Definition (Polytope)

A polytope is a convex hull of finitely many points in $\mathbb{R}^d$. Combinatorially a polytope can be defined by its face lattice.
Definition (Polyhedral Subdivision)

A polyhedral subdivision is a decomposition of $P$ into subpolytopes. A subdivision is a triangulation when each subpolytope is a simplex.
Remark

In this example, the refinement poset is the face lattice of a polytope.
Some bad triangulations are not regular or are incoherent.

Q1: The same notion of coherence? Almost certainly not.

Theorem ([GKZ94])

The refinement poset of all coherent subdivisions of $P$ is the face lattice of the fiber polytope $\Sigma(P)$.

Question

Which polytopes have incoherent triangulations? How do we detect coherence?
Definition (Hyperplane arrangement/Zonotope)

- A hyperplane arrangement $\mathcal{A} \in \mathbb{R}^{d \times n} = \{a_1^\perp, \ldots, a_n^\perp\}$.
- The zonotope $Z(\mathcal{A}) = \sum_{a_i \in \mathcal{A}} [-a_i, +a_i]$ (Minkowski sum).
- Intersections of hyperplanes $(d - k) \iff$ faces $(k)$ of $Z$. 
Definition (Monotone path graph)

- **Vertices**: Paths from the $f$-minimizing vertex $z$ to the $f$-maximizing vertex $z$.
- **Edges**: Two paths which share a common face.
Theorem (BKS94)

Every $f$-monotone path of a cube is coherent.
Reflection Arrangements

Question

- What $(A, f)$ pairs is every $f$-monotone path coherent?
Proposition

A $f$-monotone path $\gamma$ is coherent if there exists a $g \in (\mathbb{R}^d)^*$ so that:

$$\frac{g_{\gamma(1)}}{f_{\gamma(1)}} < \frac{g_{\gamma(2)}}{f_{\gamma(2)}} < \ldots < \frac{g_{\gamma(n)}}{f_{\gamma(n)}}$$
Gale Duality

- Dual hyperplane configuration is a \((n - d) \times n\) matrix.

Theorems

- Every realization \((A, f)\) of \((M, + + \ldots +)\) will be all-coherent if and only if there exists at least one such realization \((A, f)\) which is all-coherent [ADLRS00].

- If every \(f\)-monotone path refining \(\sigma\) is coherent then \(\sigma\) is coherent. [Edm15]
We can detect the presence of incoherent \( f \)-monotone paths by presence of certain minor-minimal examples (obtained by deletion and contraction).
Classification of \((\mathcal{A}, f)\) in corank 1.

- The purple \((\mathcal{A}, f)\) pair is a \textit{minimal obstruction}, all other \((\mathcal{A}, f)\) containing incoherent \(f\)-monotone paths are liftings of it.

- Really remarkable: Coherence depends only on the oriented matroid structure, not on the particular \(f\).

Theorem

\textit{When} \(n - d = 1\) \textit{there is a unique family of all-coherent} \((\mathcal{A}, f)\) \textit{pairs and all other} \((\mathcal{A}, f)\) \textit{pairs have incoherent paths.}
Classification of \((\mathcal{A}, f)\) in corank 2.

**Theorem**

*When* \(n - d = 2\) *there are two all-coherent families and 10 minimal obstructions. Of the 10 minimal obstructions 8 are single-element extensions of the corank 1 minimal obstruction.*
Other open questions

- What is the diameter of the monotone path graph?
- A sorting network corresponds to an $f$-monotone path on a Permutohedron and is nearly a great circle.

Conjecture ([AHRV07])

Let $\omega_n$ be an $n$-element uniform sorting network. For each $n$ there exists a random great circle $C_n \subset S^n$ such that $d_\infty(\omega_n, C_n) = o(n)$ in probability as $n \to \infty$. 

\[ \begin{array}{cccccccccccccccc}
6 \\
5 \\
4 \\
3 \\
2 \\
1 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array} \]
Questions?


