

2 Exponents

Problem: Show that $4^{\frac{5}{2}} = 32$.

Solution: Remember, using our rules of exponents, $4^{\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^5 = (\sqrt{4})^5$. And, since $\sqrt{4} = 2$, $4^{\frac{5}{2}} = 2^5 = 32$.

Problems to Do:

1. Simplify each to a single fraction or number:

(a)

$$\begin{aligned}\frac{\frac{7}{3}}{\frac{4}{5} + \frac{9}{2 \cdot 3} + \frac{1}{15}} &= \frac{\frac{7}{3}}{\frac{4}{5} + \frac{9}{1} \cdot \frac{3}{2} + \frac{1}{15}} && \text{(Simplify } \frac{9}{2 \cdot 3} \text{ first)} \\ &= \frac{\frac{7}{3}}{\frac{4}{5} + \frac{27}{2} + \frac{1}{15}} \\ &= \frac{\frac{7}{3}}{\frac{4 \cdot 6}{30} + \frac{27 \cdot 15}{30} + \frac{1 \cdot 2}{30}} && \text{(Find common denominator to add fractions)} \\ &= \frac{\frac{7}{3}}{\frac{24 + 405 + 2}{30}} \\ &= \frac{\frac{7}{3}}{\frac{431}{30}} \\ &= \frac{7}{3} \cdot \frac{30}{431} && \text{(Divide by multiplying by the reciprocal)} \\ &= \frac{70}{431}\end{aligned}$$

(b)

$$\begin{aligned}(2/3)^3 \div ((1 - 7/8)^2 + 1/2) &= (2/3)^3 \div ((8/8 - 7/8)^2 + 1/2) \\ &\quad \text{(Work from inner parentheses outward)} \\ &= (2/3)^3 \div ((1/8)^2 + 1/2) \\ &= (8/27) \div (1/64 + 1/2) \\ &\quad \text{(Remember: } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\text{)} \\ &= (8/27) \div (1/64 + 32/64) \\ &\quad \text{(Find a common denominator)} \\ &= (8/27) \div (33/64) \\ &= \frac{8}{27} \cdot \frac{64}{33} \\ &\quad \text{(Divide by multiplying by the reciprocal)} \\ &= \frac{512}{891}\end{aligned}$$

(c) $(2 * 10^8)^3 = (2^3 * (10^8)^3) = 8 * 10^{24}$

(d)

$$\begin{aligned}(2 * 10^4)^4 \div (4 * 10^2)^2 &= (2^4 * (10^4)^4) \div (4^2 * (10^2)^2) \\ &= (16 * 10^{16}) \div (16 * 10^4) \\ &= \frac{16 * 10^{16}}{16 * 10^4} \\ &= \frac{16}{16} \cdot \frac{10^{16}}{10^4} \\ &= 1 \cdot 10^{16-4} = 10^{12}\end{aligned}$$

2. Simplify each expression below as much as possible; your answer should have positive exponents only.

(a)

$$\begin{aligned}(a^3b^{-2})^{-3}(a^2b^{-2})(a^{-2}b^{-1})^{-1} &= (a^{-9}b^6)(a^2b^{-2})(a^2b) \\ &= a^{-9+2+2}b^{6-2+1} \\ &= a^{-5}b^5 \\ &= \frac{b^5}{a^5}\end{aligned}$$

(b)

$$\begin{aligned}(x^{-1} - y^{-1})^{-1}(x^{-1}y^{-1})^{-1} &= \left(\frac{1}{x} + \frac{1}{y}\right)^{-1} (xy) \\ &= \frac{xy}{\left(\frac{1}{x} + \frac{1}{y}\right)} \\ &= \frac{xy}{\left(\frac{y}{xy} + \frac{x}{xy}\right)} \\ &= \frac{xy}{\frac{y+x}{xy}} \\ &= \frac{(xy)(xy)}{y+x} \\ &= \frac{x^2y^2}{y+x}\end{aligned}$$

(c)

$$\begin{aligned}(x+y)^{-2}(x+y)^3(x^2y^3)^{-2} &= (x+y)^{-2+3}(x^{-4}y^{-6}) \\ &= \frac{x+y}{x^4y^6}\end{aligned}$$

(d)

$$\begin{aligned}\left(\frac{\mathbf{a}^{-2}}{\mathbf{bc}^{-4}}\right)^{-2} \left(\frac{\mathbf{a}^2}{\mathbf{b}^{-1}\mathbf{c}^{-2}}\right)^{-3} &= \left(\frac{a^4}{b^{-2}c^8}\right) \left(\frac{a^{-6}}{b^3c^6}\right) \\ &= \frac{a^{-2}}{bc^{14}} = \frac{1}{a^2bc^{14}}\end{aligned}$$

3. Please simplify each of the following expressions as much as possible.

(a) $125^{2/3} = (125^{\frac{1}{3}})^2 = (\sqrt[3]{125})^2 = 5^2 = 25$

(b) $3^{-2} = (3^2)^{-1} = 9^{-1} = \frac{1}{9}$

(c) $4^{5/2} = (4^{\frac{1}{2}})^5 = (\sqrt{4})^5 = 2^5 = 32$

(d) $81^{-4/3} = (3^4)^{-4/3} = (3^{-16})^{1/3} = \sqrt[3]{\frac{1}{3^{16}}} = \frac{1}{3^5 \sqrt[3]{3}} = \frac{1}{243 \sqrt[3]{3}}$

(e) $(-1)^{5331}$ Since 5331 is odd, $(-1)^{5331} = -1$.

(f) $3125^{2/5} = (\sqrt[5]{3125})^2 = 5^2 = 25$

4. Given that 1 meter = 39 inches, what is the distance to the Sun in kilometers if the distance in miles is assumed to be 93 million? Please show all steps leading to your answer and pay close attention to units.

We must do several conversions here. To, begin, let us write 93 million as $9.3 * 10^7$. Then, we know that there are 5280 feet in a mile. So, the distance to the sun is $9.3 * 10^7$ miles $\cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 5280 * 9.3 * 10^7$ feet. Next, we know there are 12 inches in a foot. So, the distance to the sun in inches is $12 * 5280 * 9.3 * 10^7$. We are given that 39 inches = 1 meter, so $12 * 5280 * 9.3 * 10^7$ inches $\cdot \frac{1 \text{ meter}}{39 \text{ inches}} = \frac{12 * 5280 * 9.3 * 10^7}{39}$ meters. Finally, there are 1000, or 10^3 meters in a kilometer, so, $\frac{12 * 5280 * 9.3 * 10^7}{39}$ meters $\cdot \frac{1 \text{ kilometer}}{10^3 \text{ meters}} = \frac{12 * 5280 * 9.3 * 10^7}{39 * 10^3}$ kilometers = $\frac{12 * 5280 * 9.3 * 10^4}{39}$ kilometers $\approx 15108 * 10^4$ kilometers $\approx 1.51 * 10^8$ kilometers.

5. Is it true in general that

(a) $(xy)^n = x^n y^n$? (n a positive integer) YES

(b) $(x + y)^n = x^n + y^n$?

NO. Let $x = 1$, $y = 3$, and $n = 2$. Then $(x + y)^n = (1 + 3)^2 = 4^2 = 16$, but $x^n + y^n = 1^2 + 3^2 = 1 + 9 = 10$.

(c) $1/x + 1/y = 1/(x + y)$?

NO. Let $x = 1$ and $y = 2$. $1/x + 1/y = 1 + 1/2 = 3/2$, but $1/(x + y) = 1/(1 + 2) = 1/3$.

3 Expansion

1. Please expand $(x + 4) * (x + 10)$ with the result of $x^2 + 14x + 40$.
 $(x + 4)(x + 10) = x * x + x * 10 + 4 * x + 4 * 10 = x^2 + 10x + 4x + 40 = x^2 + 14x + 40$
2. Please expand $(x - 7) * (x - 1)$ and show the result is $x^2 - 8x + 7$.
 $(x - 7)(x - 1) = x * x + x * (-1) - 7 * x - 7 * (-1) = x^2 - x - 7x + 7 = x^2 - 8x + 7$
3. Please expand $(x - 9) * (x + 9)$ and show the result is $x^2 - 81$.
 $(x - 9)(x + 9) = x^2 + 9 * x - 9 * x - 9 * 9 = x^2 + 9x - 9x - 81 = x^2 - 81$
4. What is the expansion of $(x - 8) * (x + 12)$?
 $(x - 8)(x + 12) = x * x + 12x - 8x - 8 * 12 = x^2 + 4x - 96$
5. What is the expansion of $(x + 6) * (x - 9)$?
 $(x + 6)(x - 9) = x * x - 9x + 6x + 6 * (-9) = x^2 - 3x - 73$
6. What is the expansion of $(x + 4)^2$?
 $(x + 4)^2 = (x + 4)(x + 4) = x * x + 4x + 4x + 4 * 4 = x^2 + 8x + 16$
7. What is the expansion of $(x - a) * (x + a)$?
 $(x - a)(x + a) = x * x + ax - ax - a * a = x^2 - a^2$. Note that this is consistent with the result in problem 3 above.

1. **In each case expand and collect like terms.**

- (a) $\mathbf{x^6(x^2 + 1)^2} = x^6(x^2 + 1)(x^2 + 1) = x^6(x^4 + (1 + 1)x^2 + 1 * 1) = x^6(x^4 + 2x^2 + 1) = x^{10} + 2x^8 + x^6$
- (b) $\mathbf{x(x - 9)(x + 1)} = x(x^2 + (-9 + 1)x + (-9)(1)) = x(x^2 - 8x - 9) = x^3 - 8x^2 - 9x$
- (c) $\mathbf{x^4(x^2 - 2)(x - 1)} = x^4(x^3 - x^2 - 2x + 1) = x^7 - x^6 - 2x^5 + x^4$
- (d) $\mathbf{(x + 1)^3} = (x + 1)(x + 1)(x + 1) = (x^2 + (1 + 1)x + 1 * 1)(x + 1) = (x^2 + 2x + 1)(x + 1) = x^3 + x^2 + 2x^2 + 2x + x + 1 = x^3 + 3x^2 + 3x + 1$
- (e) $\mathbf{x^3(x - 9)(x + 9)} = x^3(x^2 - 81) = x^5 - 81x^3$

1. **Please expand and simplify.**

We will use Pascal's triangle as before to figure out the coefficients.

- (a) $\mathbf{(x + y)^7}$
Row 7 of Pascal's triangle has the entries 1 7 21 35 35 21 7 1. So, $(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.

- (b) $\mathbf{(x - 3y)^5}$
Row 5 of Pascal's triangle is: 1 5 10 10 5 1. So,

$$\begin{aligned}(x - 3y)^5 &= x^5 + 5x^4(-3y) + 10x^3(-3y)^2 + 10x^2(-3y)^3 + 5x(-3y)^4 + (-3y)^5 \\ &= x^5 - 15x^4y + 10x^3(9y^2) + 10x^2(-27y^3) + 5x(81y^4) - 243y^5 \\ &= x^5 - 15x^4y + 90x^3y^2 - 270x^2y^3 + 405xy^4 - 243y^5\end{aligned}$$

(c) $(y + 2x)^4$

Row 4 of Pascal's triangle is: 1 4 6 4 1. So,

$$\begin{aligned}(y + 2x)^4 &= y^4 + 4y^3(2x)^1 + 6y^2(2x)^2 + 4y(2x)^3 + (2x)^4 \\ &= y^4 + 4y^3(2x) + 6y^2(4x^2) + 4y(8x^3) + (16x^4) \\ &= y^4 + 8xy^3 + 24x^2y^2 + 32x^3y + 16x^4\end{aligned}$$

(d)

$$\begin{aligned}(-x - y)^4 &= (-1 * (x + y))^4 = (-1)^4(x + y)^4 = (x + y)^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

(e)

$$\begin{aligned}(\mathbf{a} + 5\mathbf{b})^5 &= a^5 + 5a^4(5b) + 10a^3(5b)^2 + 10a^2(5b)^3 + 5a(5b)^4 + (5b)^5 \\ &= a^5 + 25a^4b + 10a^3(25b^2) + 10a^2(125b^3) + 5a(625b^4) + 3125b^5 \\ &= a^5 + 25a^4b + 250a^3b^2 + 1250a^2b^3 + 3125ab^4 + 3125b^5\end{aligned}$$

2. **What is the coefficient of $x^{13}y^5$ in the expansion of $(x + y)^{20}$? (think!)**

Recall that in the expansion of $(x + y)^{20}$, the exponents of each term $x^m y^n$ should sum to 20. The exponents of the term $x^{13}y^5$ only sum to $13+5 = 18$, so this is not a term in the expansion. Thus, its coefficient is zero.

3. **Please simplify each of the following as much as possible.**

(a)

$$\begin{aligned}[(\mathbf{x} + \mathbf{h})^3 - \mathbf{x}^3]/\mathbf{h} &= [(x^3 + 3x^2h + 3xh^2 + h^3) - x^3]/h \\ &= (3x^2h + 3xh^2 + h^3)/h \\ &= [h(3x^2 + 3xh + h^2)]/h \\ &= 3x^2 + 3xh + h^2\end{aligned}$$

(b)

$$\begin{aligned}((\mathbf{x} + \mathbf{a})^4 - \mathbf{x}^4)/\mathbf{a} &= [(x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4) - x^4]/a \\ &= (4x^3a + 6x^2a^2 + 4xa^3 + a^4)/a \\ &= [a(4x^3 + 6x^2a + 4xa^2 + a^3)]/a \\ &= 4x^3 + 6x^2a + 4xa^2 + a^3\end{aligned}$$

4 Factoring

1. Factor each quadratic below (there are no “duds” among them).

(a) $x^2 + x - 56$

Since we know $(x + a)(x + b) = x^2 + (a + b)x + ab$, we want to find two numbers a and b such that $a + b = 1$ and $ab = -56$. So, we see that we want $a = 8$ and $b = -7$. Then $x^2 + x - 56 = (x + 8)(x - 7)$. Of course, we can always check that we are correct by expanding our answer to see if it matches what we began with.

(b) $x^2 - 81$

You should recognize this as the difference of squares: $x^2 - 81 = x^2 - 9^2$. So, it factors as $(x - 9)(x + 9)$, as we can check.

(c) $x^2 - 6x - 91$

Remember that we want a and b so that $ab = -91$, so consider the factors of 91: $91 = 13 \cdot 7$. Since we also want $a + b = -6$, we see that we can choose $a = 7$ and $b = -13$. So, the factorization is $(x + 7)(x - 13)$.

(d) $x^2 + 7x$

We always want to factor out any common powers of x first, so we have $x^2 + 7x = x(x + 7)$. This is all we can do here.

(e) $x^3 - 6x^2 - 40x$

Again, we first factor out common powers of x : $x^3 - 6x^2 - 40x = x(x^2 - 6x - 40)$. Then we want to factor the remaining quadratic term. First consider the prime factorization of 40: $40 = 2 \cdot 2 \cdot 2 \cdot 5$, so there are several possible choices for a and b here. We could have $\{a = \pm 2, b = \pm 20\}$, $\{a = \pm 4, b = \pm 10\}$, $\{a = \pm 8, b = \pm 5\}$, etc. Since we also need $a + b = -6$, this leads us to choosing $a = 4$ and $b = -10$, to give us the factorization $x^3 - 6x^2 - 40x = x(x + 4)(x - 10)$.

2. Factor each below as much as possible. Your answers may be checked by expanding them and comparing to the original expression.

(a) $x^2 - 5x - 24 = (x - 8)(x + 3)$

(b) $x^6 - 100x^4 = x^4(x^2 - 100) = x^4(x^2 - 10^2) = x^4(x - 10)(x + 10)$

(c) $x^2 + 7x + 10 = (x + 5)(x + 2)$

(d) $x^2 - 121 = x^2 - 11^2 = (x - 11)(x + 11)$

(e) $x^2 - 12x + 35 = (x - 5)(x - 7)$

(f) $x^2 + 16$ Cannot be factored.

(g) $x^2 + 100x = x(x + 100)$

(h) $5x^3 - 125x = 5x(x^2 - 25) = 5x(x - 5)(x + 5)$

(i) $x^4 - 625 = (x^2)^2 - (25)^2 = (x^2 - 25)(x^2 + 25) = (x - 5)(x + 5)(x^2 + 25)$

(j) $100 - 4x^2 = 4(25 - x^2) = 4(5^2 - x^2) = 4(5 - x)(5 + x)$

$$(k) \quad x^4 + 25x^2 = x^2(x^2 + 25)$$

5 Logs and Exponentials

1. In what follows, suppose that it is given that the following are true: $\log(2) = .3$, $\log(3) = .48$, $\log(5) = .7$, $\log(7) = .85$. Use these facts, if applicable, and properties of logs and algebra to provide values for: (no calculators!)

(a) $\log(\mathbf{1000}) = \log(10^3) = 3$

(b) $\log(\mathbf{6}) = \log(3 \cdot 2) = \log(3) + \log(2) = .48 + .3 = .78$

(c) $\log(\mathbf{9}) = \log(3^2) = 2 \log(3) = 2(.48) = .96$

(d) $\log(\mathbf{1/2}) = \log(2^{-1}) = -1 \log(2) = -.3$

(e) $\log(\mathbf{7/5}) = \log(7) - \log(5) = .85 - .7 = .15$

(f) $\log(\mathbf{30}) = \log(3 \cdot 10) = \log(3) + \log(10) = .48 + 1 = 1.48$

(g) $\log(\mathbf{81}) = \log(3^4) = 4 \log(3) = 4(.48) = 1.92$

(h) $\mathbf{10}^{\log(2)} = 2$

(i) $\mathbf{10}^{-\log(5)} = 10^{\log(5^{-1})} = 5^{-1} = \frac{1}{5}$

(j) $\mathbf{100}^{\log(3)} = (10^2)^{\log(3)} = 10^{2 \log(3)} = 10^{\log(3^2)} = 3^2 = 9$

(k) $\mathbf{1000}^{\log(2)} = (10^3)^{\log(2)} = 10^{3 \log(2)} = 10^{\log(2^3)} = 2^3 = 8$

6 Series

1. Write out each series below.

$$(a) \sum_{i=1}^4 i^4 = 1^4 + 2^4 + 3^4 + 4^4$$

$$(b) \sum_{i=2}^5 \frac{3}{i} = \frac{3}{2} + \frac{3}{3} + \frac{3}{4} + \frac{3}{5}$$

$$(c) \sum_{k=2}^6 3^k = 3^2 + 3^3 + 3^4 + 3^5 + 3^6$$

$$(d) \sum_{k=3}^7 (2k + 3) = (2 \cdot 3 + 3) + (2 \cdot 4 + 3) + (2 \cdot 5 + 3) + (2 \cdot 6 + 3) + (2 \cdot 7 + 3)$$

$$(e) \sum_{i=1}^{12} 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

1. Please sum the following expressions:

$$(a) \mathbf{3 + 6 + 9 + 12 + 15 + \cdots + 300}$$

First, note that each term in this expression is divisible by 3, so we can rewrite this as $3(1 + 2 + 3 + 4 + 5 + \cdots + 100)$. Now, we can see that this

is just the arithmetic series $3 \sum_{i=1}^{100} i$, so the sum is $3 \left(\frac{100 \cdot 101}{2} \right) = 15150$.

$$(b) \mathbf{1 + 1/3 + 1/9 + 1/27 + \cdots + 1/2187} = \left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots + \left(\frac{1}{3}\right)^7 = \sum_{i=0}^7 \left(\frac{1}{3}\right)^i. \text{ This is a geometric series, so its sum is } \frac{1 - \left(\frac{1}{3}\right)^8}{1 - \frac{1}{3}} = \frac{3^8 - 1}{3^8 - 3^7} = \frac{6560}{4374} = \frac{3280}{2187}.$$

(c)

$$\begin{aligned} \mathbf{5 + 6 + 7 + 8 + \cdots + 80} &= \sum_{j=5}^{80} j \\ &= \sum_{j=1}^{80} j - \sum_{j=1}^4 j \\ &= \frac{80 \cdot 81}{2} - \frac{4 \cdot 5}{2} = 3240 - 10 = 3230 \end{aligned}$$

(d)

$$\begin{aligned} & \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \frac{1}{3125} + \frac{1}{15625} \\ &= \left(\frac{1}{5}\right)^1 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \left(\frac{1}{5}\right)^4 + \left(\frac{1}{5}\right)^5 + \left(\frac{1}{5}\right)^6 \\ &= \frac{1}{5} \left[\left(\frac{1}{5}\right)^0 + \left(\frac{1}{5}\right)^1 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \left(\frac{1}{5}\right)^4 + \left(\frac{1}{5}\right)^5 \right] \\ &= \frac{1}{5} \sum_{i=0}^5 \left(\frac{1}{5}\right)^i \\ &= \frac{1}{5} \cdot \frac{1 - \left(\frac{1}{5}\right)^6}{1 - \frac{1}{5}} \\ &= \frac{1}{5} \cdot \frac{5^6 - 1}{5^6 - 5^5} \\ &= \frac{1}{5} \cdot \frac{15624}{12500} = \frac{3906}{15625} \end{aligned}$$

(e) $3 + 3/2 + 3/4 + 3/8 + 3/16 + 3/32 + 3/64 + 3/128 + 3/256 + 3/512 + 3/1024$

$$\begin{aligned} &= 3 \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \dots + \left(\frac{1}{2}\right)^{10} \right] \\ &= 3 \sum_{i=0}^{10} \left(\frac{1}{2}\right)^i \\ &= 3 \left(\frac{1 - \left(\frac{1}{2}\right)^{11}}{1 - \frac{1}{2}} \right) = \frac{6141}{1024} \end{aligned}$$

2. Please sum the following series:

(a)

$$\begin{aligned} 1 + 1/3 + 1/9 + \dots + 1/729 &= \sum_{i=0}^6 \left(\frac{1}{3}\right)^i \\ &= \frac{1 - \left(\frac{1}{3}\right)^7}{1 - \frac{1}{3}} = \frac{1093}{729} \end{aligned}$$

(b)

$$\begin{aligned} 1 - 1/2 + 1/4 - 1/8 + 1/16 + \dots + 1/1024 &= \sum_{i=0}^{10} \left(-\frac{1}{2}\right)^i \\ &= \frac{1 - \left(-\frac{1}{2}\right)^{11}}{1 + \frac{1}{2}} = \frac{683}{1024} \end{aligned}$$

(c)

$$\begin{aligned}5 + 5/2 + 5/4 + \cdots + 5/256 &= 5 \sum_{i=0}^8 \left(\frac{1}{2}\right)^i \\ &= \frac{1 - \left(\frac{1}{2}\right)^9}{1 - \frac{1}{2}} = \frac{2555}{256}\end{aligned}$$

(d)

$$\begin{aligned}1/25 + 1/125 + 1/625 + \cdots + 1/78125 &= \frac{1}{25} \sum_{i=0}^5 \frac{1}{5} \\ &= \frac{1}{25} \cdot \frac{1 - \left(\frac{1}{5}\right)^6}{1 - \frac{1}{5}} = \frac{3906}{78125}\end{aligned}$$

(e)

$$\begin{aligned}6 + 8 + 10 + \cdots + 100 &= 2(3 + 4 + 5 + \cdots + 50) \\ &= 2 \sum_{k=3}^{50} k \\ &= 2 \left[\sum_{k=1}^{50} k - (1 + 2) \right] = 2544\end{aligned}$$

3. Consider the following series: $\sum_{i=1}^{100} (i^2 - (i+1)^2)$

(a) Write out the first three terms of the series and also the last two.
Terms:

$$\begin{aligned}i = 1 : & \quad 1^2 - 2^2 = -3 \\ i = 2 : & \quad 2^2 - 3^2 = -5 \\ i = 3 : & \quad 3^2 - 4^2 = -7 \\ i = 99 : & \quad 99^2 - 100^2 = -199 \\ i = 100 : & \quad 100^2 - 101^2 = -201\end{aligned}$$

(b) See if you can sum the series. Sometimes series can be summed easily by grouping the terms in a careful way. $(1^2 - 2^2) + (2^2 - 3^2) + (3^2 - 4^2) + \cdots + (99^2 - 100^2) + (100^2 - 101^2) = 1^2 + (-2^2 + 2^2) + (-3^2 + 3^2) + \cdots + (-100^2 + 100^2) - 101^2$. Now it is easy to see that the sum is just $1^2 - 101^2 = -10200$.

4. Let's look at the odd positive integers and add them up. For example, $1 + 3 + 5 + 7 + \cdots + 33$

$$\begin{array}{cccccccc}
1 & 3 & 5 & 7 & 9 & \dots & 33 \\
33 & 31 & 29 & 27 & 25 & \dots & 1 \\
\hline
34 & 34 & 34 & 34 & 34 & \dots & 34
\end{array}$$

(a) **Can you find a method for adding up the first 17 odd numbers? (other than simply adding them one by one)** We can think of this as Gauss did. Write the first 17 odd numbers forward, and then backward and let's see what we get. So, the sum is $\frac{17 \cdot 34}{2} = 17^2 = 289$ since there are 17 pairs of numbers that sum to 34, counting each number in the series twice.

(b) **Can you extend your method and find a formula for the sum of the first n odd integers?** The sum of the first n odd integers can be written as: $\sum_{k=1}^n 2k - 1$. Using the idea above, we see that the first and last number in this series should sum to $1 + (2n - 1) = 2n$. Then, our idea is that there are n pairs of numbers summing to $2n$. But, this counts each number in our series twice, so the sum is $\frac{(n)(2n)}{2} = n^2$. So, the sum of the first n odd integers is just n^2 .

5. **A very hard ball is dropped from a height of 100 feet. It strikes the ground and bounces back up, each time reaching a height of $\frac{3}{4}$ the height of the previous bounce. It bounces 12 times before coming to rest. What is the total distance it traveled? What does this have to do with series??**

When the ball is first dropped, it travels 100 feet to the ground. After it bounces, it travels up $\frac{3}{4} \cdot 100$ feet, then down $\frac{3}{4} \cdot 100$ feet before bouncing again. When the ball bounces this time, it only reaches a height of $\frac{3}{4}(\frac{3}{4} \cdot 100) = (\frac{3}{4})^2 \cdot 100$ feet before falling to the ground again. So, generalizing we see that the height the ball reaches after the n^{th} bounce is $(\frac{3}{4})^n \cdot 100$ feet. So the total distance the ball travels is: $100 + 2(\frac{3}{4}) \cdot 100 + 2(\frac{3}{4})^2 \cdot 100 + \dots + 2(\frac{3}{4})^{12} \cdot 100$ feet. Now, we can use the techniques we've learned to find the sum.

$$\begin{aligned}
100 + 2\left(\frac{3}{4}\right)100 + 2\left(\frac{3}{4}\right)^2 100 + \dots + 2\left(\frac{3}{4}\right)^{12} 100 &= 100 + 200 \sum_{n=1}^{12} \left(\frac{3}{4}\right)^n \\
&= 100 + 200 \cdot \frac{3}{4} \sum_{n=0}^{11} \left(\frac{3}{4}\right)^n \\
&= 100 + 200 \cdot \frac{3}{4} \cdot \frac{1 - \left(\frac{3}{4}\right)^{12}}{1 - \frac{3}{4}} \\
&= 100 + 200 \cdot \frac{3}{4} \cdot \frac{1987}{513} = \frac{117131}{172}
\end{aligned}$$