Workshop at a Glance
 Systems of Lines: Applications of Algebraic Combinatorics

Monday 8/10

9:00am-9:55am
ORGANIZERS, due to an unexpected cancellation
To be announced

10:30am-10:55am
PADRAIG O’CATHÁIN, Aalto University
Almost equiangular lines

11:00am-11:25am
MARKUS GRASSL, Universtität Erlangen-Nürnberg & Max Planck Institute for the Science of Light
Unextendible sets of MUBs

11:30am-11:55am
WEI-HSUAN YU, Michigan State University
Equiangular lines and spherical designs

2:00pm-2:55pm
JUSTIN ROMBERG, Georgia Tech
Solving Underdetermined Linear Systems and Overdetermined Quadratic Systems of Equations

3:30pm-3:55pm
RICHARD M. WILSON, Caltech
Combinatorial consequences and extensions of a simple inequality for matrices

4:00pm-4:25pm
YURY POLYANSKIY, MIT
On metric properties of maps between Hamming spaces and related graph homomorphisms
Tuesday 8/10

9:00am-9:55am
Steven Flammia, University of Sydney
Designs in quantum information science

10:30am-10:55am
Randy Paffenroth, WPI
Musings on sparsity detection, point-wise error bounds, and truncated norms

11:00am-11:25am
Brendan Rooney, University of Waterloo
Vector Colourings of Distance-Regular Graphs

11:30am-11:55am
Koji Momihara, Kumamoto University
Distance sets on circles

2:00pm-2:55pm
Shayne Waldron, University of Auckland
Calculating the symmetries of a tight frame

3:30pm-3:55pm
Kevin Dilks, University of Minnesota
q Gamma Nonnegativity

4:00pm-4:25pm
Robert Edman, University of Minnesota
Monotone Paths on Zonotopes
Wednesday 8/10

9:00am-9:55am
SHO SUDA, Aichi University of Education
Introduction to complex spherical designs and codes with applications

10:30am-10:55am
FERDINAND IHRINGER, Justus-Liebig-Universität Gießen
The Manickam-Miklós-Singhi Conjecture for Vector Spaces

11:00am-11:25am
KRYS TAL GUO, Simon Fraser University
Hermitian adjacency matrix of digraphs

11:30am-11:55am
TETSUJI TANIGUCHI, Hiroshima Institute of Technology
A representation of Hoffman graphs

Thursday 8/10

9:00am-9:55am
MARCO DUARTE, University of Massachusetts
Parameter Estimation in Compressive Sensing

10:30am-10:55am
SHERMIN HAMZEHEI, University of Massachusetts Amherst
Compressive Parameter Estimation via Approximate Message Passing

11:00am-11:25am
CHRISTINO TAMON, Clarkson University
Which quantum walk on graphs?

11:30am-11:55am
ADA CHAN, York University
A Walk in the Thorny Cubes

2:00pm-2:55pm
MARCUS APPLEBY, University of Sydney
Galois symmetries of a SIC

3:30pm-3:55pm
PAUL TERWILLIGER, University of Wisconsin
Q-polynomial distance-regular graphs and the q-Onsager algebra

4:00pm-4:25pm
HAJIME TANAKA, Tohoku University
A semidefinite programming approach to a cross-intersection problem with measures
Friday 8/10

9:00am-9:55am  
**Gary Greaves**, Tohoku University  
**Equiangular lines in Euclidean spaces**

10:30am-10:55am  
**Gabriel Coutinho**, University of Waterloo  
**Lines and covers of graphs**

11:00am-11:25am  
**Harmony Zhan**, University of Waterloo  
**Equiangular Lines and Covers of the Complete Graph**

11:30am-11:55am  
**Seth Chaiken**, University at Albany  
**Resistive networks, linear spaces and Tutte polynomials**
Marcus Appleby, University of Sydney
Galois symmetries of a SIC

The problem of proving (or disproving) the existence of symmetric informationally complete positive operator valued measures (SICs) has been the focus of much effort over the last 15 years, but a solution still eludes us. In this talk we describe the Galois symmetries of Weyl-Heisenberg covariant SICs (the class which has been most intensively studied). It is a striking fact that the known exact solutions are all expressible in terms of radicals, implying that the associated Galois groups must be solvable. We investigate the Galois group in more detail. We show that there are many intriguing regularities in the Galois groups of the known solutions. In particular it turns out that in every known case the number field is an Abelian extension of the real quadratic field generated by the square root of \((d - 3)(d + 1)\) (where \(d\) is the dimension). The field thus belongs to the class of number fields which are the focus of Hilbert’s (still unresolved) twelfth problem, and which have played a central, motivational role in the subsequent development of algebraic number theory. We further show that there is an intriguing interplay between the number-theoretic symmetries described by the Galois group and the unitary symmetries described by the Clifford group. We also show how a (conjectural) knowledge of the Galois group can be used to lever a merely numerical solution up to an exact one. We conclude with some speculations about the bearing all this may have on the problem of proving SIC existence.

Seth Chaiken, University at Albany
Resistive networks, linear spaces and Tutte polynomials

We survey how combinatorial theory emerges when quantities or relations of linear electrical network analysis are expressed as lines or more generally affine linear subspaces. Our story ranges over Thevenin and Norton’s theorems, load line analysis, exterior algebra and linear dependence of network behavior on parameters expressed as Tutte recursion, and a role of oriented matroids in the analysis of jump phenomena of non-linear systems.
Ada Chan, York University
A Walk in the Thorny Cubes

Let $A$ be the adjacency matrix of a graph $X$ on $n$ vertices. The continuous-time quantum walk on a graph $X$ is given by the transition operator

$$e^{-itA} = \sum_{k \geq 0} \frac{(-it)^k}{k!} A^k.$$ 

We say $X$ has perfect state transfer from $u$ to $v$ at time $\tau$ if

$$\left| \left( e^{-i\tau A} \right)_{u,v} \right|^2 = 1.$$ 

By investigating the corona products of graphs, we see that the thorny cubes do not have perfect state transfer but they satisfy a more relaxed condition called pretty good state transfer.

This is joint work with Ethan Ackelsberg, Zachary Brehm, Joshua Mundinger and Christino Tamon.

Gabriel Coutinho, University of Waterloo
Lines and covers of graphs

with: Chris Godsil, Hamed Shirazi and Harmony Zhan

The study of the relation between lines and covers of graph dates back to the work of Seidel in 70s. However many questions remain unanswered. In this talk, I will give an overview of what we know, some recent developments and some of the open questions we have.

Kevin Dilks, University of Minnesota
q Gamma Nonnegativity

A polynomial $\sum_{i=0}^n a_i t^i$ with symmetric coefficients ($a_{n-i} = a_i$) has a unique expansion $\sum_{k=0}^{|n/2|} \gamma_k t^k (1 + t)^{n-2k}$, and is said to be gamma-nonnegative if $\gamma_k \geq 0$ for all $k$. We either prove or conjecture a stronger $q$-analogue of this property for several polynomials in two variables $t,q$, whose $q = 1$ specializations are known to be gamma-nonnegative.
Parameter Estimation in Compressive Sensing

Compressive sensing (CS) implements simultaneous sensing and compression of sparse and compressible signals based on randomized dimensionality reduction. To recover a signal from its compressive measurements, standard CS algorithms seek the sparsest signal representation in some discrete basis or dictionary that agrees with the measurements. Many applications feature signals that can be represented using a small number of continuous-valued parameters. Such problems have initially been addressed in CS through the design of parametric dictionaries that collect a set of signal observations corresponding to a discretized set of parameter values. These approaches, however, suffer either from resolution limitations due to discretization or from poor performance due to the high coherence of the dictionary, the mismatch between the dictionary and the signal (which may not necessarily be sparse), or both.

This talk will introduce several techniques for compressive parameter estimation (CPE) that aim to alleviate the aforementioned issues, using the time delay estimation and frequency estimation problems common in radar imaging as running examples. First, we employ parametric dictionaries under a a structured sparsity signal model that inhibits highly coherent elements from appearing simultaneously in a signal representation. In a second approach, we use manifold models to characterize the map from parameter space to signal space and employ manifold-based interpolation for parametric dictionaries. Finally, we introduce the concept of earth mover’s distance for parametric signal representations to directly measure the performance of parameter estimation and leverage it using specially tailored algorithms. We will review the benefits and shortcomings of these proposed alternatives.

Monotone Paths on Zonotopes

A classic system of lines can be found in a triangulation of a polytope. Triangulations are the most refined subdivisions of polytopes and certain geometrically distinguished subdivisions are said to be coherent. Projections of an $n$-cube into $\mathbb{R}^d$ form a zonotope $Z$, which is a highly symmetric polytope, and paths along the edges of $Z$ from the $f$-minimizing vertex $-z$ to its opposite vertex $z$ are an analog of triangulations. An $f$-monotone path is then coherent if it lies on the boundary of a polygon obtained by projecting $Z$ to 2 dimensions. This talk motivates the question “which $(Z, f)$ pairs contain incoherent $f$-monotone paths?” and gives a remarkably simple answer when $n - d < 3$. Following these results we will discuss connections to other areas of research and ongoing work.

Designs in quantum information science

Quantum information science seeks to exploit the laws of quantum physics to perform information processing tasks like communication or computation in ways that are intractable or even impossible using classical physics. This talk will survey several applications of designs in this field – from secure communication and cryptography to “debugging” a quantum computer – and some of the unique challenges involved. No knowledge of quantum mechanics will be assumed.
Unextendible sets of MUBs

In quantum information, Mutually Unbiased Bases (MUBs) correspond to sets of pairwise complementary observables. The maximal number of such bases in a system of dimension $d$ is $d + 1$, and construction of maximal sets achieving this bound are known only if the dimension is a prime power. For other dimensions, we have a lower bound of three bases, and for infinitely many dimensions, we do not know how to improve this lower bound. For specific constructions, we can show that they do not achieve the upper bound. On the other hand, even in prime power dimensions where the maximal number of MUBs can be constructed, there are unextendible sets of smaller size. We will present results of our work in progress on this topic.

Equiangular lines in Euclidean spaces

Given some dimension $d$, what is the maximum number of lines in $\mathbb{R}^d$ such that the angle between any pair of lines is constant? This classical problem has recently enjoyed a renewed interest due to the current attention the quantum information community is giving to its complex analogue. I will report on some new developments of the theory of equiangular lines in Euclidean spaces. Among other things, I will present improvements to two long standing upper bounds for equiangular lines in dimensions 14 and 16. This talk is based on joint work with Jack Koolen, Akihiro Munemasa, and Ferenc Szöllősi.

Hermitian adjacency matrix of digraphs

The spectra of graphs and their relation to graph properties have been well-studied. For digraphs, in contrast, there are relatively fewer results. The adjacency matrix of a digraph is usually difficult to work with; it is not always diagonalizable and the interlacing theorem does not hold (in general) for adjacency matrices of digraphs. All acyclic digraphs have the same spectrum as the empty graph. This motivates the need to work with a different matrix which captures the adjacency of the digraph. To this end, we study the Hermitian adjacency matrix, where the $(u, v)$-entry is the imaginary unit $i$ if there is an arc from $u$ to $v$, $-i$ if there is an arc from $v$ to $u$, 1 if both arcs exist and 0 otherwise.

We discuss basic properties of this matrix and some interplay between the spectrum of the Hermitian adjacency matrix and properties of the digraphs, such as diameter and maximum degree.
Compressive Parameter Estimation via Approximate Message Passing

with: M. Duarte

The literature on compressive parameter estimation has been mostly focused on the use of sparsity dictionaries that encode a sampling of the parameter space; these dictionaries, however, suffer from coherence issues that must be controlled for successful estimation. We propose the use of statistical parameter estimation methods within the approximate message passing (AMP) algorithm for signal recovery. Our proposed work leverages the recently highlighted connection between statistical denoising methods and the thresholding step commonly used during recovery. As an example, we consider line spectral estimation by leveraging the well-known Root MUSIC algorithm. Numerical experiments show significant improvements in estimation performance.

Ferdinand Ihringer, Justus-Liebig-Universität Gießen

The Manickam-Miklós-Singhi Conjecture for Vector Spaces

Let $V$ be an $n$-dimensional vector space over a finite field with $q$ elements. Define a real-valued weight function on the 1-dimensional subspaces of $V$ such that the sum of all weights is zero. Let the weight of a subspace $S$ be the sum of the weights of the 1-dimensional subspaces contained in $S$. In 1988 Manickam and Singhi conjectured that if $n \geq 4k$, then the number of $k$-dimensional subspaces with nonnegative weight is at least the number of $k$-dimensional subspaces on a fixed 1-dimensional subspace.

Recently, Chowdhury, Huang, Sarkis, Shahriari, and Sudakov [1, 2] proved the conjecture of Manickam and Singhi for $n \geq 3k$. The speaker improved this result to $n \geq 2k$ for large $q$. The aim of this talk is to present recent results on the Manickam-Miklós-Singhi (MMS) conjecture on vector spaces, and to discuss its interesting connection to Erdős-Ko-Rado theorems.

In the last part of the talk we will present generalizations of the MMS conjecture to association schemes. This includes recent joint work with Karen Meagher on the MMS conjecture for designs and orthogonal arrays.

References


Koji Momihara, Kumamoto University

Distance sets on circles
with: Masashi Shinohara

An $n$-point $k$-distance set on the unit sphere $S^t \subset \mathbb{R}^{t+1}$ is a set $X$ of $n$ points on $S^t$ such that exactly $k$ Euclidean distances occur between two distinct points in $X$. In this talk, we treat distance sets on $S^1$, and show that if $k$ is small enough relative to $n$, then $X$ lies on a regular polygon. More precisely, we prove that for an $n$-point $k$-distance set $X$ on $S^1$ with $n \geq 4$, if $k < 3t$ or $3t - 2$ according as $n = 4t, 4t - 1$ or $n = 4t - 2, 4t - 3$, then $X$ lies on a regular $2k$ or $(2k + 1)$-sided polygon. Furthermore, we see that this bound cannot be improved any more. Also, we find an application of Kneser’s addition theorem to distance sets on circles.

Padraig O’Catháin, Aalto University

Almost equiangular lines

Hadamard matrices and pairwise balanced designs can be used to construct sets lines which are almost equiangular. In this talk I will outline a general construction and show how the properties of the system of lines depends on the properties of the designs used in the construction. Finally, I will show how the recent breakthrough results of Keevash on the existence of designs imply strong asymptotic existence results for almost equiangular lines.

Randy Paffenroth, WPI

Musings on sparsity detection, point-wise error bounds, and truncated norms
with: Fan Yang

In this talk we will present some ideas on using point-wise error bounds and, most recently, truncated norms to detect sparse structures in compressed sensing and low-rank matrix completion problems. Some of the results are quite preliminary, but the authors hope they will provoke discussions around the Restricted Isometry Property, incoherence, and exact recovery of sparse phenomena. In addition, we will demonstrate the applications of such techniques to problems in network analysis.
Yury Polyanskiy, MIT
On metric properties of maps between Hamming spaces and related graph homomorphisms

The classical question in coding theory is to find the maximal number of points in the Hamming space with a given lower bound on pairwise distance. The best known upper bound on this number is obtained via Delsarte’s linear programming (i.e. by bounding Schrijver $\vartheta$-function).

In this talk the following extension of the coding problem is studied: A mapping of $k$-bit strings into $n$-bit strings is called an $(\alpha, \beta)$-map if $k$-bit strings which are more than $\alpha k$ apart are mapped to $n$-bit strings that are more than $\beta n$ apart. Equivalently, an $(\alpha, \beta)$-map is a graph homomorphism between certain graphs on the hypercube. We develop and apply tools based on Schrijver’s $\vartheta$-function for testing when such homomorphisms are possible.

For $n > k$ the non-existence results on $(\alpha, \beta)$ are proved by invoking the asymptotic results on $\vartheta$-function of McEliece, Rodemich, Rumsey and Welch (1977), Samorodnitsky (2001) as well as an exact solution of Delsarte’s linear program for $d > n/2$. Among other things, these bounds show that for $\beta > 1/2$ and $n/k$ – integer, the repetition map achieving $\alpha = \beta$ is best possible. For $n < k$ a quantitative version of the no-homomorphism lemma is used together with Kleitman’s theorem, which precisely characterizes the diameter-volume tradeoff in Hamming space.

Finally, the question of constructing good linear $(\alpha, \beta)$ maps is shown to be equivalent to finding certain extremal configurations of points in (finite) projective spaces. Consequently, implications of our results for projective geometry over $\mathbb{F}_2$ is given. Time permitting we will also mention how Kneser conjecture shows obstructions to existence of $(\alpha, \beta)$-maps.

Justin Romberg, Georgia Tech
Solving Underdetermined Linear Systems and Overdetermined Quadratic Systems of Equations

Over the past decade, a rich mathematical theory has been developed around the fundamental problem of when a system of underdetermined equations can be meaningfully “inverted”. The central message of this body of work is that if a solution has a type of expected structured, it can often be found from many fewer equations from unknowns. Two examples of structure would be if the unknown entity is a vector which is sparse (has only a few “active” entries) or if it is a matrix which is low rank. We discuss some of the applications of this theory in signal processing and statistics.

Next, we will discuss how some of these structured recovery results give us new insights into solving systems of quadratic and bilinear equations. In particular, we can derive conditions on when these types of equations can be solved in the context of some common imaging and communications applications.
Brendan Rooney, University of Waterloo

Vector Colourings of Distance-Regular Graphs

with: Chris Godsil, David Roberson, Robert Samal, Antonios Varvitsiotis

A vector colouring of a graph $G$ maps the vertices of $G$ to vectors in $\mathbb{R}^m$. The goal is to map adjacent vertices to vectors that are far apart. We will look at representations of a graph on its least eigenspace as examples of vector colourings. For distance-regular graphs, these colourings are optimal.

Furthermore, by looking at a space of symmetric matrices derived from the representation, we can show that these colourings are unique. This gives us a tool for proving that various classes of distance-regular graphs are cores. We also present some computational work on strongly regular graphs that supplies evidence for a conjecture of Cameron and Kazanidis.

Brigitte Servatius, WPI

Dimension of a point-line configuration

Given a point-line configuration in the plane, what is the highest dimension it lifts to? We only know the answer for a few classical configurations.

Sho Suda, Aichi University of Education

Introduction to complex spherical designs and codes with applications

Complex mutually unbiased bases and Sic-POVMs are important objects in quantum information theory. In this talk, we review the theory on complex spherical designs and codes, and discuss these objects using complex spherical designs and codes. In particular we show that specific complex MUBs and Sic-POVMs give rise to association schemes.

Christino Tamon, Clarkson University

Which quantum walk on graphs?

Given a graph $G$ and a corresponding Hermitian matrix $M(G)$, the matrix $\exp(-itM(G))$ may be viewed as defining a certain type of walk on $G$. We describe several of such known walks, based on the choice of $M(G)$, that had been studied, their connections with algebraic combinatorics, and some open problems.

Hajime Tanaka, Tohoku University

A semidefinite programming approach to a cross-intersection problem with measures

with: Sho Suda and Norihide Tokushige

We present a semidefinite programming approach to bound the measures of cross-independent pairs in a bipartite graph. This can be viewed as a far-reaching extension of Hoffman’s ratio bound on the independence number of a graph. As an application, we solve a problem on the maximum measures of cross-intersecting families of subsets with two different product measures, which is a generalized measure version of the Erdős-Ko-Rado theorem for cross-intersecting families with different uniformities.
Tetsuji Taniguchi, Hiroshima Institute of Technology

A representation of Hoffman graphs

Hoffman graphs were introduced by Woo and Neumaier [3] to extend the results of Hoffman [1]. Hoffman proved what we would call Hoffman’s limit theorem which asserts that, in the language of Hoffman graphs, the smallest eigenvalue of a fat Hoffman graph is a limit of the smallest eigenvalues of a sequence of ordinary graphs with increasing minimum degree.


In [2], we showed that the special graph $S^-(\mathfrak{H})$ of a such a Hoffman graph $\mathfrak{H}$ is isomorphic to one of the Dynkin graphs $A_n$, $D_n$, or extended Dynkin graphs $\tilde{A}_n$ or $\tilde{D}_n$. Also we showed that, even when the Hoffman graph $\mathfrak{H}$ does not admit an integral representation, its special graph $S(\mathfrak{H})$ can be represented by one of the exceptional root lattices $E_n$ ($n = 6, 7, 8$).

In this talk, we introduce a Hoffman graph and its representation. Moreover we talk about some results and problems.

References


Paul Terwilliger, University of Wisconsin

Q-polynomial distance-regular graphs and the $q$-Onsager algebra

For a Q-polynomial distance-regular graph of $q$-Racah type, the adjacency matrix $A$ and dual adjacency matrix $A^*$ satisfy the $q$-Dolan/Grady relations. These are the defining relations for the $q$-Onsager algebra. In this talk we describe how the $q$-Onsager algebra is related to the positive part of the quantum algebra $U_q(\widehat{sl}_2)$.

Shayne Waldron, University of Auckland

Calculating the symmetries of a tight frame

I will give a brief survey of some key ideas from frame theory and how it extends to general vector spaces, e.g., every spanning sequence is a normalised tight frame for a unique inner product. I will then consider how tight frames can be constructed from abstract groups, and conversely how to calculate the symmetry group and projective symmetry group of a given frame (spanning sequence), thereby recognising whether or not it is a $G$–orbit of a small number of vectors. All interesting frames and sets of equiangular lines appear to be the orbits of a single vector.
Richard Wilson, Caltech

**Combinatorial consequences and extensions of a simple inequality for matrices**

Let $M$ be a matrix of order $n$ and of rank $r$. Let $s$ be the number of distinct off-diagonal entries of $M$ and assume that none of these values occur on the diagonal of $M$. Then

$$n \leq \binom{r+s}{s}. $$

For example, one immediate consequence is that the multiplicity of an eigenvalue $\lambda \neq 1, 0, -1$ of a $(0,1)$-matrix is less than $n - \sqrt{2n} + 2$.

Our talk will discuss applications and generalizations of this inequality. We recover known inequalities on systems of sets with restricted intersections and sets of points with restricted distances. In case the entries of $M$ are integers, we give necessary conditions for equality and, under certain conditions, can reduce the bound for $n$ to

$$n \leq \binom{r+t}{t} $$

with $t < s$.

Wei-Hsuan Yu, Michigan State University

**Equiangular lines and spherical designs**

with: Alexander Barg, Takayuki Okuda

We determine the maximum size of equiangular lines in $\mathbb{R}^n$ for $24 \leq n \leq 41$ and $n = 43$. We will discuss the relation between equiangular lines and spherical $t$-designs.

Harmony Zhan, University of Waterloo

**Equiangular Lines and Covers of the Complete Graph**

with: Chris Godsil, Gabriel Coutinho, Mirhamed Shirazi

A set of lines is equiangular if the angle between any two of them is the same. In this talk, I will discuss the relation between equiangular lines and distance-regular antipodal covers of complete graphs, and some of our recent contributions.