TRIGONOMETRY

Units: $\pi$ radians $\cong 3.14159265$ rad = 180 degrees = $180^\circ$
full (complete) circle = $2\pi = 360^\circ$

Special Values:

<table>
<thead>
<tr>
<th></th>
<th>0°</th>
<th>30° ($\pi/6$)</th>
<th>45° ($\pi/4$)</th>
<th>60° ($\pi/3$)</th>
<th>90° ($\pi/2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(\theta)$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\cos(\theta)$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\tan(\theta)$</td>
<td>0</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>1</td>
<td>$\sqrt{3}$</td>
<td>$\pm \infty$</td>
</tr>
</tbody>
</table>

Derivatives:

\[ \frac{\partial}{\partial x} \sin(x) = \cos(x) \]
\[ \frac{\partial}{\partial x} \cos(x) = -\sin(x) \]
\[ \frac{\partial}{\partial x} \tan(x) = \sec^2(x) \]

Chain rule:

\[ \frac{\partial}{\partial x} \sin(u) = \cos(u) \frac{\partial u}{\partial x} \]
example: \[ \frac{\partial}{\partial x} \sin(2x) = 2 \cos(2x) \]

Sum Rules:

\[ \sin(A \pm B) = \sin(A) \cos(B) \pm \sin(B) \cos(A) \]
\[ \cos(A \pm B) = \cos(A) \cos(B) \mp \sin(A) \sin(B) \]
\[ \tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A) \tan(B)} \]
\[ \sin^2(A) + \cos^2(B) = 1 \]
\[ \sin(2A) = 2 \sin(A) \cos(A) \]
\[ \cos(2A) = 2 \cos^2(A) - 1 = 1 - 2 \sin^2(A) = \cos^2(A) - \sin^2(A) \]

Taylor Series:

\[ f(x) = f(0) + f'(0)x + f''(0)x^2/(2!) + \ldots \text{ where prime denotes a derivative} \]

Examples:

\[ \sin(x) = x - x^3/(3!) + x^5/(5!) + \ldots \]
\[ \cos(x) = 1 - x^2/(2!) + x^4/(4!) + \ldots \]
\[ \tan(x) = x + x^3/(3!) + 2x^5/15 + \ldots \]
\[ (1 + x)^{\frac{1}{2}} = 1 + x/2 - x^2/8 + \ldots ; \]
\[ (1 + x)^b = 1 + bx + b(b-1)x^2/(2!) + \ldots \]
\[ \exp(x) = 1 + x + x^2/(2!) + x^3/(3!) + \ldots \]

These are especially important when $x$ is small (typically $x \ll 1$).
In this case, $\sin(x) \cong x$, $\cos(x) \cong 1$, etc.
COMPLEX NUMBERS

Definition:

$Z = a + jb$ is a Complex Number if:

1) $a$ and $b$ are real numbers
   
   $a \equiv \text{Re}(Z)$ is the real part of $Z$
   
   $b \equiv \text{Im}(Z)$ is the imaginary part of $Z$

2) $j$ is a solution to $j^2 + 1 = 0$ or $j^2 = -1$

Geometry:

Any complex number can be described by a point in a two-dimensional coordinate system. Consider the complex number $Z = 5.0 + j5.0$;
We could also use polar coordinates to locate $Z$:

$$Z = R \cos(\theta) + j R \sin(\theta) = R [\cos(\theta) + j \sin(\theta)],$$

where $R = (a^2 + b^2)^{\frac{1}{2}}$ and $\theta = \tan^{-1}(b/a)$.

**Derivatives:**

Consider the derivative of the function $f(\theta) = \cos(\theta) + j \sin(\theta)$:

$$\frac{\partial f(\theta)}{\partial \theta} = -\sin(\theta) + j \cos(\theta) = j [\cos(\theta) + j \sin(\theta)]$$

Thus, we find the derivative of $f(\theta)$ with respect to $\theta$ gives $j f(\theta)$. Can you think of another function that satisfies $\frac{\partial f(\theta)}{\partial \theta} = j f(\theta)$?

Consider the exponential function $\exp(c \theta) = e^{c\theta}$,

$$\frac{\partial (e^{c\theta})}{\partial \theta} = c e^{c\theta}.$$ 

So, if we let $c = j$, then this has the same property upon differentiation as $f(q)$ above.
It can be rigorously proven (by expanding both sides below in a Taylor series) that:

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \quad \text{(which is Euler’s identity)} \]

Thus, we have three ways of writing a complex number \( Z \):

\[ Z = a + jb \]
\[ Z = R[\cos(\theta) + j\sin(\theta)] \]
\[ Z = R \, e^{j\theta} \]

**Some Operations with Complex Numbers:**

Addition and Subtraction –

\[ Z_1 \pm Z_2 = (a_1 + jb_1) \pm (a_2 + jb_2) = (a_1 \pm a_2) + j(b_1 \pm b_2) \]

Multiplication –

\[ Z_1 \cdot Z_2 = (a_1 + jb_1) \cdot (a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \]

or

\[ Z_1 \cdot Z_2 = R_1 \, e^{j\theta_1} \cdot R_2 \, e^{j\theta_2} = R_1 \cdot R_2 \, e^{j(\theta_1 + \theta_2)} \]

Note how much easier multiplication is using the exponential form.

Now consider Euler’s identity above and replace \( \theta \) by \(-\theta\) to give

\[ e^{-j\theta} = \cos(\theta) - j\sin(\theta) \]

Adding these two equations together yields

\[ \cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \]

Subtracting these two equations yields

\[ \sin(\theta) = \frac{1}{2}j(e^{-j\theta} - e^{j\theta}) \].
Examples:

For what $\theta$ is $e^{j\theta} = 1$? Euler’s identity tells us that for $\theta = 0, 2\pi, -2\pi, 4\pi, -4\pi, \ldots$; $e^{j\theta} = 1$. For what values of $\theta$ will $e^{j\theta}$ equal (a) $-1$, (b) $j$, and (c) $-j$? (Try and figure it out on your own). The answers are:

(a) $\pi, -\pi, 3\pi, -3\pi, \ldots$
(b) $\pi/2, -3\pi/2, 5\pi/2, \ldots$
(c) $3\pi/2, -\pi/2, 7\pi/2, -5\pi/2, \ldots$

You know how to graph complex numbers such as $Z_1 = 2 e^{j\pi/3}$ now (I hope!). But how about $Z_2 = -2 e^{j\pi/3}$? If we write $Z_1 = a + jb$, then $Z_2 = -a - jb$ or we could write $Z_2$ as

$$Z_2 = (-1) 2 e^{j\pi/3} = (e^{\pi}) 2 e^{j\pi/3} = 2 e^{j(\pi + \pi/3)}$$

(This tells us to add $\pi$ to the old angle $\pi/3$ to find the new direction)

$$Z_2 = 2 e^{j4\pi/3}$$

For instance:
Now, how about $Z_2 = jZ_1$? Here, $jZ_1 = j(a + jb) = -b + ja$

OR

$$jZ_1 = j^2 e^{j\pi/3} = (e^{j\pi/2}) 2 e^{j\pi/3} = 2 e^{j\pi/2 + \pi/3}$$

(This tells us to add $\pi/2$ to the old angle of $\pi/3$ to find the new direction)

$$jZ_1 = 2 e^{j5\pi/6}$$
For instance:

1) Multiplication by \(-1\) rotates the “vector” by 180° (or \(\pi\))
2) Multiplication by \(j\) rotates the “vector” by 90° (or \(\pi/2\))
Complex Conjugation:

The complex conjugate of $Z = a + jb$ is defined to be $Z^* = a - jb$
OR $Z = R \ e^{j\theta}$ and $Z^* = R \ e^{-j\theta}$. Thus, to form the complex conjugate of $Z$, just change the sign of $j$ (i.e., $j \rightarrow -j$).

Note that $Z \cdot Z^*$ is always real.

When we write $Z$ in exponential form, $Z = R \ e^{j\theta}$, powers of $Z$ are easy to find:

$$Z^n = R^n \ e^{jn\theta}$$

For instance, to find the $1/3$ power of 1, we recognize that $1^{1/3} = e^{j(2\pi)1/3} = e^{j2\pi/3}$. But, 1 is also equal to $e^0$ and $e^{j4\pi}$, so $1^{1/3}$ also equals 1 and $e^{j4\pi/3}$. All together there are three possible answers, which is another way of saying that 1 has three cube-roots! (similarly four $4^{th}$-roots, five $5^{th}$-roots, and so on)