A REALLY ELEMENTARY PROOF OF THÉBAULT'S THEOREM

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An exceptionally beautiful Theorem with a very elementary statement is Thébault Theorem:

Theorem. Let P be a point on the side BC ofthe triangle ABC. Let I be the incenter of $\overset{\triangle}{ABC}$ and O_1, O_2 be thecenters of the circles c_1, c_2 which touch the sides of $\overset{\triangle}{BPA}$, $\overset{\triangle}{CPA}$ respectively and which further both touch internally the circumference c of $\overset{\triangle}{ABC}$. Then$I$ lies on the line O_1O_2.

Unfortunately, despite its elementary statement all known proofs lie beyond the grasp of a high school student or even of an average college student majoring in mathematics.

The object of this paper is to provide a really concise and elementary proof of the theorem within the frame of its "natural environment", i.e. Euclidean Geometry.

A concise history of the theorem can be found in [1]. It was proposed as a conjecture by Victor Thébault in 1938 ([2]). The first proof was given in 1983 ([3]) but only a brief summary of the 24 pages of the lengthy calculations of the proof was ever published. Most of the proofs given since then ([1], [4] and others) were heavily based on the aid of computers. Proofs using a mixture of Euclidean and projective methods can be found in [5], [6], [7]. Beyond its long history and beauty this Theorem also has an almost benchmark problem status in Groebner theory.

The proof will be given through a series of seven small claims. The first of them provides useful information about the relation among the tangency points of some circles and lines in special position. The next three claims consist of the core argument of the proof. The last four deal with purely technical details (a bunch of trivialities) settling down the question of how a picture of the situation should look like. Accordingly, the paper splits in two parts: claims 1 to 4 consisting the "CORE ARGUMENT OF THE PROOF", and claims 5 to 8 consisting the "TECHNICAL DETAILS". Although these details are necessary for a solid mathematical proof, they can be omitted without any real loss.

CORE ARGUMENT OF THE PROOF

Claim 1. *Let N_i, K_i be the common points of c_i with c and BC respectively (for $i = 1, 2$), and let us call M the midpoint of the arc BC of c, which does not contain A. Then the points M, K_i, N_i are collinear (Figure 1).*

Claim 2. *If K_iI intersects c_i again at L_i then the points A, I, N_i, L_i are concyclic ($i = 1, 2$) (Figure 2a).*

Claim 3. *The line AL_i is tangent to c_i ($i = 1, 2$) (Figure 2b).*

Claim 4. *Theorem 1 holds.*

Proof. of Claim 1 for $i = 1$ (and similarly for $i = 2$):

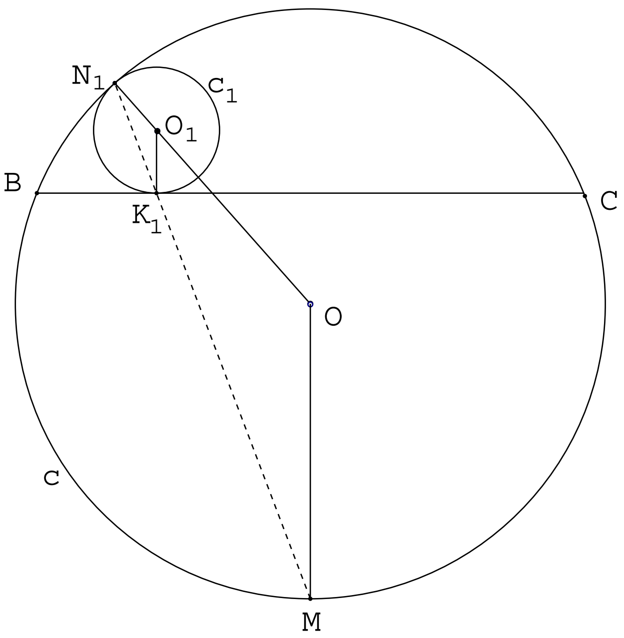

Figure 1

It is a well known result that the points O, O_1, N_1 are collinear (Figure 1), and each one of $K_1\overset{\triangle}{\widehat{O}}_1N_1$, $M\overset{\triangle}{\widehat{O}}N_1$ is isosceles, thus

(1) $$O_1\widehat{N}_1K_1 = (180^0 - N_1\widehat{O}_1K_1)/2 \text{ and } O\widehat{N}_1M = (180^0 - N_1\widehat{O}M)/2$$

But $O_1K_1 \parallel OM$ (since they are both perpendicular to BC). So

(2) $$N_1\widehat{O}_1K_1 = N_1\widehat{O}M$$

$[(1),\ (2)] \Longrightarrow O_1\widehat{N}_1K_1 = O\widehat{N}_1M \Longrightarrow O\widehat{N}_1K_1 = O\widehat{N}_1M$ and since the half-lines N_1K_1, N_1M lie at the same half-plane with respect to the line N_1O, we conclude that they coincide, thus the points K_1, N_1, M are collinear as claimed. □

Proof. of Claim 2 for $i = 1$ (and similarly for $i = 2$):

Notice that there exists a common tangent $x'N_1x$ of c_1, c at N_1, and let B, x' be at the same half-plane with respect to OO_1 (Figure 2a). It is

$N_1\widehat{A}I = N_1\widehat{A}M = x'\widehat{N}_1M \overset{\text{Claim 1}}{=} x'\widehat{N}_1K_1 = N_1\widehat{L}_1K_1$ thus the points N_1, A, I, L_1 are concyclic. □

Proof. of Claim 3 for $i = 1$ (and similarly for $i = 2$):

Since by the previous Claim, the points N_1, A, I, L_1 are concyclic we have (Figure 2b):

(3) $$N_1\widehat{L}_1A = N_1\widehat{I}A = 180^0 - N_1\widehat{I}K_1 - K_1\widehat{I}M$$

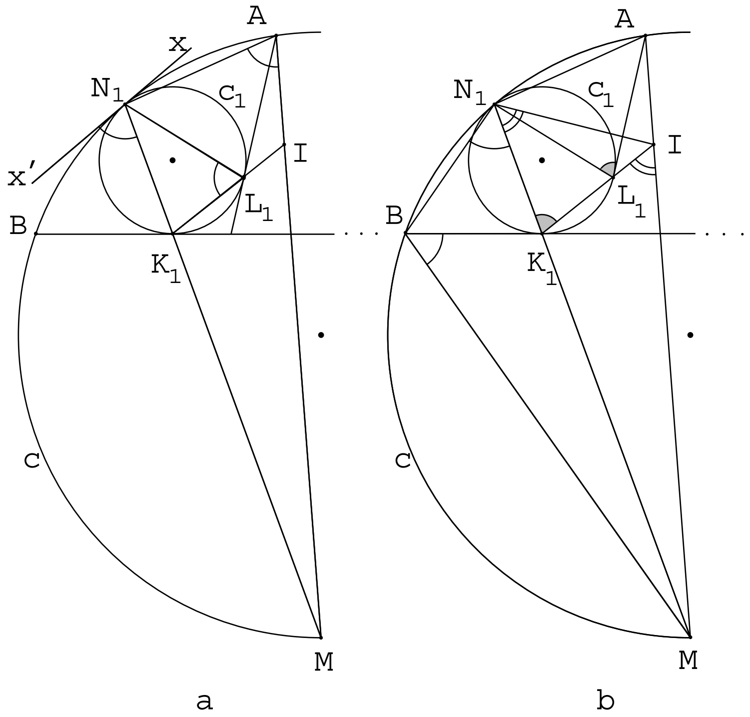

FIGURE 2

Let's observe that the line MB is tangent to the circumference of $B\overset{\triangle}{N_1}K_1$ because $M\widehat{B}K_1 = \widehat{A}/2 = B\widehat{N_1}M \stackrel{\text{Claim1}}{=} B\widehat{N_1}K_1$. Thus

$$MB^2 = MK_1 \cdot MN_1 \tag{4}$$

But a well known property of the incenter I of $A\overset{\triangle}{B}C$ is that

$$MI = MB \tag{5}$$

(4),(5) imply $MI^2 = MK_1 \cdot MN_1$. This means that MI is tangent to the circumference of $K_1\overset{\triangle}{N_1}I$, thus

$$K_1\widehat{I}M = K_1\widehat{N_1}I \tag{6}$$

[(3), (6)] imply $N_1\widehat{L_1}A = 180^0 - N_1\widehat{I}K_1 - K_1\widehat{N_1}I$ and looking at $K_1\overset{\triangle}{N_1}I$ the last relation can be written as $N_1\widehat{L_1}A = N_1\widehat{K_1}L$. But this means that AL_1 is tangent to c_1, QED. □

Proof. of Claim 4:

Since K_1L_1, K_2L_2 are the tangent chords of c_1, c_2, with the sides of $B\overset{\triangle}{P}A, C\overset{\triangle}{P}A$ respectively, we have (Figure 3): $O_1P \perp K_1L_1, O_2P \perp K_2L_2$ and $O_1P \perp O_2P$.

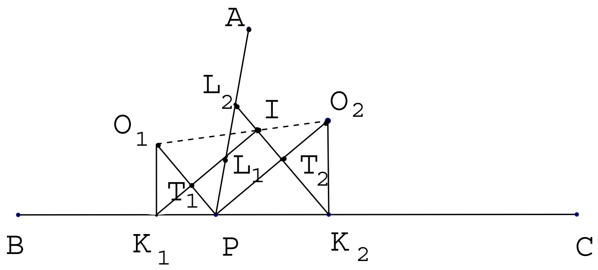

FIGURE 3

Calling $T_1 = O_1P \cap K_1L_1$ and $T_2 = O_2P \cap K_2L_2$, we have then that IT_2PT_1 is a (rectangular) parallelogram, and so

$$IT_2 = T_1P \tag{7}$$

We also have that the acute angles $O_1\widehat{P}K_1, P\widehat{O_2}K_2$ of the triangles O_1K_1P, O_2K_2P are equal. This implies that the triangles are similar. Since T_1, T_2 are the projections of the vertices of the right angles to the hypotenuses of these triangles, their similarity implies $O_1P/T_1P = O_2P/T_2O_2$ and then (7) gives

$$O_1P/IT_2 = O_2P/O_2T_2 \tag{8}$$

But O_1P is parallel to IT_2. Then according to the converse of Thales' Theorem, relation (8) implies that O_1, I, O_2 are collinear as wanted. □

TECHNICAL DETAILS

The above proof of Claim 4 (i.e. of Thébault's Theorem) is all perfect except for a tiny detail; namely, that one cannot take it for granted that (for $i = 1, 2$) the tangent AL_i of A to c_i coincides with the tangent AP of A to c_i. It could be the case that the tangents AL_i, AP of A to c_i are different for one or even both indices $i = 1, 2$! That this is not so is the end result of the four Claims (5-8) below.

Claim 5. *I lies outside one of c_1, c_2 and inside the other (Figure 4), or it lies on both of them.*

Note that whenever the point I lies on both c_1, c_2, it is necessary a common tangent point for both circles with the line AP. Of course in this case Thébault's Theorem is a triviallity and we are not dealing further with it anywhere in this paper. By the way, note also that in proving Claims 3, 4 we dealt only with circle c_1, in which case according to a silent convention the point I is interior to the circle and to the segment AP. But nothing changes if we deal with the circle c_2 as well.

Claim 6. *The line AL_i intersects the line BC in a point of the segment BC (Figure 5).*

Claim 7. *The tangent line from A to c_i other than AP ($i = 1, 2$) does not intersect the line BC in a point of the segment BC.*

Claim 8. *Line AP coincides with $AL_i, (i = 1, 2)$.*

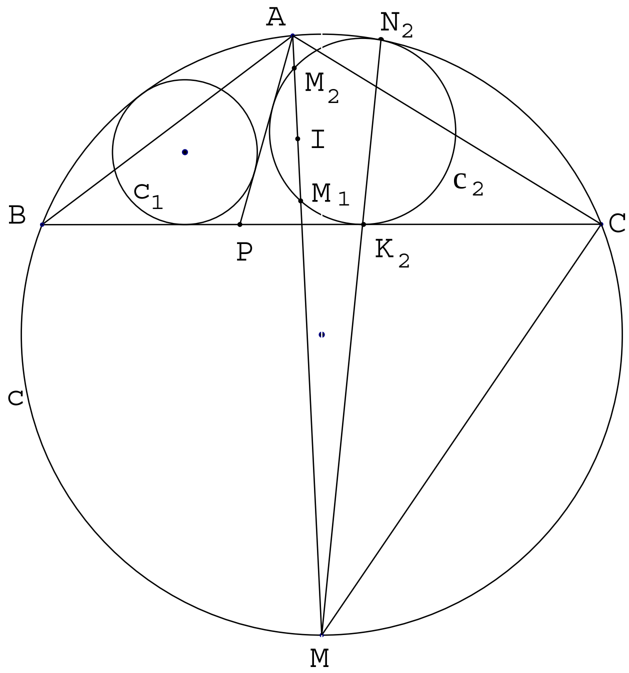

FIGURE 4

Proof. of Claim 5:

Of course it is true that I lies on the segment AM.

The half-line AIM belongs to one of $B\widehat{A}P, C\widehat{A}P$, let this be $C\widehat{A}P$, as in Figure 4. Then, since c_1 lies in the exterior of $C\widehat{A}P$, the point I lies outside c_1.

Notice now that line AC leaves the points N_2, K_2 of c_2 in different half-planes, thus it has to intersect c_2. The part of c_2 in the half-plane with respect to AC in which K_2 lies, belongs entirely in the angle $P\widehat{A}C$ touching both sides of the angle. Thus the half-line AIM which belongs to the $P\widehat{A}C$ has to intersect c_2; say at points M_1, M_2.

Since by Claim 1 the point K_2 lies in the segment N_2M, the power of M with respect to c_2 is

$$MM_1 \cdot MM_2 = MK_2 \cdot MN_2 \tag{9}$$

Noticing now that $M\widehat{C}K_2 = M\widehat{C}B = M\widehat{A}B = C\widehat{A}B/2 = C\widehat{A}M = C\widehat{N}_2M$, we conclude that CM is tangent to the circumference of $\overset{\triangle}{CN_2K_2}$ which translates to

$$MK_2 \cdot MN_2 = MC^2 = MI^2 \tag{10}$$

(the last equality is a well known one).

(9),(10) imply that $MM_1 \cdot MM_2 = MI^2$ which in turn assures us that I is a point of the segment M_1M_2. Thus I lies inside c_2.

Whenever the line AM coincides with the line AP (Figure 7), if we call I' the tangent point of line AP with c_1, then $MI'^2 = MK_1 \cdot MN_1$. But the triangles

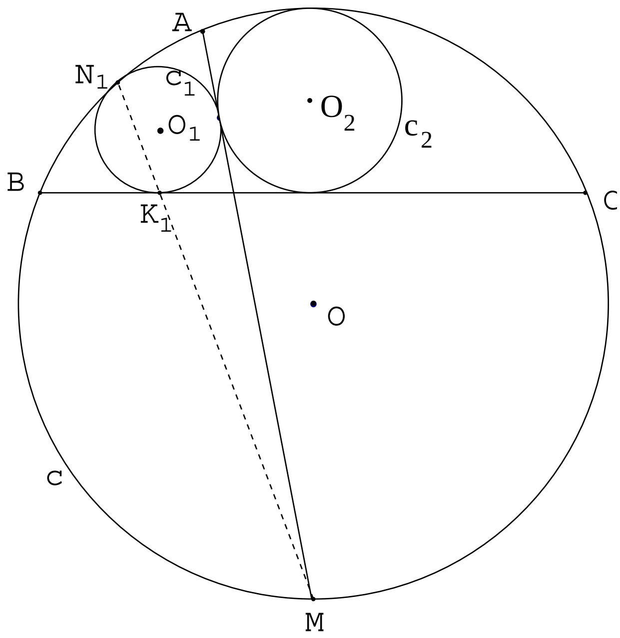

FIGURE 5

MBK_1, MBN_1 are similar, thus $MK_1 \cdot MN_1 = MB^2$. We conclude $MB^2 = MI'^2$. Now note that as already mentioned, the incenter of $\overset{\triangle}{ABC}$ satisfies $MI^2 = MB^2$ implying $MI^2 = MI'^2$. So I has to coincide with I'.Then the second common point L_1 of the line K_1I with c_1 is I itself as claimed. We similarly prove that I coincides with L_2. □

For what follows we assume that I is interior to c_2 and exterior to c_1 as in Figure 4.

Proof. of Claim 6:

i) Line MN_1 splits c_1 into two arcs, one of them lying at the same half plane with respect to MN_1 as A does (Figure 6a). Call this arc c^1 and the half plane π. Let also E be the interior of c_1 lying in π.

I is a point of π, and by Claim 5 it is exterior to E. Thus trivially c^1 intersects the segment K_1I in one more point other than K. Of course this is the second point of intersection of c_1 with the line K_1I, called L_1.

Observe that the segment K_1I is an interior segment of $\overset{\triangle}{AK_1T}$ (where T is the intersection point of the segments BC and AIM). Thus the line AL_1 intersects the line K_1T in a point of the segment K_1T, so it intersects line BC at a point of the segment BC.

ii) Line AM splits c_2 into two arcs c^2, c^3 and let K_2 lie in c^3 (Figure 6b). Since I lies on the chord M_1M_2 of c_2 the line K_2I intersects c^2 in a point. This is of course the second common point of K_2I and c, called L_2. But c^2 lies in the interior of the

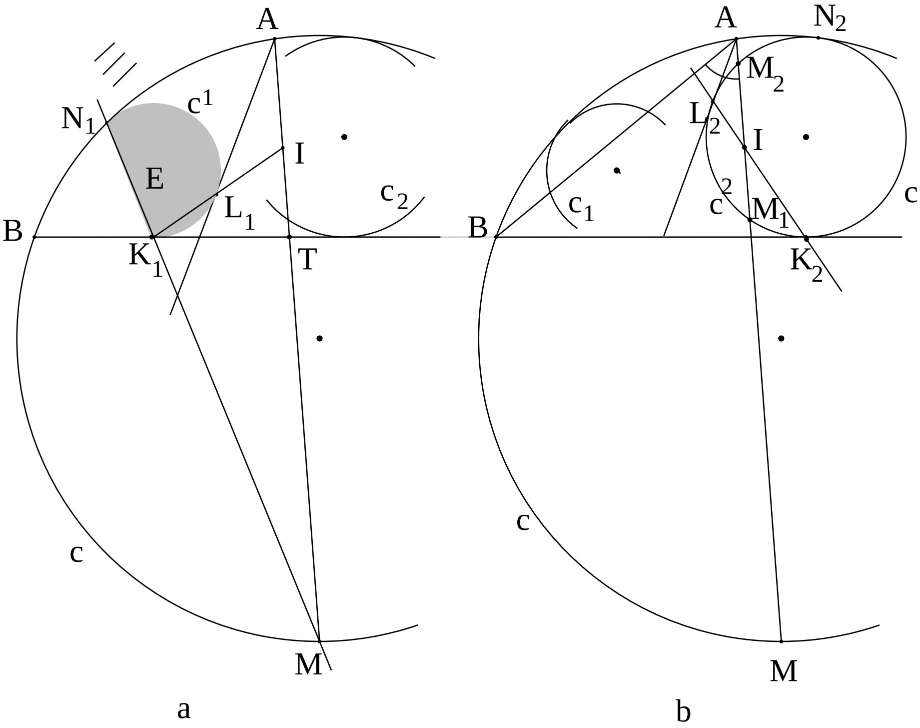

FIGURE 6

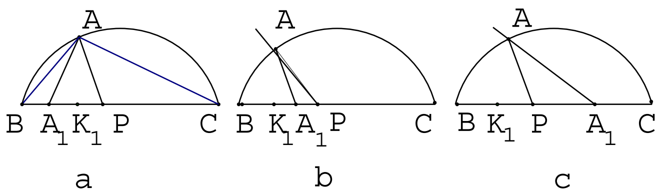

FIGURE 7

angle $B\widehat{A}T$ of $\overset{\triangle}{BAT}$. So L_2 does so. Then of course the line AL_2 intersects BC in a point of the segment BT, thus in a point of the segment BC. □

Proof. of Claim 7 for $i = 1$ (and similarly for $i = 2$):

If the second tangent from A to c_1 is parallel to BC we are done.

So let's assume that the second tangent from A to c_1 intersects the line BC at a point A_1.

-If we moreover assume for a moment that A_1 is a point of the segment BK_1 (Figure 7a), then since K_1 is an interior point of $\overset{\triangle}{AA_1P}$, the circle c_1 would be the incircle of $\overset{\triangle}{AA_1P}$. But $\overset{\triangle}{AA_1P}$ lying inside c makes impossible for its incircle c_1 to be tangent to c; a contradiction by the assumption of the Theorem.

-If instead we assume that A_1 is a point of the segment K_1P or of PC (Figures 7b,7c), then since K_1 is an exterior point of $\overset{\triangle}{AA_1P}$, the circle c_1 would be an excircle of $\overset{\triangle}{AA_1P}$. But c_1 also lies in the half-plane with respect to line BC as A does. So it touches the half-line PA (in the first case) or the half-line A_1A (in the second case) at a point beyond A, thus exterior to c. So then c_1 cannot be tangent to c internally as the assumption of the Theorem demands; a contradiction. QED □

Proof. of Claim 8:

The result is immediate because of Claims 3,6,7. □

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