The Manickam-Miklós-Singhi Conjecture for Vector Spaces

Ferdinand Ihringer

Justus Liebig University Giessen, Germany

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History

- 1984: Bier defines the \textit{i-th distribution invariant} of an \textit{association scheme}.
- 1987: Bier and Delsarte generalize this concept to \textit{distribution numbers} of association schemes.
- 1986: Manickam, a student of Eiichi Bannai, publishes his PhD thesis on “Distribution Invariants of Association Schemes”.
- 2014: Simeon Ball tells the speaker about this conjecture during a car drive to a \textbf{pub}.
The MMS Conjecture for Sets

1. Consider $M = \{1, \ldots, 10\}$.
2. Let $f : M \rightarrow \mathbb{R}$ a weighting of $M$ with $\sum_{x \in M} f(x) = 0$.

Question

How many 5-element subsets $S$ of $M$ have nonnegative weight, i.e. how many such $S$ satisfy $\sum_{x \in S} f(x) \geq 0$?
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**Answer**

At least $\binom{10}{5}/2 = 126$. If $S \subseteq M$ has negative weight, then its complement $\complement S$ has positive weight.

Too simple. Let’s change the question...
The MMS Conjecture for Sets

1. Consider \( M = \{1, \ldots, 10\} \).
2. Let \( f : M \to \mathbb{R} \) a **weighting** of \( M \) with \( \sum_{x \in M} f(x) = 0 \).

**Question**

*How many 3-element subsets \( S \) of \( M \) have nonnegative weight, i.e. how many such \( S \) satisfy \( \sum_{x \in S} f(x) \geq 0 \)?
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**Question**

*How many 3-element subsets $S$ of $M$ have nonnegative weight, i.e. how many such $S$ satisfy $\sum_{x \in S} f(x) \geq 0$?*

**Answer** (Marino, Chiaselotti (2002), Hartke, Stolee (2014))

At least $\binom{7}{3} = 35$. 
Some Examples

We have \( \binom{10}{3} = 120 \) subsets with 3 elements.

**Example**

Put the weight 1 on 1, \ldots, 9 and the weight \(-9\) on 10. Then we have \( \binom{9}{3} = 84 \) nonnegative 3-element subsets.

**Example**

Put the weight \(-1\) on 1, \ldots, 9 and the weight 9 on 10. Then we have \( \binom{9}{2} = 36 \) nonnegative 3-element subsets.

**Example**

Put the weight 3 on 1, \ldots, 7 and the weight \(-7\) on 8, 9, 10. Then we have \( \binom{7}{3} = 35 \) nonnegative 3-element subsets.

The last example is the unique smallest example (Marino, Chiaselotti, Hartke, Stolee).
The Manickam-Miklós-Singhi Conjecture for Sets

Conjecture (Manickam-Miklós-Singhi)

Let $M = \{1, \ldots, n\}$, $n \geq 4k$, and $f : M \rightarrow \mathbb{R}$ a weighting with

$$\sum_{x \in M} f(x) = 0.$$ 

Then the set $Y$ of nonnegative $k$-element subsets of $M$ satisfies

$$|Y| \geq \binom{n-1}{k-1}.$$
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<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>Bound on $n$</th>
<th>$k = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bier, Manickam</td>
<td>1987</td>
<td>$\approx k^{2k+1}$</td>
<td>$4 \cdot 10^{19}$</td>
</tr>
<tr>
<td>Manickam, Miklós</td>
<td>1988</td>
<td>$(k - 1)(k^k + k^2) + k$</td>
<td>$9 \cdot 10^{10}$</td>
</tr>
<tr>
<td>Bhattacharya</td>
<td>2003</td>
<td>$2^{k+1}e^k k^{k+1}$</td>
<td>$5 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Tyomkyn</td>
<td>2012</td>
<td>$k^2(4e \log k)^k$</td>
<td>$10^{16}$</td>
</tr>
<tr>
<td>Alon, Huang, Sudakov</td>
<td>2012</td>
<td>$\min{33k^2, 2k^3}$</td>
<td>$2000$</td>
</tr>
<tr>
<td>Frankl</td>
<td>2013</td>
<td>$\frac{3}{2}k^3$</td>
<td>$1500$</td>
</tr>
<tr>
<td>Chowdhury, Sarkis, Shahriari</td>
<td>2015</td>
<td>$8k^2$</td>
<td>$800$</td>
</tr>
<tr>
<td>Pokrovskiy</td>
<td>2015</td>
<td>$10^{46}k$</td>
<td>$10^{47}$</td>
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Let \( M = \{1, \ldots, n\}, \ n \geq 4k, \) and \( f : M \to \mathbb{R} \) a weighting with \( \sum_{x \in M} f(x) = 0 \). Then the set \( Y \) of nonnegative \( k \)-element subsets of \( M \) satisfies

\[
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One can also try to solve the problem for small \( k \).

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<tr>
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<tr>
<td>Trivial</td>
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<td></td>
<td>( k = 2 )</td>
</tr>
<tr>
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<td>2002</td>
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<td>2014</td>
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</tr>
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Let $M = \{1, \ldots, n\}$, $n \geq 4k$, and $f : M \rightarrow \mathbb{R}$ a weighting with $\sum_{x \in M} f(x) = 0$. Then the set $Y$ of nonnegative $k$-element subsets of $M$ satisfies

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One published paper claims a complete solution for all $n \geq 4k$. 
Vector Spaces

- A vector space of dimension \( n \) over a finite field with \( q \) elements: \( V \).
- \( S \) is a subspace of \( V \) of dimension \( k \): \( k \)-space.
- 1-spaces: points, \( (n - 1) \)-spaces: hyperplanes, \( \mathcal{P} \): all points of \( V \).
- The number of \( k \)-spaces in an \( n \)-space: \( \binom{n}{k} \).

Conjecture (Manickam-Miklós-Singhi)

Let \( n \geq 4k \), and \( f : \mathcal{P} \to \mathbb{R} \) a weighting with \( \sum_{x \in \mathcal{P}} f(x) = 0 \). Then the set \( Y \) of nonnegative \( k \)-spaces of \( V \) satisfies

\[
|Y| \geq \binom{n - 1}{k - 1}.
\]
Some Examples

Example

Let $P$ be a point. Put the weight $\binom{n}{1} - 1$ on $P$, and $-1$ on all the other points. Then exactly the $\binom{n-1}{k-1}$ $k$-spaces through $P$ have nonnegative weight.

Example

Let $H$ be a hyperplane. Put the weight $-1$ on all points not in $H$, and $q^{n-1}/\binom{n-1}{1}$ on all points in $H$. Then exactly the $\binom{n-1}{k}$ $k$-spaces in $H$ have nonnegative weight.
A Strengthened Conjecture

Conjecture

Let $n \geq k$, and $f : \mathcal{P} \to \mathbb{R}$ a weighting with $\sum_{x \in \mathcal{P}} f(x) = 0$. Then the set $Y$ of nonnegative $k$-spaces of $V$ satisfies

$$|Y| \geq \min \left\{ \left\lceil \frac{n-1}{k-1} \right\rceil, \left\lceil \frac{n-1}{k} \right\rceil \right\}$$

with equality if and only if $Y$ is either the set of all $k$-spaces through a point or the set of all $k$-spaces in a hyperplane.

It is enough to show the conjecture for $n \geq 2k$ as the case $n < 2k$ follows from duality.
More History

The conjecture is true if $k$ divides $n$.

**Theorem (Manickam and Singhi (1988))**

*If $k$ divides $n$, then the smallest set of nonnegative $k$-spaces $Y$ is an Erdős-Ko-Rado set of maximum size, i.e. a set of pairwise non-trivially intersecting $k$-spaces of maximum size.*
The conjecture is true if $k$ divides $n$.

**Theorem (Manickam and Singhi (1988))**

If $k$ divides $n$, then the smallest set of nonnegative $k$-spaces $Y$ is an **Erdős-Ko-Rado set** of maximum size, i.e. a set of pairwise non-trivially intersecting $k$-spaces of maximum size.

**Proof.**

As $k$ divides $n$, there exists a partition $S$ of $\mathcal{P}$ into $k$-spaces. Let $S$ be a $k$-space with negative weight. Suppose $S \in \mathcal{S}$. We have

$$\sum_{x \in M} f(x) = \sum_{T \in S} \sum_{x \in T} f(x) = 0,$$

so at least one element $T \in S \setminus \{S\}$ has positive weight. Double counting over all $S$ with $S \in \mathcal{S}$ shows $|Y| \geq \binom{n-1}{k-1}$ with equality if and only if $Y$ is an Erdős-Ko-Rado set of maximum size.
Contemporary History

Theorem (Chowdhury, Sarkis, Shahriari (2015))

If \( n \geq 3k \), then

\[
|Y| \geq \binom{n-1}{k-1}
\]

with equality if and only if \( Y \) is the set of all \( k \)-spaces through a fixed point.

Theorem (Huang, Sudakov (2015))

If \( n \geq ck \) for sufficiently large \( c \), then

\[
|Y| \geq \binom{n-1}{k-1}
\]
Two Ideas

The ideas used by Chowdhury, Sarkis, and Shahriari. Many of the following only holds for \( n \geq 2k + 1 \).

1. If \( k \) does not divide \( n \), then one can still use something similar to a spread to imitate the Manickam-Singhi double count. This shows

\[
|Y| \geq (1 - O(1/q)) \left\lceil \frac{n - 1}{k - 1} \right\rceil.
\]

2. An eigenvalue trick shows

\[
|Y| \geq (1 - O(1/q)) \left\lceil \frac{n - 1}{k - 1} \right\rceil.
\]

Combining both ideas shows the result for \( n \geq 3k \) and \( q \geq 2 \).
The First Idea

**Theorem (Beutelspacher (1975))**

Let $n = rk + \delta$, $r \in \mathbb{Z}$, $\delta < k$. Then one can partition $\mathcal{P}$ into one $(k + \delta)$-space and $k$-spaces.

Chowdhury, Sarkis, and Shahriari use this to show that if there exists a $k$-space $S$ with negative weight, then there are

$$\left(1 - \frac{2}{q^{n-2k-\delta+1}}\right) \left\lceil \frac{n-1}{k-1} \right\rceil$$

$k$-spaces with positive weight which intersect $S$ trivially.
The Second Idea

- Let $W$ be the **incidence matrix** whose rows are indexed by the $k$-spaces and whose columns are indexed by the points, i.e.

$$W_{PS} = \begin{cases} 1 & \text{if } P \text{ is a point of } S, \\ 0 & \text{otherwise.} \end{cases}$$

- Let $A$ be the **distance-$(k-1)$-adjacency matrix** of $k$-spaces, i.e. the symmetric matrix indexed by $k$-spaces with

$$A_{ST} = \begin{cases} 1 & \text{if } \dim(S \cap T) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- If we view the weight function $f$ as a vector, then it is well-known that $b = Wf$ is an **eigenvector** of $A$, i.e. the vector $b$ indexed by the $k$-spaces with the weights of the $k$-spaces as its entries is an eigenvector of $A.$
The Second Idea

- We know that the weight vector $b$ of the $k$-spaces is an eigenvector of the distance-$(k - 1)$-adjacency matrix $A$ with eigenvalue $\lambda$. This shows for a $k$-space $C$

$$\sum_{\dim(S \cap C) = 1} b_S = (Ab)_C = \lambda b_C.$$  

- If $C$ is a highest weight $k$-space, then this shows that at least $\lambda$ $k$-spaces, which meet $C$ in exactly a point, have nonnegative weight. Fortunately,

$$\lambda \geq \left( 1 - \frac{3}{q^{n-2k+1}} \right) \left[ \frac{n-1}{k-1} \right].$$
Both Ideas Together

Recall $n = rk + \delta$.

- For each negative $k$-space, there are
  
  $$\left(1 - \frac{2}{q^{n-2k-\delta+1}}\right) \binom{n-1}{k-1}$$

  nonnegative $k$-spaces disjoint to this $k$-space.

- The highest weight $k$-space meets at least
  
  $$\left(1 - \frac{3}{q^{n-2k+1}}\right) \binom{n-1}{k-1}$$

  nonnegative $k$-spaces in a point.

- This shows the conjecture for $n \geq 3k$ and $q \geq 2$ (C.–S.–S.).

- Similar arguments: $n \geq k$ and $q$ large. Here $q$ depends on $k$. (I., accepted).

- Similar arguments: $(n, k) = (5, 2)$ and $q \geq 2$ (C.–S.–S., unpublished).
2-(n(m - 1) + 1, m, 1)-Designs

- A set \( M = \{1, 2, \ldots, n(m - 1) + 1\} \).
- The design consists of a family \( X \) of \( m \)-sets (blocks).
- Two different elements of \( M \) are contained in exactly one block.
- Put weights on the elements of \( M \) as before.

**Question**

How many elements of \( X \) have nonnegative weight, i.e. how many such \( S \in X \) satisfy \( \sum_{x\in S} f(x) \geq 0 \)?
2-\((n(m - 1) + 1, m, 1)\)-Designs

**Theorem (Huang, Sudakov (2015))**

*If* \( n \gtrapprox 10m^2 \), *then the set* \( Y \) *of all nonnegative sets satisfies* \(|Y| \geq n\) *with equality if* \( Y \) *is the set of all objects on a fixed element.*

**Theorem (Meagher, I.)**

*If* \( n \gtrapprox m^2 + 2m^{3/2} \), *then ...
2-\((n(m - 1) + 1, m, 1)\)-Designs

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*If \(n \gtrapprox m^2 + 2m^{3/2}\), then ...*

Example

- The theorem does not hold for some designs with a particular substructure and \(n = m^2 - m + 1\).
- The design has to contain a projective plane of order \(m - 1\) *missing* one line. This “partial” projective plane is not extendable to a projective plane.
- We only know of existence for \(m = 3\).
Orthogonal Arrays $\text{OA}(m, n)$

**Theorem (Huang, Sudakov (2015))**

If $n \gtrapprox 10m^2$, then the set $Y$ of all nonnegative sets satisfies $|Y| \geq n$ with equality if $Y$ is the set of all objects on a fixed element.

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Orthogonal Arrays \(OA(m, n))

Theorem (Huang, Sudakov (2015))

If \( n \gtrapprox 10m^2 \), then the set \( Y \) of all nonnegative sets satisfies \(|Y| \geq n\) with equality if \( Y \) is the set of all objects on a fixed element.

Theorem (Meagher, I.)

If \( n \gtrapprox m^2 + 2m^{3/2} \), then ...

Example

- We have a counterexample for \( OA(m, (m-1)^2) \).
- We only know of existence for \( m = 3 \).
- For \( m = 3 \) counterexample is tight.
Let \((X, \mathcal{R})\) be a \(d\)-class association scheme.

Let \(\langle j \rangle, V_1, \ldots, V_d\) be its common eigenspaces.

Call a vector \(\nu\) \textbf{general} if it contains no zeros.
Let \((X, \mathcal{R})\) be a \(d\)-class association scheme.

Let \(\langle j \rangle, V_1, \ldots, V_d\) be its common eigenspaces.

Call a vector \(v\) **general** if it contains no zeros.

**Definition (Bier and Delsarte (1984/1988))**

The **\(i\)-th distribution invariant** of an association scheme is defined as

\[
\min_{w \in V_i \text{ general}} |\{x : w^T \chi_x > 0\}|.
\]

In the usual ordering of common eigenspaces, all the presented problems belong to the first distribution invariant of the corresponding scheme.
Thank You!