2113	2113	2113	2113	2113	2113	2113	2113	2113
66	2114	2114	2114	2114	2114	2114	2114	2114	2114	2114	2114	2114	2114	2114

ϵ	.139	.140	.141
45	90	90	90
46	91	91	91
47	96	96	96
48	99	99	99
49	100	100	100
50	101	101	101
51	102	102	102
52	107	107	107
53	108	108	108
54	1512	1512	1512
55	1513	1513	1513
56	1514	1514	1514
57	1515	1515	1515
58	1740	1740	1740
59	1741	1741	1741
60	1742	1742	1742
61	1743	1743	1743
62	1984	1984	1984
63	1985	1985	2111
64	2112	2112	2112
65	2113	2113	2113
66	2114	2114	2114

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