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EMERGENT 1D ISING BEHAVIOR IN AN ELEMENTARY CELLULAR AUTOMATON MODEL

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The fundamental nature of an evolving one-dimensional (1D) Ising model is investigated with an elementary cellular automaton (CA) simulation. The emergent CA simulation employs an ensemble of cells in one spatial dimension, each cell capable of two microstates interacting with simple nearest-neighbor rules and incorporating an external field. The behavior of the CA model provides insight into the dynamics of coupled two-state systems not expressible by exact analytical solutions. For instance, state progression graphs show the causal dynamics of a system through time in relation to the system's entropy. Unique graphical analysis techniques are introduced through difference patterns, diffusion patterns, and state progression graphs of the 1D ensemble visualizing the evolution. All analyses are consistent with the known behavior of the 1D Ising system. The CA simulation and new pattern recognition techniques are scalable (in both dimension, complexity, and size) and have many potential applications such as complex design of materials, control of agent systems, and evolutionary mechanism design.

Keywords: Ising; cellular automata; automaton.

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1. Introduction

Fundamentally, the Ising model consists of an ensemble of members that can exist in one of two possible states (e.g., up or down). The state of a particular member depends upon the states of other members of the ensemble (nearest, next-nearest neighbors, etc. or averaged overall members) as well as its coupling to an external field that is conjugate to the microstate. Advantages of this model are the clarity of its construction, even in multiple dimensions, and its wide range of applicability such as magnetic spins (the original system considered by Ising), financial markets, robotic control, chemical reactions, etc., to name a few.

However, there are a number of difficulties with an Ising approach,[1] namely, there is no systematic formulation for determining the Ising partition function.

Despite the conceptual simplicity of the Ising approach, analytic solutions exist only in one dimension and in two dimensions without an external field. These difficulties warrant the continued and active efforts in applying and studying the Ising model.

A relatively new approach, sometimes referred to as "emergent theory," has as its fundamental principle that simple rules of interaction between members of an ensemble can lead to complex behavior. This method has simple rules of member interaction stand in for complex analytical models and allows an evolution of cellular ensembles to be easily monitored. This principle as it applies to an Ising system is the primary focus of this work.

1.1. *Historical review*

The 2004 study by Jarkko Kari[2] serves as an excellent tutorial in cellular automaton (CA) theory. This survey begins with CA models for reproduction in 1962 by E. F. Moore up to Stephen Wolfram's seminal work in 2002.[3,4] However, perhaps the most influential foundational work on CA theory was published by John von Neumann.[5]

CAs were shown to be useful in simulating the Navier–Stokes equations in the 1986 study by Frisch.[10] Entropy considerations of CAs were discussed in 1992 by Hurd.[11] Conservative CA systems were investigated in 2002 by Boccara.[12]

Textbooks written on physical investigations utilizing CAs include the 1998 work by Chopart[13] and the 2002 work by Stephen Wolfram.[4]

To the best of our knowledge, the earliest application of CA to statistical mechanics was done in 1983 by Stephen Wolfram.[6] While this work was a comprehensive investigation of the topic, it only alluded to the Ising model. Several papers were independently written in response to Wolfram's by many authors.[7–9]

The approaches taken by these later works eschewed the principles described by Wolfram, viz., to keep CA models as simple as possible and to allow the complexity to emerge despite this simplicity. The CAs in later works were very complex upon construction. Simpler CAs equivalent to the Ising model and their fundamental nature have not been investigated.

1.2. *Findings*

We show that a 1D Ising system can be realized in the form of an elementary cellular automaton. Several interesting symmetries of the Ising system were revealed by the computational experiments performed in this work. Percolations through the system, studied by the difference and diffusion patterns, are found to exhibit complex, stochastic behavior. In addition, we introduce state progression graphs of these finite computational systems that lend new interpretations of the evolution of entropy and raise new questions of their graph theoretical structure. These graphs show the causal dynamics of a system through time in relation to the system's entropy and provide a unique tool in the computational study of interacting systems.

1.3. *Map of the paper*

In what follows, each emergent methodology for studying the Ising system will be encapsulated in its own section. Each methodology is introduced in an overview subsection, wherein a method is briefly elucidated in terms of its functioning and meaning. The following subsection details a procedural construction of the method for its reproduction. Each method's section is concluded with a discussion of results gathered through its use. The methods considered as a whole are discussed in the concluding section of the paper.

2. An Elementary Cellular Automaton Model

2.1. *Overview*

The emergent model used in this work is expressed as a cellular automaton (CA). The progression rules in Fig. 1 describe the evolution of a cell's state influenced solely by its nearest neighbors, and not by a global influence. Ambivalent cases are reconciled by symmetry considerations. Essentially, if a member's neighbors agree in orientation, then the member will acquiesce.

The elementary CA model shown in Fig. 1 maps onto the 1D Ising model with only nearest-neighbor interactions. The "J" coupling between states (spins) are not adjusted here (equivalent to being fixed to unity).

An Ising system with a global influence can be constructed by altering the progression rule described above. With a global influence, a member more easily acquiesces to neighbors that are in agreement with it.

2.2. *Procedure*

The model employs a Mathematica platform to implement and study the elementary cellular automata. The progression rules described graphically above may be defined as mathematical functions. The value $a(t, i)$ for a member at position i on time step t can be defined as

$$a(t, i) = f\{a(t-1, i-1), a(t-1, i), a(t-1, i+1)\}\,, \tag{1}$$

where various progression rules correspond to different choices of the function f. The CA model given in Fig. 1 corresponds to

$$f(x, y, 0) = 0 \quad \text{and} \quad f(x, y, 1) = 1\,, \tag{2}$$

Fig. 1. Representation of a cellular automaton rule set that is isomorphic to an Ising system in isolation. The three top cells represent the cell of interest and its nearest neighbors. Open (0) and filled (1) cells denote one of the two possible states. The lone bottom cell shows the resulting state of the cell of interest.

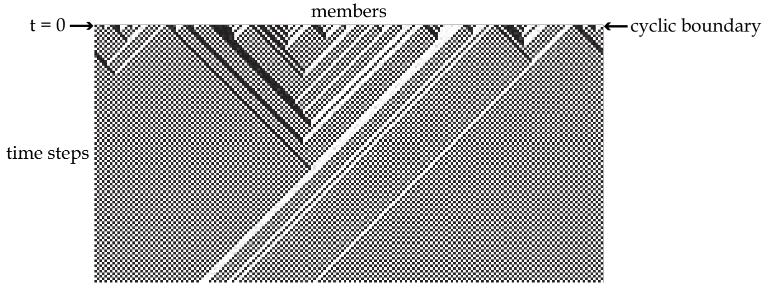

Fig. 2. The emergent behavior of the CA in Fig. 1 given a disordered initial condition of 200 members long played out for 100 progressions. The spatial dimension runs left to right; the temporal, top to bottom. The spatial dimension has cyclical boundary conditions.

where x and y represent either the value 1 or 0. This progression of number sets can be fully enumerated as

$$\{0,0,0\} \to 0\,, \qquad \{0,0,1\} \to 1\,, \qquad \{0,1,0\} \to 0\,, \qquad \{0,1,1\} \to 1\,,$$
$$\{1,0,0\} \to 0\,, \qquad \{1,0,1\} \to 1\,, \qquad \{1,1,0\} \to 0\,, \qquad \{1,1,1\} \to 1\,.$$

The 1D cellular automata in this study have cyclical boundary conditions, where the left-most cell is treated as being physically adjacent to the right-most cell.

2.3. *Discussion*

The emergent behavior of an Ising system at zero field, depicted in Fig. 2, exhibits grain boundary behavior. Small grains progressively coalesce into larger grains. While most of the members tend toward the homogeneity of alternating neighborly states, certain grains are able to persist indefinitely. Throughout the time evolution of states, the total number of black cells remains constant, as do the white. This is an expected consequence of conservation considerations given that the system is isolated.

For scenarios involving a global influence, the CA progression rule-set in Fig. 1 is altered. Figure 3(a) depicts the weakest augmentation of a global influence. Grain boundary behavior is again seen. Of course, the system eventually reaches a state in accordance with the global influence. The time it takes for a system to reach saturation of one state is studied later.

Figures 3(b) and 3(c) depict the expected evolution with progressively increasing global (field). Note that each panel in Fig. 3 represents a unique, augmented rule-set.

3. Difference Patterns

3.1. *Overview*

Difference patterns facilitate analysis of how a single bit of information percolates through an Ising system. Given the emergent behavior of a system seeded by a

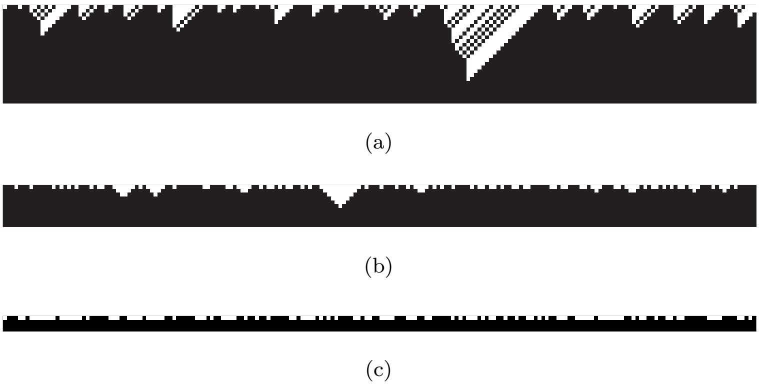

Fig. 3. From (a) to (c) are progressively augmented CAs given a disordered initial condition 200 members long. They are each cut off once they fully acquiesce to the external influence.

particular initial condition, the seeding may be altered by a single cell variation and contrasted with the emergent behaviors of the two instances: hence, the term "difference pattern".

The transient characteristics of a CA can be readily studied using difference patterns. As the transient behavior of information flow through a difference pattern halts, so too does the global transient behavior of the CA.

3.2. *Procedure*

A difference pattern may be constructed as follows:

(i) seed a CA; record the seeding and its emergence;
(ii) alter the seeding accordingly; record a new emergence;
(iii) subtract the two emergences.

The transient period of a difference pattern may be discerned by the following algorithm:

(i) construct a difference pattern of sufficient time length;
(ii) copy the difference pattern;
(iii) alter the copy by shifting each time instance by one spatial unit;
(iv) alter the original by removing its initial time instance;
(v) take the difference of the original and the copy;
(vi) measure the number of time steps that have a null value from the final time instance backward;
(vii) subtract this number from the total time steps.

3.3. *Discussion*

Figure 4 depicts erratic percolations that fluctuate with regard to direction, speed, and breadth. A difference pattern's "speed" refers to the horizontal distance a

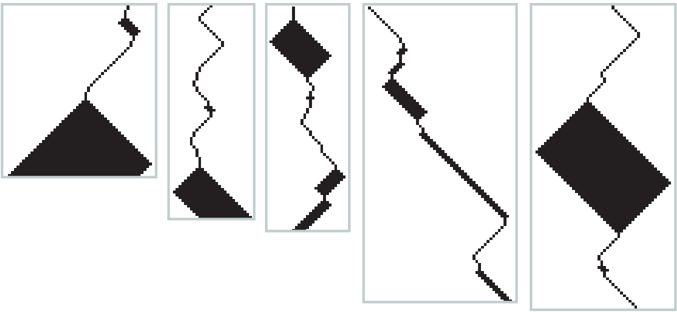

Fig. 4. Difference pattern of Ising system with no global influence. Each panel evolved from a different initial condition. Each evolution has been cut off where it begins to be predictable.

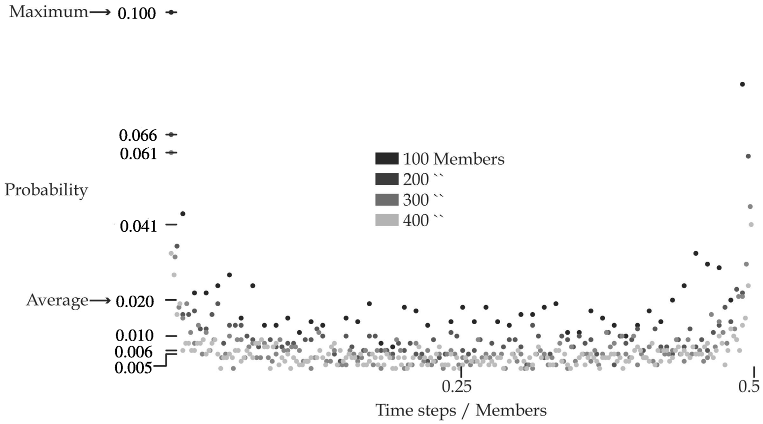

Fig. 5. Statistical analysis of the number of progressions required before a difference pattern becomes predictable (steady state) with no global influence and various members. All simulations were performed for 1000 timesteps. The upper four probabilities ticked off are the maximum values for each experiment. The lower four probabilities are the average values.

pattern travels in a single time step. Its "breadth" refers to the horizontal width at a particular instant.

The erratic behavior of a percolation eventually extinguishes, giving way to perpetual predictability. Figure 5 depicts a statistical analysis of the time it takes for an isolated Ising system to reach a steady state. The probability is defined as the number of evolutions that reached steady state at a particular time step normalized by the total number of evolutions observed. At a time step equal to half the number of members in the system, there is a time horizon beyond which the system will be at a steady state with absolute certainty. The probability of reaching a steady state spikes at a time near zero and a time near the horizon. The probability density function appears to be of a stochastic nature.

Figure 6 repeats the analysis shown in Fig. 5 but on a system with a small external field. Regardless of the number of members in the system, the probability

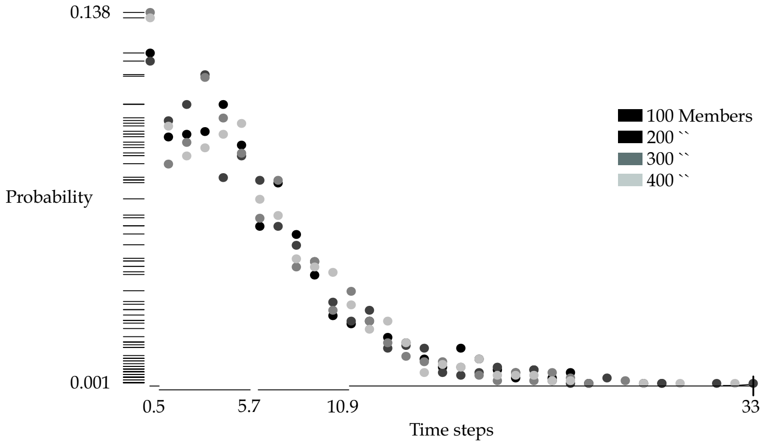

Fig. 6. Statistical analysis of the number of progressions required before the difference pattern becomes predictable with minimal global influence and various members. The statistical analysis of each system with a fixed number of members was performed over 1000 experiments. The average time to equilibrium and the standard deviation are shown on the abscissa. The extreme values of the data are shown on the ordinate.

density function (PDF) of the transient period decreases through what appears to be an inflection point and then asymptotically approaches zero. The deviation of the PDF around its expected value is not negligible close to time equal to zero.

4. Diffusion Patterns

4.1. *Overview*

Whereas a difference pattern is made by contrasting two instances differing by a single change in initial conditions, diffusion patterns are made by numerous such contrasts. The result is an intuitive visual representation of the statistical characteristics of how information flows through a system.

4.2. *Procedure*

A diffusion pattern may be constructed by flipping one member of the initial condition at a time and then combining the emergence of each case. This process may be described algorithmically as follows:

(i) construct a seed; record the emergence;
(ii) copy the original seed and alter it by one bit; record its emergence;
(iii) alter the original seed by a different bit than the previous; record its emergence;
(iv) repeat until every bit has been treated;
(v) sum all of the emergences.

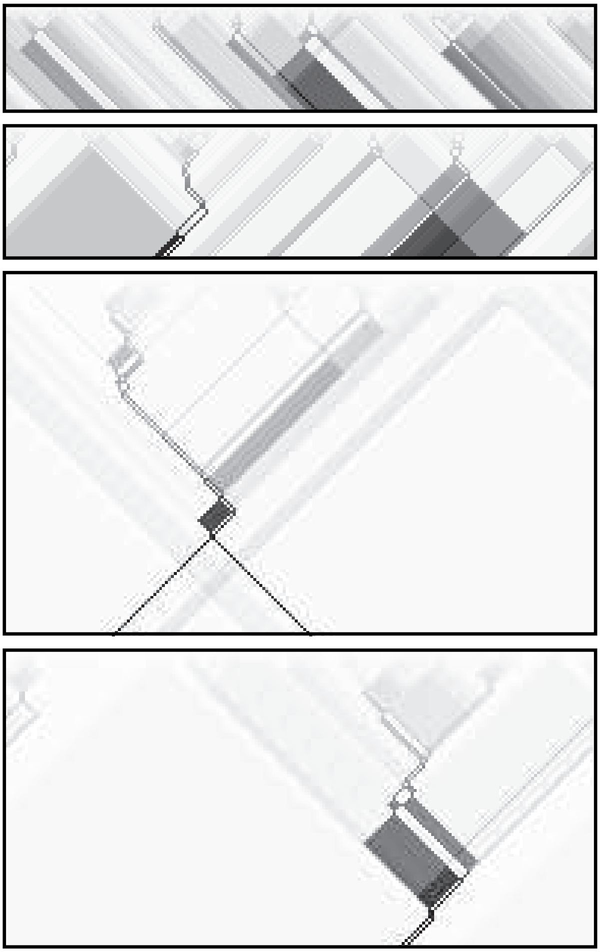

Fig. 7. Diffusion patterns of the Ising system in isolation, 200 members, 100 progressions, constructed by summing difference patterns focused on different initial members. Progressing down the figure are cases with heightening predilections for the information flows to converge to a single stream.

4.3. *Discussion*

Figure 7 depicts several diffusion patterns of the Ising system in isolation. Progressing down the figure are cases with heightening predilections for the information flows to converge to a single stream.

5. State Progression Graphs

5.1. *Overview*

State progression graphs show how systems advance through their states of affairs. The entropy of a system is intuited by a mere glance of a state graph. State graphs may be constructed over a finite set of members, constituting a particular system.

More interestingly, a finite set of members may be considered to constitute not an actual system in reality, but rather a contiguous subset of a system isolated artificially by an observer. Consequentially, a state graph made up of the union of subsets represents the system independent of observation.

5.2. *Procedure*

State progression graphs are produced by applying a rule-set to a prescribed state with a particular number of members and recording the resultant causal relationship. After collecting all of the possible causal relationships for a system of a given size, a graph like the one shown in Fig. 8 can be produced.

State graphs irrelevant to observation are made similarly, but require a mapping of analogous states of affairs. For instance, a system of four members might take on the state (0, 0, 0, 1). This state is equivalent to (0, 0, 1) in a system of three members. Of course, the required mapping is only a statement of the equivalence between binary numbers with arbitrarily many leading zeros.

5.3. *Discussion*

In the vernacular graph theory, a "community" is a connected subgraph.

Figure 8 depicts the state graph of the Ising system with eight members in isolation, consisting of several communities. The simplest community is trivial, wherein a state gives rise to itself. The next least trivial community has a periodic nature shown in Fig. 8(d). A variation on the latter community has certain states that can only be accessed as initial conditions, proceeded by periodicity shown in Figs. 8(b) and 8(c). The most interesting community is displayed in Fig. 8(a).

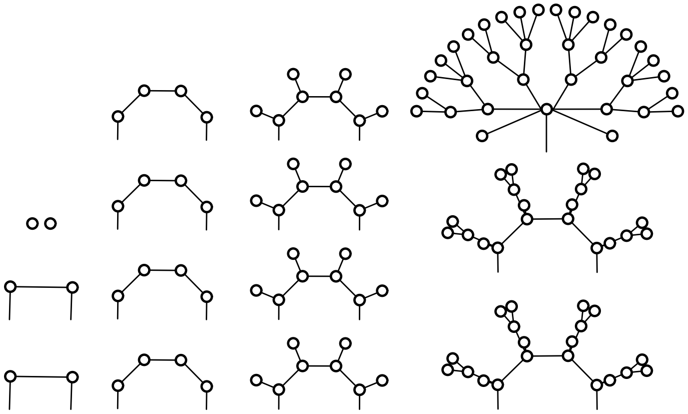

Fig. 8. State progression graphs for an eight-member 1D Ising system in isolation. Each community is symmetric; so, only half of each is shown.

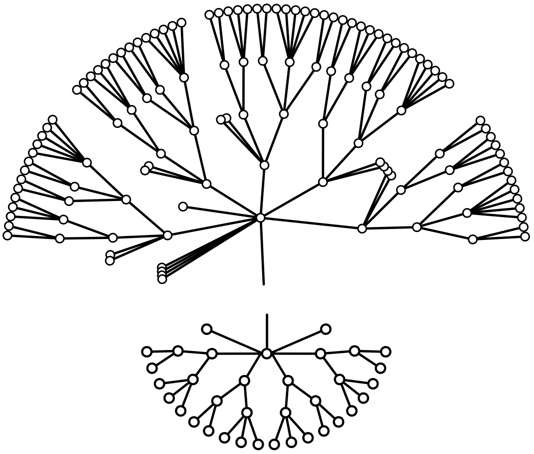

Fig. 9. Interesting communities of ten (top) and eight (bottom) member state progression graphs of the Ising system with no global influence.

Within the interesting community, the system tends toward periodicity between two states that may be reached in numerous different ways. The community is symmetric over reflection about the two fluctuating states. The final two states over which it oscillates are symmetric over a shift by one spatial unit.

The less interesting communities result in a periodic progression over a number of states equal to the number of members in the system or lesser factors of that number. In the instance of Fig. 8, there are eight members in the system; so, the majority of the communities result in periodicity over eight or four states. Interesting communities only arise in systems with an even number of members.

Figure 9 depicts interesting communities of eight and ten members. Both are symmetric over reflection about the two fluctuating states.

Figure 10 depicts interesting communities of an Ising system with eight members and various levels of global influence.

Figure 11 depicts the observation-independent state graph of an Ising system with eight members in isolation. An "observation-independent" CA with rule-set r is the union of N evolutions each with i members, $\bigcup_i^N \mathrm{CA}(r, i)$. Further investigations into the nature of these graphs is necessary and will be pursued in a future work.

6. Conclusions

The 1D Ising system's fundamental nature was partially unfurled with several new interpretation tools. The scope of this investigation was to introduce these tools and present a case for their merit. No analytical models, or discretizations thereof, were used in the study. Nor were any artificial complexities imposed on the cellular automata. Instead, cellular automata were studied in their own right as systems equivalent to Ising systems.

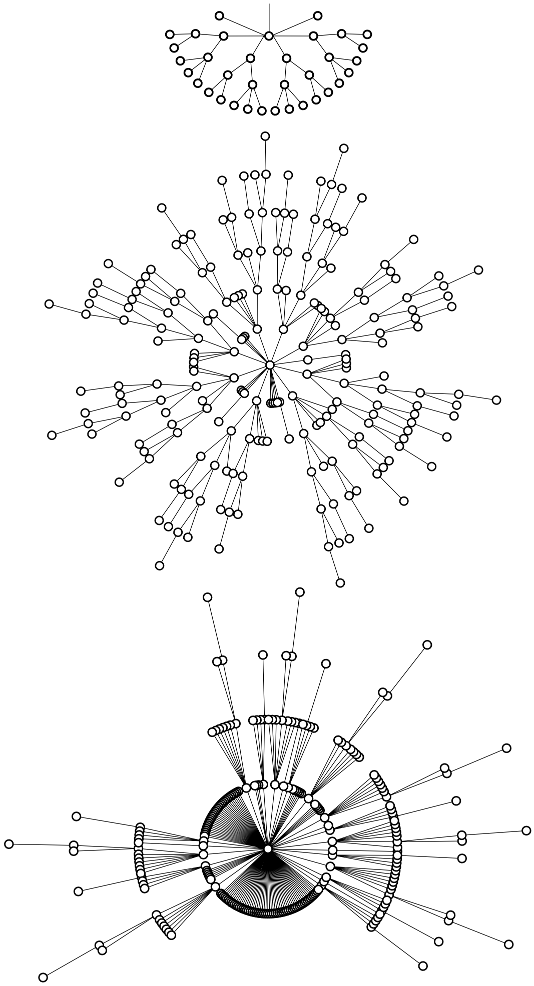

Fig. 10. Interesting communities of eight member state progression graphs with increasing external field from zero (top) to higher values (bottom). The highest global influence scenario is omitted.

Despite the simplicity of the systems studied, each interpretation revealed underlying complexity and interesting symmetries. More questions have been raised by this investigation than were answered. What follows are suggestions for future study:

- Rigorous characterization of the transient behavior of the 1D Ising system;
- Develop physical implications of diffusion patterns;
- Graph theoretical analysis of state progression graphs;
- Generalization of the cellular automata in terms of dimensionality and range of neighborly influence.

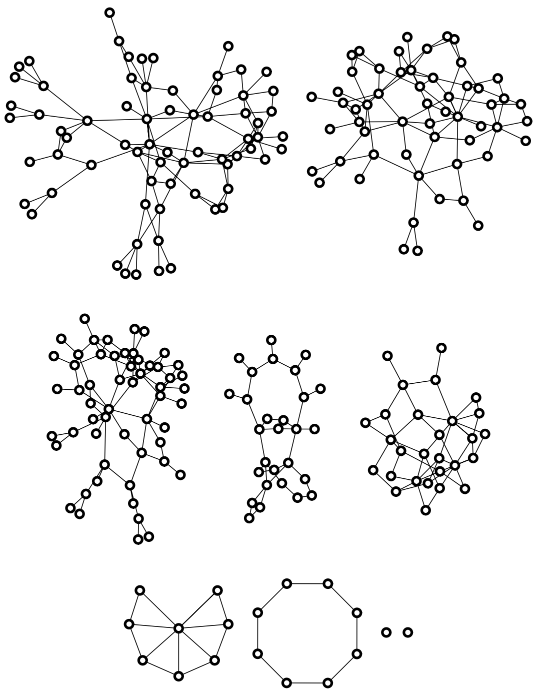

Fig. 11. Observation-independent Ising system with eight members in isolation.

For those who pursue further research in these matters, it is of the utmost importance to internalize the new intuition that "simple rules do not necessarily lead to simple behavior."

Acknowledgments

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