1. The PTC Cell

Let C be a centrally symmetric convex cell in the plane with at least 6 vertices. We take the center to be the origin. The number of vertices is even and will be denoted by $\{v_1, \ldots, v_k, -v_1, \ldots, -v_k\}$ reading clockwise around the figure. A *PTC cell* is such a convex cell with an equivariant set of 6 distinguished vertices, $\{a_1, a_2, a_3, -a_1, -a_2, -a_3\}$.

Lemma 1.1. *There is, up to rotation, a one to one correspondence between the set of pseudo-triangles T and the set of PTC cells C_T such that the vector paths between the distinguished vertices of C_T are translations and half-turns of the pseudo arcs of T.*

Proof. Let T be a pseudo-triangle with pseudo-vertices $\{v_1, v_2, v_3\}$. So the pseudo-arcs $\{a_{12}, a_{23}, a_{31}\}$ lie in the interior of the triangle $[v_1, v_2, v_3]$. Propagate this

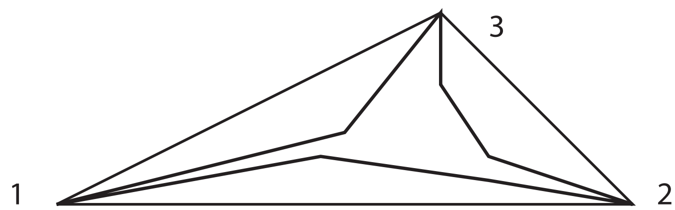

Figure 1. A Pseudo-Triangle.

triangle into its euclidian tessellation together with its pseudo-arcs. We call this tessellation the PT-tessellation. The translation subgroup of the tessellation group

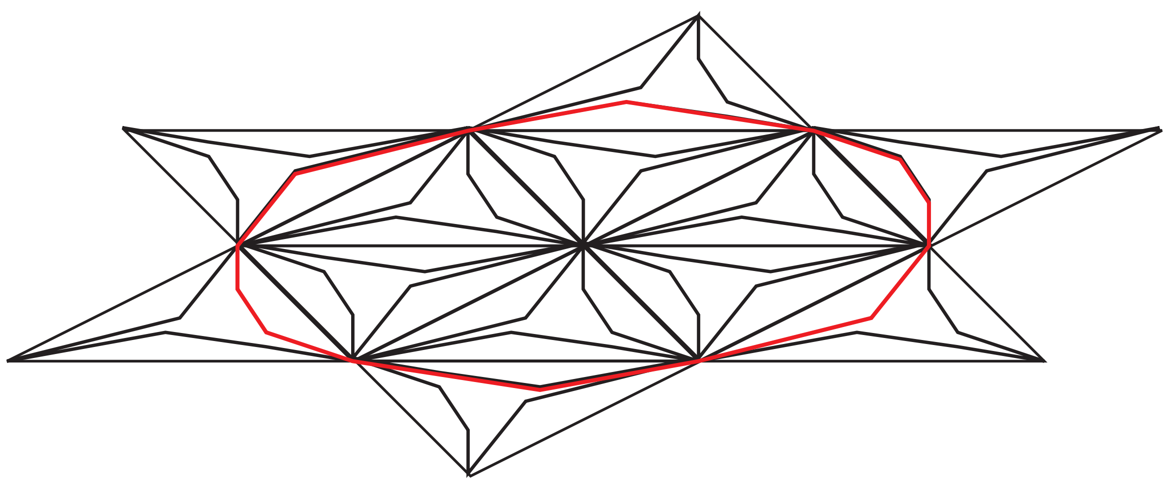

Figure 2. The PT-Tesselation.

acts transitively on the pseudo-vertices.

The PTC-cell is defined by the arcs in red in Figure 2. To show that the cell is convex we need only consider the angles at the translates of the pseudo-vertices. It is clear that the shaded region in Figure 3 has an angle at the pseudo-vertex

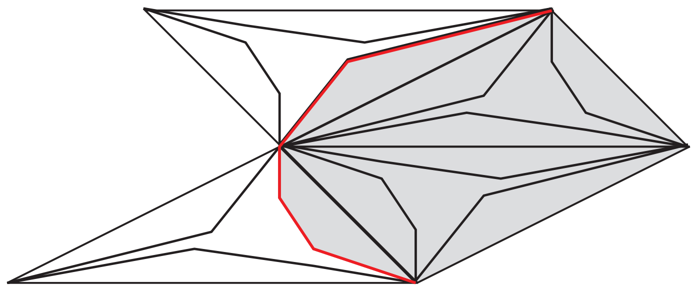

FIGURE 3.

which is equal to the angle subtended by the shaded area at the corresponding pseudo-vertex in Figure 4, which is less than 180° since the straight angle is made

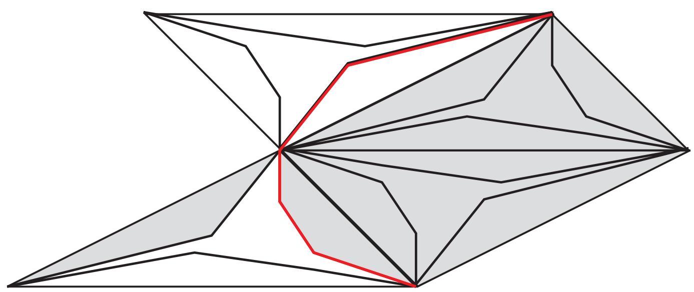

FIGURE 4.

up of different representatives of the angles of the triangle $[v_1, v_2, v_3]$.

Contrarywise, suppose we have any PTC-cell. The line segments joining the consecutive distinguished vertices form a centrally symmetric hexagon whose diameters meet at the origin and divide the hexagon into 6 congruent triangles which it is again convenient to propagate into a tessellation. The three arcs of the PTC-cell can be translated into one of these triangles forming a non-convex figure with three distinguished vertices at the vertices of the triangle. Since the arcs joining these vertices came from a convex cell, their pairwise angles are greater thatn 180° with respect to the exterior of the cell, that is, with respect to the interior of the non-convex region. To conclude that this figure is a pseudo-triangle, we must check that the three arcs do not intersect, which we may do by again inspecting Figures 3 and 4.

□

2. PSEUDO-TRIANGLES AND POINTED RECIPROCAL VERTICES.

We now give a lemma which describes a correspondence between the local behavior of pseudo-triangles and their reciprocal vertices.

Let $\mathbf{\Delta v}_1, \ldots \mathbf{\Delta v}_k$ be the edge vectors of a pseudo-triangle T, indices and edge directions oriented counter-clockwise. Let $\{s_1, \ldots, s_k\}$ be a stress on T. In order for a reciprocal figure to be defined, the stress ought to be part of a resolvable stress on a larger framework, but $\{s_1, \ldots, s_k\}$ need not itself be resolvable.

A consecutive pair s_i, s_{i+1} (indices modulo k) is said to have a *proper* sign change if the vertex v_i lies on the interior of a pseudo-arc and $\mathrm{sign}(s_i) = \mathrm{sign}(s_{i+1})$, if the vertex v_i is a pseudo-vertex of the pseudo-triangle and $\mathrm{sign}(s_i) = -\mathrm{sign}(s_{i+1})$.

The vectors of the reciprocal vertex of T, $\hat{T}$, are $\{s_1\Delta\mathbf{v}_1, \ldots, s_k\Delta\mathbf{v}_k\}$.

Lemma 2.1. *Given a framework together with a resolvable stress s and a pseudo-triangle T. Suppose that T is a face in the rotation system which governs the reciprocal.*

The following are equivalent:

(1) *the cyclic ordering of $\{s_1\Delta\mathbf{v}_1, \ldots, s_k\Delta\mathbf{v}_k\}$ around the reciprocal vertex is $(k, \ldots, 2, 1)$.*
(2) *the reciprocal figure is pointed at $\hat{T}$*
(3) *there is exactly one improper sign change on $\{s_1, \ldots, s_k\}$.*

Proof. The edge vectors of the pseudo-triangle T are oriented counterclockwise. We orient the edge vectors of the PTC-cell consistent with T for the edge vectors along those three arcs which are translates of pseudo-arcs of T, and the reverse for those which are translates of $-T$. Then the edge vectors of the PTC-cell are oriented clockwise. Thus, proceeding clockwise around the PTC-cell we encounter the edge vectors $\Delta\mathbf{v}1, \ldots \Delta\mathbf{v}_k$ in the correct cyclic order twice, with proper sign changes, and with opposite signs for antipodal edge vectors, see Figure 5.

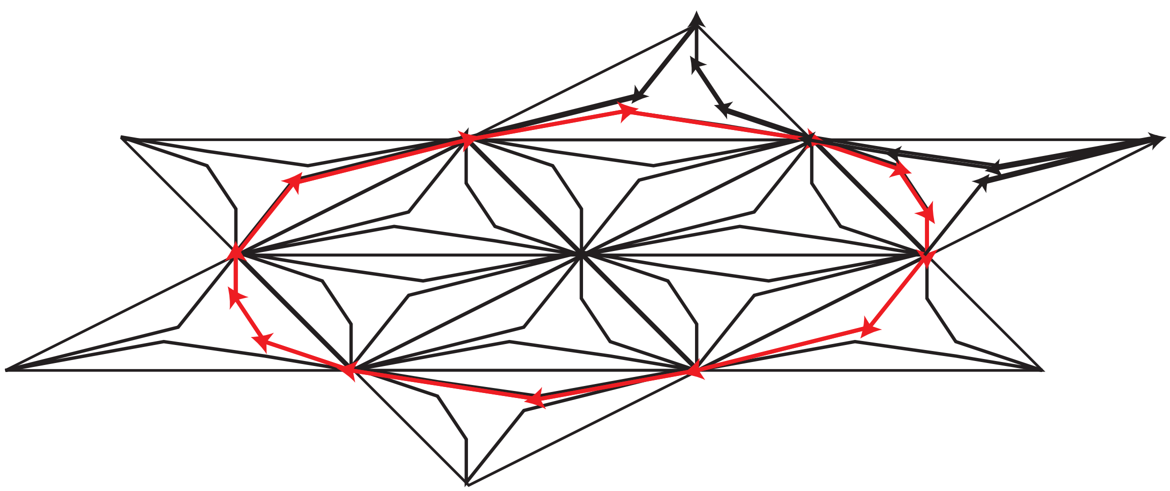

FIGURE 5. Orienting the PCT-cell.

Up to positive scaling, the vectors of the reciprocal vertex $\hat{T}$ correspond to a selection from each antipodal pair in the PTC-cell. Since we cannot choose the entire circumference, there must be at least one improper sign change.

If there is exactly one improper sign change, we have chosen a connected section of the circumference of the PTC-cell, so the edges around $\hat{T}$ occur in the correct cyclic order.

If the vectors occur in the correct cyclic order, then, at the first improper sign change we are forced to omit half the boundary vectors of the PTC-cell, and are obliged to keep the other half. Considering the radial vectors, it is obvious that $\hat{T}$ is pointed.

If $\hat{T}$ is pointed, then the we have selected a connected section of the PTC-cell, and thus have only one improper sign change.

$\square$