Monotone Paths on Zonotopes.

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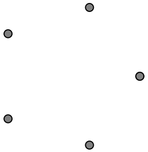
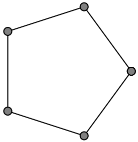
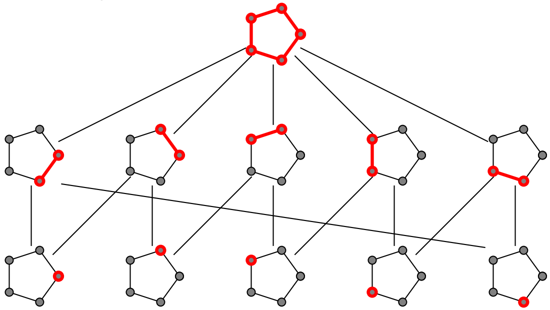
Definition (Polytope)

A polytope is a convex hull of finitely many points in $\mathbb{R}^d$.
Combinatorially a polytope can be defined by its face lattice.

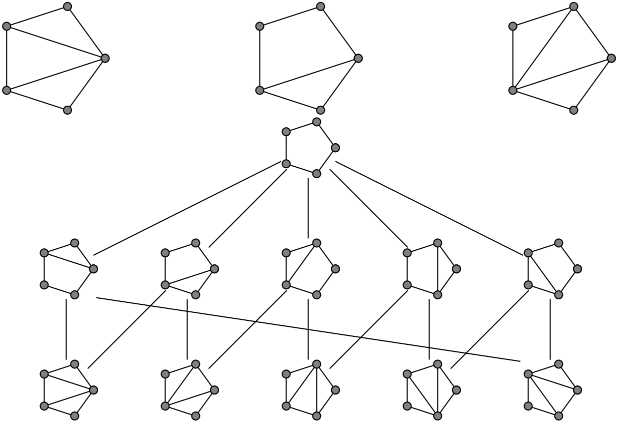


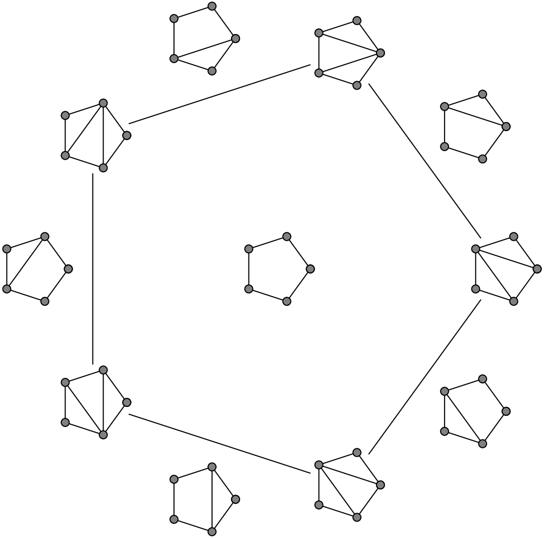




Definition (Polyhedral Subdivision)

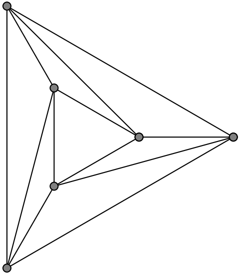
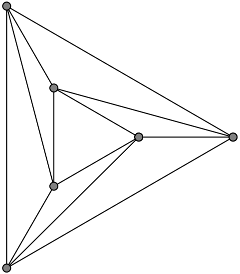
A polyhedral subdivision is a decomposition of P into subpolytopes. A subdivision is a triangulation when each subpolytope is a simplex.





Remark

In this example, the refinement poset is the face lattice of a polytope.

- Some bad triangulations are not *regular* or are *incoherent*.
- Q1: The same notion of coherence? Almost certainly not.

Theorem ([GKZ94])

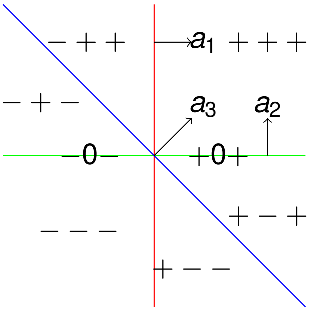
The refinement poset of all coherent subdivisions of P is the face lattice of the fiber polytope $\Sigma(P)$.

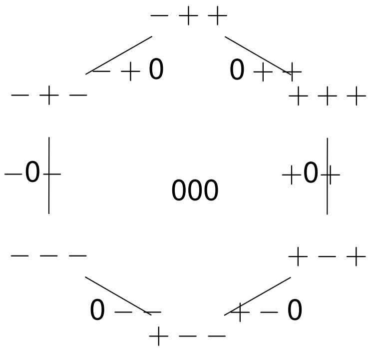
Question

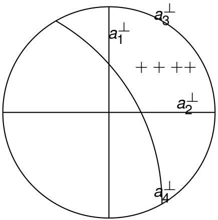
Which polytopes have incoherent triangulations? How do we detect coherence?

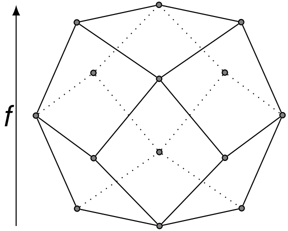
Definition (Hyperplane arrangement/Zonotope)

- *A hyperplane arrangement* $\mathcal{A} \in \mathbb{R}^{d \times n} = \{a_1^\perp, \ldots, a_n^\perp\}$.
- *The zonotope* $Z(\mathcal{A}) = \sum_{a_i \in \mathcal{A}} [-a_i, +a_i]$ *(Minkowski sum)*
- *Intersections of hyperplanes* $(d-k) \iff$ *faces* (k) *of* Z.



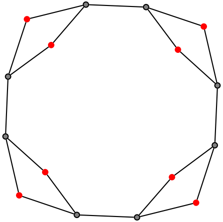


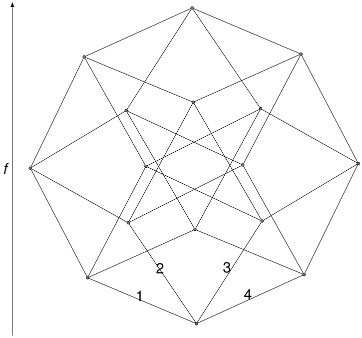


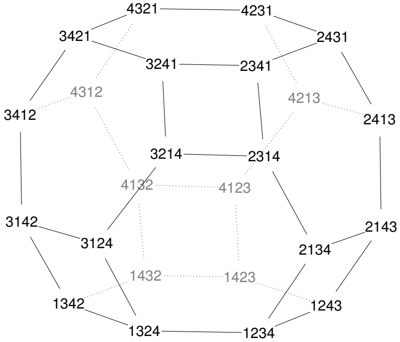


Definition (Monotone path graph)

- *Vertices: Paths from the f-minimizing vertex $-z$ to the f-maximizing vertex z.*
- *Edges: Two path which share a common face.*



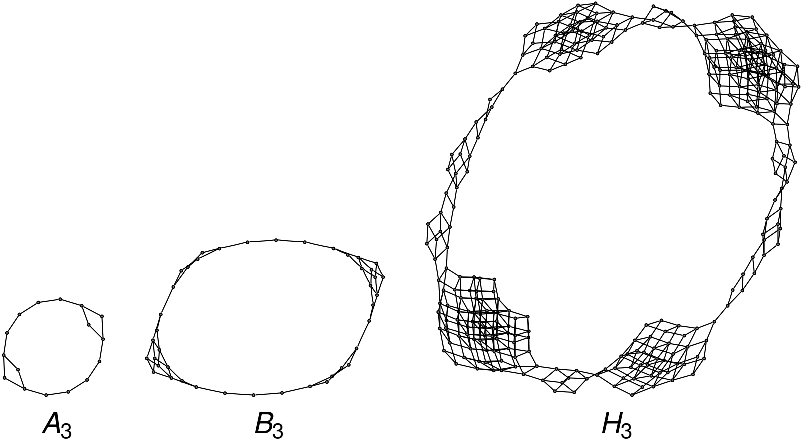




Theorem ([BKS94])

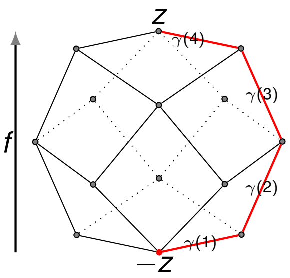
Every f-monotone path of a cube is coherent.

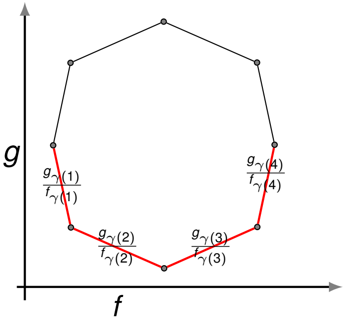
Reflection Arrangements



Question

- *What $(\mathcal{A}, f)$ pairs is every f-monotone path coherent?*



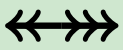
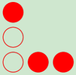


Proposition

A f-monotone path γ is coherent if there exists a $g \in \left(\mathbb{R}^d\right)^*$ so that:

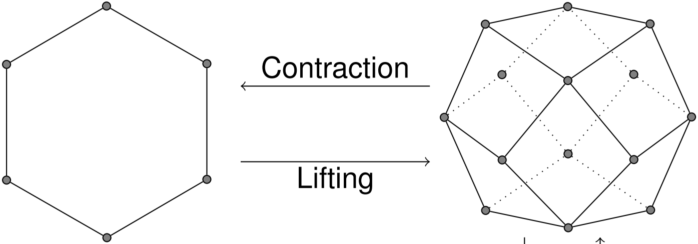
$$\frac{g_{\gamma(1)}}{f_{\gamma(1)}} < \frac{g_{\gamma(2)}}{f_{\gamma(2)}} < \dots < \frac{g_{\gamma(n)}}{f_{\gamma(n)}}$$

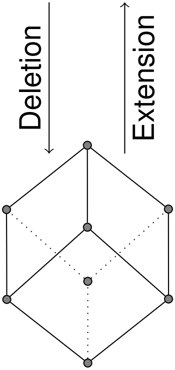
Gale Duality

- Dual hyperplane configuration is a $(n-d) \times n$ matrix.
-  $\Longleftrightarrow$ 
-  $\Longleftrightarrow$ 
-  $\Longleftrightarrow$ 

Theorems

- Every realization $(\mathcal{A}, f)$ of $(\mathcal{M}, ++\ldots+)$ will be all-coherent if and only if there exists at least one such realization $(\mathcal{A}, f)$ which is all-coherent [ADLRS00].
- If every f-monotone path refining σ is coherent then σ is coherent. [Edm15]



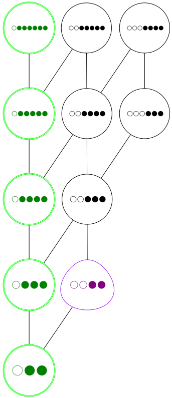


Proposition

We can detect the presence of incoherent f-monotone paths by presence of certain minor-minimal examples (obtained by deletion and contraction).

Classification of $(\mathcal{A}, f)$ in corank 1.

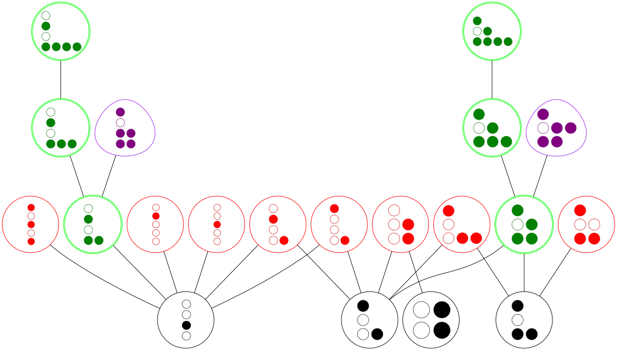
- The purple $(\mathcal{A}, f)$ pair is a *minimal obstruction*, all other $(\mathcal{A}, f)$ containing incoherent f-monotone paths are liftings of it.
- Really remarkable: Coherence depends only on the oriented matroid structure, not on the particular f.



Theorem

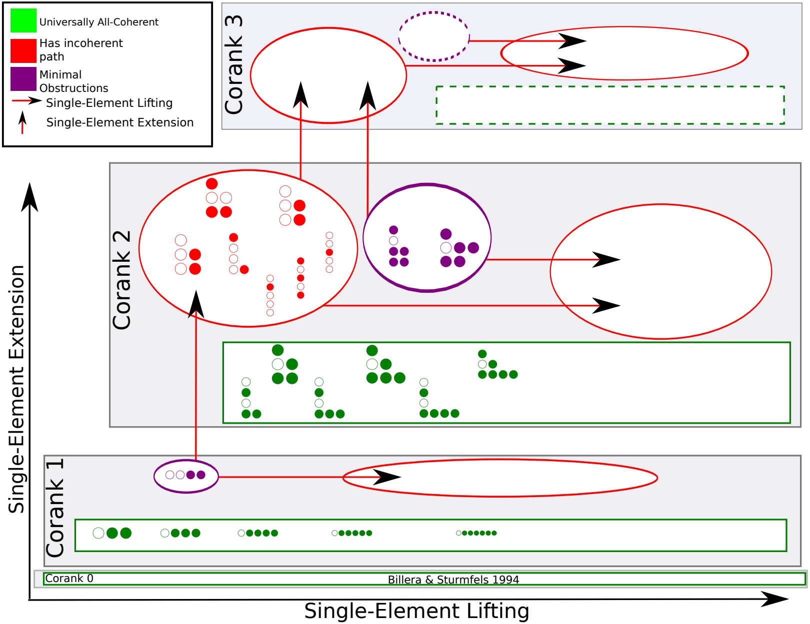
When $n - d = 1$ there is a unique family of all-coherent $(\mathcal{A}, f)$ pairs and all other $(\mathcal{A}, f)$ pairs have incoherent paths.

Classification of $(\mathcal{A}, f)$ in corank 2.



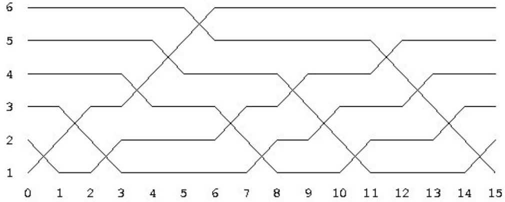
Theorem

When $n - d = 2$ there are two all-coherent families and 10 minimal obstructions. Of the 10 minimal obstructions 8 are single-element extensions of the corank 1 minimal obstruction.


Universally All-Coherent
Has incoherent path
Minimal Obstructions
Single-Element Lifting
Single-Element Extension
Corank 3
Corank 2
Corank 1
Corank 0
Billera & Sturmfels 1994
Single-Element Extension
Single-Element Lifting

Other open questions

- What is the diameter of the monotone path graph?
- A sorting network corresponds to an f-monotone path on a Permutohedron and is nearly a great circle.



Conjecture ([AHRV07])

Let ω_n be an n-element uniform sorting network. For each n there exists a random great circle $C_n \subset S^n$ such that $d_\infty(\omega_n, C_n) = o(n)$ in probability as $n \to \infty$.

Questions?

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