Calculating the symmetries of a tight frame

Shayne Waldron

Mathematics Department, University of Auckland

August 11th, 2015
Outline

- A brief (personal) history of tight frames.
- Tight frames for multivariate orthogonal polynomials.
- Theory of frames (canonical coordinates).
- Equivalence of frames up to similarity
- Equivalence of frames up to projective unitary equivalence.
- Symmetries of frames.
Finite dimensional linear algebra

- Vectors written (uniquely) in terms of bases - for “efficient calculation”
- Nice (natural) bases important for vector spaces with additional structure, e.g., spaces of polynomials
- Assumes that when there is a natural spanning set, then there is a natural basis
- Orthonormal bases are considered to be the best possible
Shifts and dilations of $\psi$ the Haar Wavelet (1907) give an orthonormal expansion for signals $f \in L_2(\mathbb{R})$ in terms of signals localised in both time and frequency.

- These signals are not smooth.
Time–Frequency analysis

- Tight frame expansions, i.e., with $\psi_{m,n}(t) = 2^{-m/2} \psi(2^{-m} t - n)$,

$$f = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle f, \psi_{m,n} \rangle \psi_{m,n}$$

given by smoother wavelets $\psi$ (1990's onwards), e.g., Daubechies wavelet. Here the functions $(\psi_{m,n})$ need not be orthogonal

- The time–frequency shifts applied to the mother wavelet $\psi$ do not form group (multiresolution analysis).

- There is a parallel theory where they do, i.e., Weyl–Heisenberg systems (the group is generated by a shifts and modulations)
Let $u_1, u_2, u_3$ be three equally spaced unit vectors in $\mathbb{R}^2$. For a given nonzero vector $f \in \mathbb{R}^2$, what is the sum of its orthogonal projections onto these vectors?

1. $\sum_{j=1}^{3} \langle f, u_j \rangle u_j = 0$ (since $u_1 + u_2 + u_3 = 0$)
2. $\sum_{j=1}^{3} \langle f, u_j \rangle u_j = \frac{3}{2} f$, $\forall f \in \mathbb{R}^2$. 
Tight frames in finite dimensional spaces

The following sets of vectors \( \{v_j\}_{j=1}^3 \) form tight frames for \( \mathbb{R}^2 \)
i.e., give decompositions of the form

\[
f = \sum_{j=1}^{3} \langle f, v_j \rangle v_j, \quad \forall f \in \mathbb{R}^2.
\]

This is technically similar to an orthogonal expansion, except it has more terms (redundancy).
An example where there is no natural basis

Let $T = \text{conv}(V)$ be a simplex in $\mathbb{R}^d$ with $d + 1$ vertices $V$, with corresponding barycentric coordinates $\xi = (\xi_v)_{v \in V}$, and define the Jacobi inner product

$$\langle f, g \rangle_\nu := \int_T fg \xi^{\nu-1}, \quad \nu = (\nu_v)_{v \in V} > 0.$$ 

e.g., for $d = 2$, $T = \text{conv}\{e_1, e_2, 0\}$, $\nu - 1 = (\alpha, \beta, \gamma)$

- $\xi_0(x, y) = 1 - x - y$
- $\xi_{e_2}(x, y) = y$
- $\xi_{e_1}(x, y) = x$

$$\langle f, g \rangle_\nu = \int_0^1 \int_0^{1-x} f(x, y)g(x, y) x^\alpha y^\beta (1 - x - y)^\gamma \, dy \, dx$$
Multivariate Jacobi polynomials

The **Jacobi polynomials** of degree \( k \) are

\[
P^\nu_k := \{ f \in \Pi_k : \langle f, p \rangle^\nu = 0, \forall p \in \Pi_{k-1} \}.
\]

This space has

\[
\dim(P^\nu_k) = \binom{k + d - 1}{d - 1}.
\]

Each polynomial in \( P^\nu_k \) is uniquely determined by its leading term, e.g., for \( \xi_0^2 + \) lower order terms, the leading term is

\[
\{(1 - x - y)^2\} \downarrow = x^2 - 2xy + y^2.
\]
Orthogonal and biorthogonal systems

We describe the known representations for $P_k^\nu$ in terms of the leading terms (for the case $d = 2$, $k = 2$).

**Biorthogonal system** (Appell 1920’s): partial symmetries

$$x^2, \ xy, \ y^2.$$ 

**Orthogonal system** (Prorial 1957, et al): no symmetries

$$x^2 + y^2 + 2xy, \ x^2 - y^2, \ x^2 - y^2 - 4xy.$$ 

For the *three* dimensional space of all quadratic Jacobi polynomials on the triangle, we want an orthonormal basis with leading terms determined by the *six* polynomials

$$x^2, \ xy, \ y^2, \ x(1 - x - y), \ y(1 - x - y), \ (1 - x - y)^2.$$
Scaling to obtain a tight frame

Theorem

(W, Peng) Let $\mathcal{H}$ be a Hilbert space of dimension $d$, and

$$n = \begin{cases} \frac{1}{2}d(d + 1), & \mathcal{H} \text{ real} \\ d^2, & \mathcal{H} \text{ complex} \end{cases}$$

Then for a generic sequence of vectors $v_1, \ldots, v_n$ there are unique positive scalars $c_j$ so that $(c_jv_j)$ is a normalised tight frame for $\mathcal{H}$.

- You will recognise this $n$ as the maximal number of equiangular lines in $\mathcal{H} = \mathbb{R}^d, \mathbb{C}^d$.
- For our six quadratic orthogonal polynomials in a space of dimension three, there should be a unique scaling to a tight frame.
A miracle

This works, even when it shouldn’t!

Let $\phi^\nu_\alpha$ be the orthogonal projection of $\xi^\alpha/(\nu)_\alpha$, $|\alpha| = n$ onto $P^\nu_n$, which is given by

$$\phi^\nu_\alpha = \frac{(-1)^n}{(n + |\nu| - 1)_n} \sum_{\beta \leq \alpha} \frac{(n + |\nu| - 1)|\beta|(-\alpha)_\beta}{(\nu)_\beta} \xi^\beta \alpha^n \xi^\beta_\beta.$$

Theorem

(W, Xu, Rosengren) The Jacobi polynomials on a simplex have the tight frame representation

$$f = (|\nu|)_{2n} \sum_{|\alpha| = n} \frac{(\nu)_\alpha}{\alpha!} \langle f, \phi^\nu_\alpha \rangle \phi^\nu_\alpha, \quad \forall f \in P^\nu_n,$$

where the normalisation is $\langle 1, 1 \rangle_\nu = 1$. 
A nice example

The group of symmetries of the triangle (the dihedral group \( G = D_3 \cong S_3 \)) induces a representation on the quadratic Legendre polynomials \( P_2 \) on the triangle, and so (by algebra) we can construct

\[
f = (2\sqrt{5} - 5\sqrt{2})\left(\xi_v^2 + \xi_w^2 + \xi_u^2 - \frac{1}{2}\right) + 15\sqrt{2}\left(\xi_v^2 - \frac{4}{5}\xi_v + \frac{1}{10}\right) \in P_2
\]

a single polynomial whose orbit under \( G \) consists of \textit{three} polynomials which form an orthonormal basis for \( P_2 \).

Contour plots of \( f \) and those of its orbit showing the triangular symmetry.
Let $\mathcal{H}$ be a finite dimensional Hilbert space ($\mathbb{R}^d$ or $\mathbb{C}^d$), and $\Phi = (v_1, \ldots, v_n)$ be a sequence of vectors in $\mathcal{H}$.

- $\Phi$ is **frame** for $\mathcal{H}$ if it spans $\mathcal{H}$.
- $\Phi$ is a **normalised tight frame** (or **Parseval frame**)$\text{ if }$

\[
f = \sum_j \langle f, v_j \rangle v_j, \quad \forall f \in \mathcal{H}.
\]

$\Phi$ is a normalised tight frame $\iff$ its Gramian

\[
P_\Phi = [\langle v_k, v_j \rangle]_{j,k=1}^n
\]

is an orthogonal projection matrix.

Moreover, the columns of $P_\Phi$ (as a subspace of $\mathbb{C}^n$) are a copy of $\Phi$. 
Normalised tight frames and orthogonal projections

- The orthogonal projection of a normalised tight frame is a normalised tight frame (for its span).
- Every normalised tight frame is the orthogonal projection, namely the orthogonal projection $P_\Phi$ of the standard basis $(e_j)$.
Every frame is similar to a unique normalised tight frame

We say that frames $\Phi$ and $\Psi$ (with the same index set) are similar, if $\Psi = A\Phi$ for some invertible $A$, and are unitarily equivalent if $A$ is unitary.

- Normalised tight frames are similar if and only if they are unitarily equivalent.
- Frames are unitarily equivalent if and only if their Gramians are equal.

The frame operator $S = S_\Phi : \mathcal{H} \to \mathcal{H}$ of a frame $\Phi = (v_j)$

$$Sf := \sum_j \langle f, v_j \rangle v_j,$$

is positive definite, and

- The canonical tight frame $(S^{-\frac{1}{2}}v_j)$ is the unique normalised tight frame (up to unitary equivalence) which is similar to $\Phi.$
The dual frame

Let $\Phi = (v_j)$ be a frame. The unique coefficients $c_j = c_j(f)$ with

$$f = \sum_j c_j v_j$$

and $\sum_j |c_j|^2$ minimal are

$$c_j = \langle f, \tilde{v}_j \rangle, \quad \tilde{v}_j = S^{-1}v_j$$

where $(\tilde{v}_j)$ is called the **dual frame**.

- The coefficients $c_j(f)$ are linear functions of $f$
- They do not depend on the inner product
Canonical coordinates

Let $\Phi = (v_j)$ be a spanning set for an $F$–vector space ($F$ a subfield of $\mathbb{C}$ closed under conjugation), and denote the ($F$–linear) dependencies of $\Phi$ by

$$\text{dep}(\Phi) = \{ a \in F^n : \sum_j a_j v_j = 0 \}$$

The **canonical coordinates** $c_j = c_j(f)$ of $f$ with respect to $\Phi$ are the unique coefficients with $\sum_j |c_j|^2$ minimal, and $f = \sum_j c_j v_j$, and the **canonical Gramian** is

$$P_\Phi = [c_j(v_k)]$$

- The coefficients $c_j(f)$ are linear functions of $f$
- $P_\Phi$ is the orthogonal projection matrix with kernel $\text{dep}(\Phi)$
- The columns of $P_\Phi$ are a normalised tight frame which is similar to $\Phi$
- If $f = \sum_j a_j v_j$, then $(c_j(f)) = P_\Phi a.$
Every spanning set for vector space can be viewed as a normalised tight frame

If $\Phi = (v_j)$ is a spanning set for a vector space $X$, then there is a unique inner product on $X$ for which $\Phi$ is a normalised tight frame, i.e.,

$$\langle f, g \rangle := \langle c(f), c(g) \rangle.$$  

**Example**

$P_\Phi = (v_j)$ can be calculated from an orthogonal basis for $\text{dep}(\Phi)$. Suppose that there is just one dependence

$$v_1 + v_2 + \cdots + v_n = 0, \quad a = (1, 1, \ldots, 1)$$

Then $P_\Phi = I - \frac{aa^*}{\langle a, a \rangle}$, which is the vertices of a regular simplex.

- Here we started with a spanning sequence for a vector space and converted it into a set of equiangular lines. Perhaps other sets of equiangular lines could be obtained this way.
Example
Let $\omega$ be a primitive $n$–root of unity. The cyclotomic field $\mathbb{Q}[\omega]$ is a vector space of dimension $\varphi(n)$ over $\mathbb{Q}$ (which is closed under complex conjugation). Here $\varphi(n)$ is the Euler phi function,

$$\varphi(n) = \text{the number of primitive } n\text{–th roots of unity.}$$

There is no natural basis, e.g., for $n = 4$, the primitive roots $\pm i$ are $\mathbb{Q}$–linearly dependent, and so do not form a basis.
The roots $\Phi = (1, \omega, \omega^2, \ldots, \omega^{n-1})$ are a natural spanning set (invariant under the Galois action). The corresponding canonical coordinates have many nice properties:

- $P_\Phi = \frac{1}{n} \sum_{j \in \mathbb{Z}_n^*} \chi_j \chi_j^*$,

where $\chi_j = (1, \omega^j, \ldots, \omega^{(n-1)j})$ are the irreducible characters of $\mathbb{Z}_n$.

- Multiplication by $\omega$ corresponds to a cyclic shift of the canonical coordinates

- Multiplication is given by cyclic convolution of the canonical coordinates

- The $n$–the roots of unity are an equal–norm tight frame.
The symmetry group of a frame

Intuitively, a symmetry of a frame $\Phi = (v_j)_{j \in J}$ for $\mathcal{H}$ is a linear map which permutes its vectors, i.e.,

$$v_{\sigma j} = L_{\sigma} v_j, \quad \forall j$$

where $\sigma \in S_J$ and $L_{\sigma} \in \mathcal{GL}(\mathcal{H})$.

For technical reasons, we define the **symmetry group** as a group of permutations

$$\text{Sym}(\Phi) := \{ \sigma \in S_J : v_{\sigma j} = L_{\sigma} v_j, \ \forall j \}.$$ 

Since a linear map is defined by its action on a spanning set, the representation

$$\text{Sym}(\Phi) \rightarrow \mathcal{GL}(\mathcal{H}); \sigma \mapsto L_{\sigma}$$

is easily determined.
Properties

- Similar frames (such as the dual and canonical tight frames) have the same symmetry group.

- The complement of a frame $\Phi$ is the frame with canonical Gramian $I - P_\Phi$. A frame and its complement have the same symmetry group.

Example

The tight frame

$$\Phi = (1, 1, \ldots, 1)$$

for $\mathbb{R}^1$ has $\text{Sym}(\Phi) = S_n$. Its complement, the vertices of a regular simplex, therefore has symmetry group $S_n$. 
Calculation of the symmetry group

Each permutation $\sigma \in S_J$ corresponds to a permutation matrix $P_\sigma$

$$P_\sigma e_j := e_{\sigma j}.$$  

Since each frame $\Phi$ is similar to the columns of its canonical Gramian $P_\Phi$, we have

$$\sigma \in \text{Sym}(\Phi) \iff P_\sigma^* P_\Phi P_\sigma = P_\Phi.$$  

Thus the symmetry group of a finite sequence of vectors can be calculated (in principle).
When the action of $\text{Sym}(\Phi)$ is transitive

Let $G$ be a finite abstract group, with a unitary action on $\mathcal{H}$. We say $\Phi$ is a $G$–frame (or group frame) if 

$$\Phi = (gv)_{g \in G},$$

i.e., $\Phi$ is the $G$–orbit of a single vector $v$.

Since the action is unitary

- The frame operator $S$ commutes with (the action of) $G$.
- The dual and canonical tight frames of a $G$–frame are $G$–frames.
- The complement of a tight $G$–frame is a tight $G$–frame.
The Gramian of a $G$–frame is a $G$–matrix, i.e.,

$$\langle gv, hv \rangle = \langle h^* gv, v \rangle = \langle g^{-1} hv, v \rangle = \mu(g^{-1} h).$$

**Example**

For $G$ a cyclic group, a $G$–matrix is a *circulant matrix*. As with circulant matrices, $G$–matrices can be diagonalised using the characters of the group as eigenfunctions.

**Theorem**

*Let $G$ be a finite group. Then $\Phi = (\phi_g)_{g \in G}$ is a $G$–frame (for its span $\mathcal{H}$) if and only if its Gramian is a $G$–frame.*

Further, $G$–frames can also be identified with elements of the group algebra $\mathbb{C}G$. 
Irreducible $G$–frames

If the (unitary) action of $G$ is **irreducible**, i.e., every nonzero orbit spans $\mathcal{H}$, then $(gv)_{g \in G}$ is a tight $G$–frame for every $v \neq 0$.

**Example**

The vertices of a simplex, Platonic solid, or the $n$ equally spaced unit vectors in $\mathbb{R}^2$ are irreducible $G$–frames.

These can be constructed from their corresponding symmetry groups (as abstract groups).

The vertices of the platonic solids are distinguished from other orbits of the symmetry group (which have more vectors), but the fact a vertex is stabilised by a nontrivial subgroup. This idea leads to **highly symmetric tight frames**.
The dihedral group

Consider the irreducible action of the **dihedral group**

\[ G = D_3 = \langle a, b : a^3 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle, \]

on \( \mathbb{R}^2 \) as symmetries of three equally spaced unit vectors (the Mercedes–Benz frame).

- There are *uncountably* many unitarily inequivalent \( D_3 \)-frames for \( \mathbb{R}^2 \). This is always the case for \( G \) *nonabelian*.
- For the abelian subgroup \( C_3 = \langle a \rangle \) of rotations all \( C_3 \)-frames are unitarily equivalent.
Harmonic frames

For $G$ abelian there are finitely many tight $G$–frames, the **harmonic frames**. These can be obtained by selecting rows of the character table, e.g., for $C_3$

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}, \quad \omega := e^{\frac{2\pi i}{3}}
$$

gives two unitarily inequivalent harmonic frame for $C^2$

\[
\begin{align*}
\{ & \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} \omega \\ \omega^2 \end{bmatrix}, & \begin{bmatrix} \omega^2 \\ \omega \end{bmatrix} \} \text{ (real)} \\
\{ & \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} 1 \\ \omega \end{bmatrix}, & \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix} \} \text{ (complex)}
\end{align*}
\]

- Equivalently (taking columns) one can restrict the characters to a subset $J$ of $G$

$$
\Phi_J = (\xi|J)_{\xi \in \hat{G}}
$$

Here $\hat{G}$ is the character group of $G$ ($\hat{G} \cong G$).
We say $J, K \subset G$ are

- **translates** if $K = J - b$, $b \in G$
- **multiplicatively equivalent** if $K = \sigma J$, $\sigma : G \to G$ an automorphism

**Theorem**

(W, Chien) Let $J, K$ be subsets of a finite abelian group $G$. Then

1. If $J$ and $K$ are translates, then $\Phi_J$ and $\Phi_K$ are projectively unitarily equivalent.

2. If $J$ and $K$ are multiplicatively equivalent, then $\Phi_J$ and $\Phi_K$ are unitarily equivalent after reordering (by an automorphism)

This essentially allows one to classify all harmonic frames up to (projective) unitary equivalence.
Theorem

Let $G$ be a finite group of order $n$, and $\Phi = (\xi | J)_{\xi \in \hat{G}}$ be the harmonic frame of $n$ vectors for $\mathbb{C}^d$ given by $J \subset G$, $|J| = d$. Then

1. $\Phi$ has distinct vectors if and only if $J$ generates $G$.
2. $\Phi$ is a real frame if and only if $J$ is closed under taking inverses.
3. $\Phi$ is equiangular if and only if $J$ is an $(n, d, \lambda)$–difference set for $G$, i.e., each nonidentity element of $G$ can be written as a difference $j_1j_2^{-1}$ of two elements $j_1, j_2 \in J$ in exactly $\lambda$ ways.
**Theorem**

*(Characterisation).* Let there be a unitary action of a finite group $G$ on $\mathcal{H} = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, an orthogonal direct sum of irreducible $G$–invariant subspaces (irreducible $FG$–modules). Then

\[(gv)_{g \in G}, \quad v = v_1 + \cdots + v_m, \quad v_j \in V_j\]

is a tight $G$–frame for $\mathcal{H}$ if and only if

\[v_j \neq 0, \quad \forall j, \quad \frac{\|v_j\|^2}{\|v_k\|^2} = \frac{\dim(V_j)}{\dim(V_k)}, \quad j \neq k,\]

and when $V_j \neq V_k$ are $FG$–isomorphic, $(gv_j)_{g \in G}$ and $(gv_k)_{g \in G}$ are orthogonal, i.e.,

\[\sum_{g \in G} \langle v_j, gv_j \rangle gv_k = 0. \tag{1}\]
Moreover, if $V_j$ is absolutely irreducible, then (1) can be replaced by

\[ \langle \sigma v_j, v_k \rangle = 0, \]  

where $\sigma : V_j \to V_k$ is any $FG$–isomorphism.

Continuing this line of reasoning, allows one to determine the minimal number of generators for a $G$–invariant tight frame (or spanning set).
Projective unitary equivalence

We say frames $\Phi = (v_j)$ and $\Psi = (w_j)$ are **projectively unitarily equivalent** if

$$w_j = c_j U v_j, \quad \forall j$$

where $U$ is unitary and $c_j$ are unit modulus scalars. In other words we now view the vectors as (weighted) lines. The term *fusion frame* is also used.

Unitary equivalence is characterised by the Gramian (inner products). These are *not* projective unitary invariants

$$\langle c_j U v_j, c_k U v_k \rangle = c_j \overline{c_k} \langle v_j, v_k \rangle.$$
The $m$–products

The norm and inner product squared are projective unitary invariants, e.g.,

$$|\langle c_j U v_j, c_k U v_k \rangle|^2 = \langle c_j U v_j, c_k U v_k \rangle \langle c_k U v_k, c_j U v_j \rangle$$
$$= c_j \overline{c_k} \langle U v_j, U v_k \rangle c_k \overline{c_j} \langle U v_k, U v_j \rangle$$
$$= |\langle v_j, v_k \rangle|^2.$$

In the same way, the $m$–products

$$\Delta(v_{j_1}, v_{j_2}, \ldots, v_{j_m}) := \langle v_{j_1}, v_{j_2} \rangle \langle v_{j_2}, v_{j_3} \rangle \cdots \langle v_{j_m}, v_{j_1} \rangle$$

are projective unitary invariants.

These (finitely many) $m$–products determine $\Phi = (v_j)$ up to projective unitary equivalence.
The **frame graph** of a sequence of vectors \((v_j)\) is the graph with vertices \(\{v_j\}\) (or the indices \(j\) themselves) and

an edge between \(v_j\) and \(v_k\), \(j \neq k\) \iff \langle v_j, v_k \rangle \neq 0\).

Cycles (with a direction) correspond to nonzero \(m\)–products.

**Theorem**

A finite frame \(\Phi\), with frame graph \(\Gamma\), is determined up to projective unitary equivalence by a determining set for the \(m\)–products, e.g.,

1. The \(2\)–products.

2. The \(m\)–products, \(3 \leq m \leq n\), corresponding to a fundamental cycle basis (for the cycle space of \(\Gamma\)) formed from a spanning tree (forest) \(T\) for \(\Gamma\).

In particular, if \(M\) is the number of edges of \(\Gamma \setminus T\), then it is sufficient to know all of the \(2\)–products, and \(M\) of the \(m\)–products, \(3 \leq m \leq n\).
Example SICs

The frame graph of a SIC, or any set of equiangular lines is the complete graph.

By taking the star graph with internal vertex say $v_1$, it follows that a set of equiangular lines is determined by the 2-products and the 3–products (triple products) involving the point $v_1$. For $\mathcal{H} = \mathbb{R}^2$ this can be interpreted in terms of two graphs.
Example MUBs

Let $\Phi = (\nu_j)$ be the two MUBs for $\mathbb{C}^2$ given by

$$
\Phi = \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right), \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{array} \right) \right),
\left( \begin{array}{cccc} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{array} \right). 
$$

The frame graph $\Gamma$ of $\Phi$ is the 4–cycle $(\nu_1, \nu_3, \nu_2, \nu_4)$. Thus this MUB is determined up to projective unitary equivalence by the 2–products and the 4–product given by this cycle.

This case is an anomaly.

Corollary

A frame consisting of three or more MUBs is determined by to projective unitary equivalence by its 2–products and 3–products.
If there is a natural spanning sequence $\Phi$ for a vector space, then it can be viewed as a tight frame.

- The symmetry group and projective symmetry group of a spanning sequence can be calculated.

- Many interesting tight frames such as SICs and MUBs come as group orbits.
Thank you for your attention

If you want to learn more about finite tight frames, then ...

Shayne F. D. Waldron

An Introduction to
Finite Tight Frames, Draft

– Monograph –

August 7, 2015

Springer