Asymptotic formulas for perturbations in the eigenfrequencies of the full Maxwell equations due to the presence of imperfections of small diameter

Habib Ammari a,* and Darko Volkov b
a Centre de Mathématiques Appliquées, CNRS UMR 7641 & École Polytechnique, 91128 Palaiseau cedex, France
E-mail: ammari@cmapx.polytechnique.fr
b Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
E-mail: dvolkov@math.rutgers.edu

Abstract. We consider electromagnetic media that consist of a homogeneous (constant coefficient) electromagnetic material with a finite number of small dielectric imperfections. For such media, we provide a rigorous derivation of asymptotic formulas for perturbations in the eigenfrequencies of the full Maxwell equations caused by the presence of the dielectric imperfections.

Keywords: Maxwell’s equations, eigenvalues, asymptotics, small dielectric imperfections

1. The main result

Let \( \Omega \) be a bounded, open subset of \( \mathbb{R}^3 \), with a smooth boundary. For simplicity we take \( \partial \Omega \) to be \( C^\infty \), but this regularity condition could be considerably weakened. We suppose that \( \Omega \) contains a finite number of imperfections, each of the form \( z_j + \rho B_j \), where \( B_j \subset \mathbb{R}^3 \) is a bounded, smooth \( (C^\infty) \) domain containing the origin. The total collection of imperfections thus takes the form \( I_\rho = \bigcup_{j=1}^m (z_j + \rho B_j) \).

The points \( z_j \in \Omega, j = 1, \ldots, m, \) that determine the location of the imperfections are assumed to satisfy

\[
0 < d_0 \leq |z_j - z_l| \quad \forall j \neq l, \\
0 < d_0 \leq \text{dist}(z_j, \partial \Omega) \quad \forall j.
\]

As a consequence of this assumption it follows immediately that

\[
m \leq 6|\Omega|/\pi d_0^3
\]

We also assume that \( \rho > 0 \), the common order of magnitude of the diameters of the imperfections, is sufficiently small that these are disjoint and that their distance to \( \mathbb{R}^3 \setminus \Omega \) is larger than \( d_0/2 \). Let \( \mu^0 > 0 \) and \( \varepsilon^0 > 0 \) denote the permeability and the permittivity of the background medium; we shall assume

* Corresponding author.

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that these are positive constants. Let $\mu^j > 0$ and $\varepsilon^j > 0$ denote the permeability and the permittivity of the $j$-th inhomogeneity, $z_j + \rho B_j$; these are also assumed to be positive constants. Using this notation we introduce the piecewise constant magnetic permeability

$$
\mu_\rho(x) = \begin{cases} 
\mu^0, & x \in \Omega \setminus \overline{T}_\rho, \\
\mu^j, & x \in z_j + \rho B_j, 
\end{cases} 
$$

(2)

If we allow the degenerate case $\rho = 0$, then the function $\mu_0(x)$ equals the constant $\mu^0$. The piecewise constant electric permittivity, $\varepsilon_\rho(x)$ is defined analogously. The eigenvalue problem for the full Maxwell equations in the presence of imperfections consists of finding $\omega_\rho$ such that there exists a nontrivial electric field $E_\rho$ that is solution to

$$
\nabla \times \left( \frac{1}{\mu_\rho} \nabla \times E_\rho \right) = \omega_\rho^2 \varepsilon_\rho E_\rho, \quad \nabla \cdot (\varepsilon_\rho E_\rho) = 0 \quad \text{in } \Omega, 
$$

(3)

with

$$
E_\rho \times \nu = 0 \quad \text{on } \partial \Omega. 
$$

(4)

Eqs (3), (4) may alternatively be formulated as follows

$$
\nabla \times (\nabla \times E_\rho) = \omega_\rho^2 \mu^0 \varepsilon^0 E_\rho, \quad \nabla \cdot (E_\rho) = 0 \quad \text{in } \Omega \setminus \overline{T}_\rho, 
$$

$$
\nabla \times (\nabla \times E_\rho) = \omega_\rho^2 \mu^j \varepsilon^j E_\rho, \quad \nabla \cdot (E_\rho) = 0 \quad \text{in } z_j + \rho B_j, 
$$

$$
E_\rho \times \nu \text{ is continuous across } \partial(z_j + \rho B_j),
$$

$$
\frac{1}{\mu^0} (\nabla \times E_\rho)^+ \times \nu - \frac{1}{\mu^j} (\nabla \times E_\rho)^- \times \nu = 0 \quad \text{on } \partial(z_j + \rho B_j),
$$

$$
\varepsilon^0 E_\rho^+ \cdot \nu - \varepsilon^j E_\rho^- \cdot \nu = 0 \quad \text{on } \partial(z_j + \rho B_j),
$$

$$
E_\rho \times \nu = 0 \quad \text{on } \partial \Omega.
$$

Here $\nu$ denotes the outward unit normal to $\partial(z_j + \rho B_j)$ (and to $\partial \Omega$); superscripts “+” and “−” indicate the limiting values as we approach $\partial(z_j + \rho B_j)$ from outside $z_j + \rho B_j$, and from inside $z_j + \rho B_j$, respectively.

It is well known that all eigenfrequencies of the Maxwell equations are real, of finite multiplicity, have no finite accumulation points, and there are corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega)$, see [6,10]. These results are shown to be still valid for a large class of domains $\Omega$ and electromagnetic parameters ($\varepsilon$, $\mu$), see [4]. We find it relevant to mention that it is not clear how to view the Maxwell operator as a self-adjoint operator in $L^2(\Omega) \times L^2(\Omega)$. Difficulties arise since the Maxwell system is not elliptic and is not semi-bounded. To overcome these difficulties, we embed the Maxwell operator in an elliptic boundary value problem using a Hodge decomposition, see [4,19].

Let $\omega_0$ be an eigenfrequency of multiplicity $n$ for the Maxwell equations in the absence of any imperfections. Then there exist $n$ nonzero solutions $(E_0^i)_{i=1}^n$ to

$$
\nabla \times (\nabla \times E_0^i) = \omega_0^2 \mu^0 \varepsilon^0 E_0^i, \quad \nabla \cdot (E_0^i) = 0 \quad \text{in } \Omega, 
$$

(5)
such that

$$\int_{\Omega} \varepsilon_0^i E_0^i \cdot E_0^l = \delta_{il}, \quad \forall i, l = 1, \ldots, n,$$

where $\delta_{il}$ is the Kronecker symbol. Before our main results we need to introduce more notations. Let $\gamma^k$, $0 \leq k \leq m$, be a set of positive constants. In effect, $\{\gamma^k\}$ will either be the set $\{\varepsilon^k\}$ or the set $\{\mu^k\}$. For any fixed $1 \leq j_0 \leq m$, let $\gamma_j$ denote the coefficient given by

$$\gamma_j(x) = \begin{cases} \gamma^0, & x \in \mathbb{R}^3 \setminus \overline{B_{j_0}}, \\ \gamma^j_{j_0}, & x \in B_{j_0}. \end{cases}$$

$\phi_l$, $1 \leq l \leq 3$, denotes the solution to

$$\nabla \phi \cdot \gamma(y) \nabla \phi_l = 0 \quad \text{in } \mathbb{R}^3,$$

$$\phi_l - y_l \to 0 \quad \text{as } |y| \to \infty.$$ 

This problem may alternatively be written

$$\begin{cases} \Delta \phi_l = 0 \quad \text{in } B_{j_0}, \text{ and in } \mathbb{R}^3 \setminus \overline{B_{j_0}}, \\ \phi_l \text{ is continuous across } \partial B_{j_0}, \\ \frac{\gamma^0_{j_0}}{\gamma^j_{j_0}} (\partial_\nu \phi_l)^+ - (\partial_\nu \phi_l)^- = 0 \quad \text{on } \partial B_{j_0}, \\ \phi_l(y) - y_l \to 0 \quad \text{as } |y| \to +\infty. \end{cases}$$

It is therefore obvious that the function $\phi_l$ only depends on the coefficients $\gamma^0$ and $\gamma^j_{j_0}$ through the ratio $c = \gamma^0 / \gamma^j_{j_0}$. The existence and uniqueness of this $\phi_l$ can be established using single layer potentials with suitably chosen densities. It is essential here, that the constant $c$, by assumption, cannot be 0 or a negative real number. We now define the polarization tensor, $M_{j_0}(c)$ of the inhomogeneity $B_{j_0}$ (with aspect ratio $c$)

$$M_{kl}^{j_0}(c) = c^{-1} \int_{B_{j_0}} \partial_{y_k} \phi_l \, dy.$$ 

It is quite easy to see that the tensor $M_{kl}(c)$ is symmetric; since $c$ is a positive real number, it is furthermore positive definite, see [5,7].

Our main results are summarized in the following theorem.

**Theorem 1.1.** Suppose $\omega_0$ is an eigenfrequency of multiplicity $n$ for the Maxwell equations in absence of inhomogeneities, and let $(E_0^i)_{i=1}^n$ denote the corresponding eigenfunctions defined in (5)–(7). For $\rho$ small enough there exist $n$ eigenfrequencies $(\omega_\rho^i)_{i=1}^n$ (counted according to multiplicity) for the Maxwell equations.
2. Preliminary results

We leave the details to the reader. Homogeneities may be derived by a fairly straightforward iteration of the arguments we present, however the term \(O(\rho^4)\) is bounded by \(C\rho^4\), where the constant \(C\) depends on the eigenfrequency \(\omega_0\), the domains \((B_j)_{j=1}^n, \Omega\), the constants \(d_0, (\mu^j, \varepsilon^j)_{j=0}^n\), but is otherwise independent of the location of the set of points \((z_j)_{j=1}^n\).

The asymptotic formula (10) generalizes those in [2], where only two-dimensional Maxwell equations with TE (and TM) symmetries were considered and those derived by Ozawa [12–17] for the Laplace system in the presence of small elastic inhomogeneities of different Lamé coefficients. This will be discussed in a forthcoming paper. For reasons of brevity we restrict a significant part of the proof of our asymptotic formula to the case of a single inhomogeneity \((m = 1)\). The proof in the case of multiple inhomogeneities may be derived by a fairly straightforward iteration of the arguments we present, however we leave the details to the reader.

2. Preliminary results

Throughout this paper, we shall use quite standard \(L^2\)-based Sobolev spaces to measure regularity. The notation \(H^s\) is used to denote those functions who along with all their derivatives of order less than or equal to \(s\) are in \(L^2\). \(H^1_0\) denotes the closure of \(C^{\infty}_{0}\) in the norm of \(H^1\). Sobolev spaces with negative indices are in general defined by duality, using \(L^2\)-inner product. We shall only need one such space, namely \(H^{-1}\), which is defined as the dual of \(H^1_0\). The standard notation \(H(\text{curl}, \Omega)\) is used to denote those functions that, along with their curls, are in \(L^2(\Omega)\). \(TH_{\text{div}}^{-1/2}(\partial \Omega)\) denotes the space of tangential vector fields on \(\partial \Omega\) that lie in \(H^{-1/2}(\partial \Omega)\) and whose surface divergences also lie in \(H^{-1/2}(\partial \Omega)\). \(TH_{\text{curl}}^{-1/2}(\partial \Omega)\) denotes its dual: \(TH_{\text{curl}}^{-1/2}(\partial \Omega) = (TH_{\text{div}}^{-1/2}(\partial \Omega))'\), see [18]. Let us also introduce the two Hilbert spaces

\[ X_{\rho}(\Omega) = \{ u \in L^2(\Omega), \nabla \cdot (\varepsilon_\rho u) = 0 \text{ in } \Omega \}, \]

and

\[ Z_{\rho}(\Omega) = \{ u \in L^2(\Omega), \nabla \times u \in L^2(\Omega), \nabla \cdot (\varepsilon_\rho u) = 0 \text{ in } \Omega, u \times \nu = 0 \text{ on } \partial \Omega \}, \]

which are respectively equipped with the norms

\[ (u, v)_{X_{\rho}(\Omega)} = \int_{\Omega} \varepsilon_\rho u \cdot v, \]
and

$$(u, v)_{Z_\rho(\Omega)} = \int_\Omega \varepsilon_\rho u \cdot v + \int_\Omega \nabla \times u \cdot \nabla \times v.$$  

It will turn out to be convenient to first establish two auxiliary results. By adapting a compactness result, stated and proved in Proposition 3 in Appendix 1 from [3], the following compactness result for $Z_\rho(\Omega)$ holds.

**Lemma 2.1.** Suppose $(\rho_l)$ is a sequence that converges to 0, or is constant, and suppose $(u_l)$ is a sequence with $u_l \in Z_{\rho_l}(\Omega)$, such that

$$\|u_l\|_{L^2(\Omega)} + \|\nabla \times u_l\|_{L^2(\Omega)} \leq C.$$  

Then $(u_l)$ has a strongly convergent subsequence in $L^2(\Omega)$.

**Proof.** The boundary condition for the vector fields involved in [3] is that their normal component vanishes on the boundary. It can be easily realized that a similar compactness result hold for vector fields whose tangential component has to be zero. We leave the details to the reader. \qed

The second lemma asserts that the inner product $\int_\Omega \frac{1}{\mu_\rho} \nabla \times u \cdot \nabla \times v$ generates a norm which is equivalent to the $Z_\rho(\Omega)$-norm.

**Lemma 2.2.** The following inequalities hold for all $u$ in $Z_\rho(\Omega)$:

$$C_1\|u\|_{Z_\rho(\Omega)}^2 \leq \int_\Omega \frac{1}{\mu_\rho} |\nabla \times u|^2 \leq C_2\|u\|_{Z_\rho(\Omega)}^2,$$  

(11)

where $C_1$ and $C_2$ are constants that are independent of $\rho$.

**Proof.** It suffices to prove that there exists a constant $C$ that is independent of $\rho$ such that, for any $u \in Z_\rho(\Omega)$,

$$\|u\|_{H(\text{curl,} \Omega)} \leq C\|\nabla \times u\|_{L^2(\Omega)}.$$  

(12)

We argue by contradiction. Suppose (12) fails to be true. Then there is a bounded sequence $(\rho_l)$ of positive numbers and a sequence $(u_{\rho_l})$ in $Z_{\rho_l}(\Omega)$ such that

$$\|u_{\rho_l}\|_{L^2(\Omega)} \geq \frac{1}{l}\|\nabla \times u_l\|_{L^2(\Omega)}.$$  

(13)

By extraction of a subsequence we can assume that $\rho_l$ converges to a nonnegative number $\rho^*$. It is also possible to assume that $\|u_l\|_{L^2(\Omega)} = 1$. From Lemma 2.1 it follows that there is a subsequence of $(u_l)$, also denoted by $(u_l)$ for ease of notation, that is strongly convergent in $L^2(\Omega)$ and weakly convergent in $H(\text{curl,} \Omega)$ to a vector field $u^*$ with $\|u^*\|_{L^2(\Omega)} = 1$. We infer from (13) that $\nabla \times u^* = 0$, thus $u^*$ is equal to a gradient of a function $\varphi$ that exhibits the following properties

$$\nabla \cdot (\varepsilon_\rho^* \nabla \varphi) = 0 \quad \text{in} \ \Omega \quad \text{and} \quad \nabla \varphi \times \nu = 0 \quad \text{on} \ \partial \Omega.$$  

It follows that $\nabla \varphi = 0$ which contradicts $\|u^*\|_{L^2(\Omega)} = 1$. \qed
We will need the following existence and uniqueness result.

**Lemma 2.3.** For any \( f \in X_\rho(\Omega) \), there exists a unique \( u_\rho \in Z_\rho(\Omega) \) solution of

\[
\nabla \times \left( \frac{1}{\mu_\rho} \nabla \times u_\rho \right) = \varepsilon_\rho f, \quad \nabla \cdot (\varepsilon_\rho u_\rho) = 0 \quad \text{in } \Omega,
\]

with

\[
u_\rho \times \nu = 0, \quad \text{on } \partial \Omega.
\]

Furthermore,

\[
\|u_\rho\|_{Z(\Omega)} \leq C \|f\|_{X(\Omega)}.
\]

where the constant \( C \) is independent of \( \rho \).

**Proof.** The natural weak formulation of the boundary value problem (14), (15) is that \( u_\rho \) be in \( Z_\rho(\Omega) \) and satisfy

\[
a_\rho(u_\rho, v) = l_\rho(v), \quad \forall v \in Z_\rho(\Omega),
\]

where \( a_\rho(u, v) \) denotes the sesquilinear form

\[
a_\rho(u, v) = \int_\Omega \frac{1}{\mu_\rho} \nabla \times u \cdot \nabla \times v,
\]

and the continuous linear functional \( l_\rho \) is defined by \( l_\rho(v) = \int_\Omega \varepsilon_\rho f \cdot v \). From (11) it follows immediately by the Lax–Milgram lemma that the variational problem (17) has a unique solution \( u_\rho \in Z_\rho(\Omega) \) satisfying estimate (16). \( \square \)

Let the continuous linear operator \( R_\rho : X_0(\Omega) \to X_\rho(\Omega) \) be defined by \( R_\rho(f) = (1/\varepsilon_\rho)f \). Define the operator \( T_\rho : X_\rho(\Omega) \to X_\rho(\Omega) \) by \( T_\rho(f) = u_\rho \) where \( u_\rho \) is the unique solution of (14), (15) given by Lemma 2.3. We begin with establishing some properties of the operator \( T_\rho \).

**Lemma 2.4.** The following properties of the operator \( T_\rho \) hold.

(a) \( T_\rho : X_\rho(\Omega) \to X_\rho(\Omega) \) is a self-adjoint compact positive operator;
(b) \( T_\rho \) is (uniformly) bounded;
(c) The set \( (1/\omega^2_\rho) \), where \( (\omega^2_\rho) \) is the set of eigenfrequencies of the Maxwell equations (3), (4), is the set of eigenvalues of \( T_\rho \);
(d) For any uniformly bounded sequence \( f_\rho \in X_\rho(\Omega) \), there exists a subsequence \( f_\rho' \) and \( f_0 \in X_0(\Omega) \) such that \( \|T_\rho(f_\rho') - R_\rho(f_0)\|_{X_\rho'(\Omega)} \to 0 \) as \( \rho' \to 0 \);
(e) \( \|T_\rho(f_\rho) - R_\rho T_0(f_0)\|_{X_\rho(\Omega)} \to 0 \) if \( \|f_\rho - R_\rho f_0\|_{X_\rho(\Omega)} \to 0 \) for any \( f_\rho \in X_\rho(\Omega) \) and \( f_0 \in X_0(\Omega) \).
Proof. (a) it is easy to see that $T_\rho : X_\rho(\Omega) \to X_\rho(\Omega)$ is a self-adjoint positive operator. Its compactness follows from the compact embedding of $Z_\rho(\Omega)$ into $L^2(\Omega)$, see, for example, [21]; (b) follows from estimate (16); (c) if $E_\rho \in Z_\rho(\Omega)$ is a nonzero solution to the Maxwell equations (3), (4) then $T_\rho(E_\rho) = (1/\omega_\rho^2)E_\rho$ and so, $1/\omega_\rho^2$ is an eigenvalue of $T_\rho$. The converse is also immediate. Point (d) can be easily obtained by using once again estimate (16) and applying Lemma 2.1.

The proof of point (e) is slightly more delicate. Since $\epsilon_\rho f_\rho$ is divergence free, the following equality is true for all $v$ in $H(\text{curl}, \Omega)$ such that $\nu \times \nu = 0$ on $\partial \Omega$:

$$
\int_\Omega \frac{1}{\mu_\rho} \nabla \times T_\rho(f_\rho) \cdot \nabla \times v = \int_\Omega \epsilon_\rho f_\rho \cdot v. \tag{18}
$$

Thus, using Lemma 2.2, $T_\rho(f_\rho)$ is bounded in $H(\text{curl}, \Omega)$. A subsequence $T_\rho(f'_{\rho})$ converges weakly to an element $g$ in $H(\text{curl}, \Omega)$, this convergence is strong in $L^2(\Omega)$ due to Lemma 2.1. At the limit we obtain

$$
\int_\Omega \frac{1}{\mu_0} \nabla \times g \cdot \nabla \times v = \int_\Omega f_0 \cdot v \tag{19}
$$

which proves that $g = R_\rho T_0(f_0)$, which does not depend on the choice of the subsequence $\rho'$. In addition, it is clear that $R_\rho T_0(f_0) - R_\rho T_0(f_0)$ tends to 0 in $L^2(\Omega)$, which concludes the proof of (e). \qed

Now, direct application of classical theorems [8, Theorems 11.4 and 11.5, pp. 343, 344] on spectral properties of a sequence of operators satisfying properties (a), (b), (d), and (e) that have been stated in Lemma 2.4 allow us in view of point (c) to describe in the following lemma the relative asymptotic properties of a sequence of operators satisfying properties (a), (b), (d), and (e) that have been stated in Lemma 2.4. These properties allow us to describe in the following lemma the relative asymptotic properties of the eigenfrequencies for the Maxwell equations (3), (4) in the presence of small imperfections to the eigenfrequencies for the homogeneous Maxwell equations (5), (6).

**Lemma 2.5.** Let $\omega_0$ be an eigenfrequency of multiplicity $n$ for (5), (6), with the basis of eigenfunctions $(E^{(i)}_0)_{i=1}^n$:

$$
E^{(i)}_0 \in X_0(\Omega), \tag{20}
$$

$$
\nabla \times (\nabla \times E^{(i)}_0) = \omega_0^2 \mu_0^{\epsilon_0} E^{(i)}_0, \quad \nabla \cdot (E^{(i)}_0) = 0 \quad \text{in } \Omega, \tag{21}
$$

$$
E^{(i)}_0 \times \nu = 0 \quad \text{on } \partial \Omega, \tag{22}
$$

$$
\int_\Omega \epsilon_0 E^{(i)}_0 \cdot E^{(i)}_0 = \delta_i. \tag{23}
$$

For $\rho$ small enough there exist $n$ eigenfrequencies $(\omega^{(i)}_\rho)_{i=1}^n$ (counted according to multiplicity) of the Maxwell equations (3), (4) in the presence of small imperfections and $n$ nonzero solutions $(E^{(i)}_\rho)_{i=1}^n$ to:

$$
E^{(i)}_\rho \in X_\rho(\Omega), \tag{24}
$$

$$
\nabla \times \frac{1}{\mu_\rho} (\nabla \times E^{(i)}_\rho) = (\omega^{(i)}_\rho)^2 \epsilon_\rho E^{(i)}_\rho, \quad \nabla \cdot (\epsilon_\rho E^{(i)}_\rho) = 0 \quad \text{in } \Omega, \tag{25}
$$

$$
E^{(i)}_\rho \times \nu = 0 \quad \text{on } \partial \Omega, \tag{26}
$$

$$
\int_\Omega \epsilon_\rho E^{(i)}_\rho \cdot E^{(i)}_\rho = \delta_i, \tag{27}
$$

such that, for any $i = 1, \ldots, m$, $\omega^{(i)}_\rho \to \omega_0$ as $\rho \to 0$. 

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3. Proof of the asymptotic formulas

3.1. The intermediate fields \( u^i_\rho \)

For \( i = 1, \ldots, n \), define \( u^i_\rho \) to be the unique solution in \( Z_\rho(\Omega) \) of

\[
\nabla \times \left( \frac{1}{\mu_\rho} \nabla \times u^i_\rho \right) = E^i_0, \quad \nabla \cdot (\varepsilon_\rho u^i_\rho) = 0 \quad \text{in } \Omega,
\]

with

\[
u \times u^i_\rho = 0 \quad \text{on } \partial \Omega.
\]

Notice that \( T_\rho R_\rho E^i_0 = u^i_\rho \). Since \( T_\rho R_\rho E^i_0 = (1/(\omega_0^2 \varepsilon_0))E^i_0 \) it seems that \( u^i_\rho - (1/(\omega_0^2 \varepsilon_0))E^i_0 \) is small in norm. To obtain our main result, it is in fact crucial not only to prove that \( u^i_\rho - (1/(\omega_0^2 \varepsilon_0))E^i_0 \) is indeed small in norm but also to give a first order expansion of that difference.

As stated earlier, we restrict our proof to the case of a single inhomogeneity (\( m = 1 \)). We suppose that this inhomogeneity is centered at the origin, so it is of the form \( \rho B \). Assumption (1) simply becomes \( 0 \in \Omega \) with \( 0 < d_0 \leq \text{dist}(0, \partial \Omega) \). The general case may be verified by a fairly direct iteration of the argument we present here, adding one inhomogeneity at a time. Let \( q^* \) be the unique (scalar) solution to

\[
\begin{align*}
\Delta q^* &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\
\varepsilon^0(\partial_\nu q^*)^+ - \varepsilon^1(\partial_\nu q^*)^- &= -(\varepsilon^0 - \varepsilon^1)E^i_0(0) \cdot \nu \quad \text{on } \partial B, \\
q^* \text{ is continuous across } \partial B, \\
\lim_{|y| \to +\infty} q^*(y) &= 0.
\end{align*}
\]

The reader is referred to [3] for the existence and uniqueness of \( q^* \). The following asymptotic estimate can be derived for \( \nabla q^* \), see [3]:

\[
\nabla q^*(y) = O \left( \frac{1}{|y|^3} \right).
\]

Denote \( \tilde{\Omega} = \Omega / \rho \) in the rescaling \( y = x / \rho \). We will need the approximation \( q^*_\rho \) of \( q^* \) in \( \tilde{\Omega} \) defined as follows:

\[
\begin{align*}
\Delta q^*_\rho &= 0 \quad \text{in } \tilde{\Omega} \setminus \overline{B}, \\
q^*_\rho \text{ is continuous across } \partial B, \\
\varepsilon^0(\nabla q^*_\rho \cdot \nu)^+ - \varepsilon^1(\nabla q^*_\rho \cdot \nu)^- &= -(\varepsilon^0 - \varepsilon^1)E_0(\rho y) \cdot \nu \quad \text{across } \partial B, \\
q^*_\rho(y) &= q^*(y) \quad \text{on } \partial \tilde{\Omega}.
\end{align*}
\]
We form an equation for the difference \((q^*_\rho - q^*)(\frac{x}{\rho})\) in order estimate it. We have:

\[
\begin{cases}
\Delta(q^*_\rho - q^*) = 0 & \text{in } \rho B \text{ and in } \Omega \setminus \text{cl} B, \\
q^*_\rho - q^* \text{ is continuous across } \partial(\rho B), \\
\varepsilon^0((\nabla_y q^*_\rho - \nabla_y q^*) \cdot \nu) - \varepsilon^1((\nabla_y q^*_\rho - \nabla_y q^*) \cdot \nu) = -(\varepsilon^0 - \varepsilon^1)(E^i_0(\rho y) - E^i_0(0)) \cdot \nu \\
(q^*_\rho - q^*)\left(\frac{x}{\rho}\right) = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(30)

Since \((q^*_\rho - q^*)(\frac{x}{\rho})\) lies in \(H^1_0(\Omega)\) and, recalling that \(E^i_0\) is divergence free, we get the following identity

\[
-\int_{\Omega \setminus \text{cl} B} \varepsilon^0|\nabla (q^*_\rho - q^*)|^2 - \int_{\rho B} \varepsilon^1|\nabla (q^*_\rho - q^*)|^2 = \int_{\rho B} (\varepsilon^0 - \varepsilon^1)(E^i_0(x) - E^i_0(0)) \cdot \nabla (q^*_\rho - q^*). 
\]  

(31)

Since \(E^i_0\) is a smooth vector field, the supremum of \((E^i_0(x) - E^i_0(0))\) for \(x \in \rho B\) is of order \(\rho\) thus the right-hand side of (31) is of order \(\rho^{5/2}\). Consequently, the following lemma holds.

**Lemma 3.1.**

\[
\sup_{\Omega \setminus \text{cl} B} \varepsilon^0|\nabla (q^*_\rho - q^*)|(\frac{x}{\rho}) \leq C\rho^{5/2}. 
\]  

(32)

Now, let \(S\) denote the space

\[
S = \left\{ u : \frac{u}{1 + \nu^2} \in L^2(\mathbb{R}^3), \nabla \times u \in L^2(\mathbb{R}^3), \nabla \cdot u = 0 \text{ in } \mathbb{R}^3 \setminus \text{cl} B, \nabla \cdot u = 0 \text{ in } B \right. \\
\left. \text{and } \varepsilon^0(u \cdot \nu)^+ = \varepsilon^1(u \cdot \nu)^- \text{ across } \partial B \right\}.
\]

We introduce the vector field \(e^*\) by the requirements that \(e^* \in S\) and

\[
\int_{\mathbb{R}^3 \setminus \text{cl} B} \frac{1}{\mu^0} \nabla \times e^* \cdot \nabla \times v + \int_B \frac{1}{\mu^1} \nabla \times e^* \cdot \nabla \times v = \left(\frac{1}{\mu^0} - \frac{1}{\mu^1}\right) \int_{\partial B} (\nabla \times E^i_0)(0) \times \nu \cdot v,
\]

for any \(v \in S\). For existence and uniqueness of such a vector field we refer the reader to [3] where it is also proven that \(e^*\) satisfies the following properties

\[
\begin{align*}
\Delta e^* &= 0 & \text{in } \mathbb{R}^3 \setminus \text{cl} B \text{ and in } B, \\
\nabla \cdot e^* &= 0 & \text{in } \mathbb{R}^3 \setminus \text{cl} B \text{ and in } B, \\
\frac{1}{\mu^0} (\nabla \times e^*)^+ + \nu - \frac{1}{\mu^1} (\nabla \times e^*)^- \times \nu &= - \left(\frac{1}{\mu^0} - \frac{1}{\mu^1}\right) (\nabla \times E^i_0)(0) \times \nu & \text{across } \partial B, \\
\varepsilon^0(e^* \cdot \nu)^+ &= \varepsilon^1(e^* \cdot \nu)^- & \text{across } \partial B, \\
e^* \times \nu & \text{is continuous across } \partial B, \\
e^*(y) &= O(|y|^{-1}) & \text{uniformly as } |y| \to +\infty.
\end{align*}
\]
We will need to use the field \( e^*_\rho(y) \) defined for \( y \) in \( \tilde{\Omega} \) by:

\[
\begin{aligned}
\Delta e^*_\rho &= 0 \quad \text{in } \tilde{\Omega} \setminus B \text{ and in } B, \\
\nabla \cdot e^*_\rho &= 0 \quad \text{in } \tilde{\Omega} \setminus B \text{ and in } B, \\
\frac{1}{\mu_B} (\nabla \times e^*_\rho)^+ \times \nu - \frac{1}{\mu_0} (\nabla \times e^*_\rho)^- \times \nu &= -\left( \frac{1}{\mu_B} - \frac{1}{\mu_0} \right) (\nabla \times E_\rho^0(0)) \times \nu \quad \text{across } \partial B, \\
e^0 (e^*_\rho \cdot \nu)^+ = e^1 (e^*_\rho \cdot \nu)^- \quad \text{across } \partial B, \\
e^*_\rho \times \nu \text{ is continuous across } \partial B, \\
e^*_\rho(y) \times \nu = -\frac{1}{\rho} \nabla q^*_\rho(y) \times \nu \quad \text{on } \partial \tilde{\Omega}.
\end{aligned}
\]

Back to the \( x \) variable, \((e^*_\rho - e^*)(\tilde{x})\) satisfies in \( \Omega \):

\[
\begin{aligned}
\Delta (e^*_\rho - e^*) &= 0 \quad \text{in } B \\
\nabla \cdot (e^*_\rho - e^*) &= 0 \quad \text{in } B \\
\frac{1}{\mu_B} \nabla \times (e^*_\rho - e^*)^+ \times \nu - \frac{1}{\mu_0} \nabla \times (e^*_\rho - e^*)^- \times \nu &= 0 \quad \text{across } \partial B, \\
\varepsilon^0 (e^*_\rho - e^*)^+ \cdot \nu = \varepsilon^1 (e^*_\rho - e^*)^- \cdot \nu \quad \text{across } \partial B, \\
(e^*_\rho - e^*) \times \nu &= -e^* \times \nu - \frac{1}{\rho} \nabla q^*_\rho \left( \frac{x}{\rho} \right) \times \nu \quad \text{on } \partial \Omega.
\end{aligned}
\]

In order to estimate \( e^*_\rho - e^* \) we need the following lemma.

**Lemma 3.2.** There is a constant \( C \) depending only on \( \Omega \) such that for all \( w \) in \( TH_{\text{curl}}^{-1/2}(\partial \Omega) \) there is a vector field \( \overline{w} \) in \( H(\text{curl}, \Omega) \) that satisfies \( \nu \times (\nu \times \overline{w}) = w \) on \( \partial \Omega \) together with

\[
\| \overline{w} \|_{H(\text{curl}, \Omega)} \leq C \| w \|_{TH_{\text{curl}}^{-1/2}(\partial \Omega)}
\]

and \( \nabla \cdot (\varepsilon_\rho \overline{w}) = 0 \).

**Proof.** This is a well known result except for the “\( \nabla \cdot (\varepsilon_\rho \overline{w}) = 0 \)” requirement. Let us then pick an extension \( \overline{w}_2 \) of \( w \) that does not necessarily satisfy this last requirement. Then find \( q \) in \( H_0^1(\Omega) \) such that

\[
\nabla \cdot (\varepsilon_\rho \nabla q) = \nabla \cdot (\varepsilon_\rho \overline{w}_2).
\]

Necessarily

\[
\| \nabla q \|_{L^2(\Omega)} \leq \| \overline{w}_2 \|_{L^2(\Omega)}.
\]

It suffices then to set \( \overline{w} = \overline{w}_2 - \nabla q \). \( \square \)
Analogously to Lemma 3.1 we have:

**Lemma 3.3.** \((e_\rho^* - e^*)\) is well defined in \(H(\text{curl}, \Omega)\) and satisfies

\[
\left\| (e_\rho^* - e^*) \left( \frac{x}{\rho} \right) \right\|_{H(\text{curl}, \Omega)} \leq C \rho^{3/2}.
\]

**Proof.** We first prove that

\[
\| \nu \times e_\rho^* \left( \frac{x}{\rho} \right) \|_{TH_{\div}^{-1/2}(\partial \Omega)} \quad \text{and} \quad \frac{1}{\rho} \| \nu \times \nabla y q_\rho^* \|_{TH_{\div}^{-1/2}(\partial \Omega)}
\]

are of order \(\rho^2\) and \(\rho^{3/2}\), respectively. Let \(S\) be a sphere strictly containing \(\Omega\). There is a constant \(C\) such that any smooth tangential vector field \(\nu\) on \(\partial \Omega\) can be extended as a smooth vector field in \(S \setminus \Omega\) such that \(\nu \times v = 0\) on \(\partial S\) and

\[
\|v\|_{L^2(S \setminus \Omega)} + \|\nabla \times v\|_{L^2(S \setminus \Omega)} \leq C \|v \times v\|_{TH_{\div}^{-1/2}(\partial \Omega)}.
\]

Then an integration by parts yields

\[
\left| \int_{\partial \Omega} (e_\rho^* \times \nu) \cdot \nu \times (\nu \times v) \right| \leq \left| \int_{S \setminus \Omega} \nabla_x \times e_\rho^* \left( \frac{x}{\rho} \right) \cdot v \right| + \left| \int_{S \setminus \Omega} e_\rho^* \left( \frac{x}{\rho} \right) \cdot \nabla_x \times v \right| \leq C \left[ \left( \int_{S \setminus \Omega} \left| \nabla_x \times e_\rho^* \left( \frac{x}{\rho} \right) \right|^2 \ dx \right]^{1/2} + \left( \int_{S \setminus \Omega} \left| e_\rho^* \left( \frac{x}{\rho} \right) \right|^2 \ dx \right]^{1/2} \right\|v \times v\|_{TH_{\div}^{-1/2}(\partial \Omega)}.
\]

But

\[
\left( \int_{\mathbb{R}^3 \setminus \Omega} \left| \nabla_x \times e_\rho^* \left( \frac{x}{\rho} \right) \right|^2 \ dx \right]^{1/2} = \rho^{1/2} \left( \int_{\mathbb{R}^3 \setminus \Omega} \left| \nabla y e_\rho^*(y) \right|^2 \ dy \right]^{1/2} \leq C \rho^2,
\]

and

\[
\left( \int_{\mathbb{R}^3 \setminus \Omega} \left| e_\rho^* \left( \frac{x}{\rho} \right) \right|^2 \ dx \right]^{1/2} = \rho^{3/2} \left( \int_{\mathbb{R}^3 \setminus \Omega} \left| e_\rho^*(y) \right|^2 \ dy \right]^{1/2} \leq C \rho^2,
\]

which shows that

\[
\| \nu \times e_\rho^* \left( \frac{x}{\rho} \right) \|_{TH_{\div}^{-1/2}(\partial \Omega)} \leq C \rho^2.
\]

Now we want to estimate the term \(\frac{1}{\rho} \| \nu \times \nabla y q_\rho^*(y) \|_{TH_{\div}^{-1/2}(\partial \Omega)}\). We start out by estimating

\[
\frac{1}{\rho} \| \nu \times \nabla y q_\rho^*(y) \|_{TH_{\div}^{-1/2}(\partial \Omega)}.
\]
An integration by parts yields
\[
\left| \int_{\partial \Omega} \left( (\nabla_y q^*)(y) \times \nu \right) \cdot (\nu \times \nu \times v) \right| \leq \left| \int_{S \setminus \Omega} (\nabla_y q^*) \left( \frac{x}{\rho} \right) \nabla \times v \right|
\]
\[
\leq C \left( \int_{S \setminus \Omega} \left| (\nabla_y q^*) \left( \frac{x}{\rho} \right) \right|^2 \, dx \right)^{1/2} \|v \times \nu\|_{TH_{av}^{1/2}(\partial \Omega)}.
\]
But
\[
\left( \int_{\mathbb{R}^3 \setminus \Omega} \left| (\nabla_y q^*) \left( \frac{x}{\rho} \right) \right|^2 \, dx \right)^{1/2} = \rho^{3/2} \left( \int_{\mathbb{R}^3 \setminus \Omega} \left| (\nabla_y q^*) (y) \right|^2 \, dy \right)^{1/2} \leq C \rho^3,
\]
which shows that
\[
\left\| \frac{1}{\rho^2} \nu \times (\nabla_y q) \left( \frac{x}{\rho} \right) \right\|_{\mathbb{H}_{av}^{1/2}(\partial \Omega)} \leq C \rho^2.
\] (35)
To estimate \( \frac{1}{\rho^2} \|v \times \nabla_y(q\rho - q)(y)\|_{\mathbb{H}_{av}^{1/2}(\partial \Omega)} \), we extend this time the tangential field \( v \) in \( \Omega \) instead of \( S \setminus \Omega \). Using an integration by parts followed by (32), we can write
\[
\left| \int_{\partial \Omega} \left( (\nabla_y (q\rho - q^*)) (y) \times \nu \right) \cdot (\nu \times \nu \times v) \right| \leq \left| \int_{\Omega} (\nabla_y (q\rho - q^*)) \left( \frac{x}{\rho} \right) \nabla \times v \right|
\]
\[
\leq C \left( \int_{\Omega} \left| (\nabla_y (q\rho - q^*)) \left( \frac{x}{\rho} \right) \right|^2 \, dx \right)^{1/2} \|v \times \nu\|_{TH_{av}^{1/2}(\partial \Omega)} \leq C \rho^{5/2} \|v \times \nu\|_{TH_{av}^{1/2}(\partial \Omega)}.
\] (36)
Combining (35) with (36) we conclude
\[
\left\| \frac{1}{\rho^2} \nu \times (\nabla_y q^*) \left( \frac{x}{\rho} \right) \right\|_{\mathbb{H}_{av}^{1/2}(\partial \Omega)} \leq C \rho^{3/2}.
\]
Next pick \( l_\rho \) in \( H(\text{curl}, \Omega) \) such that \( \nabla \cdot (\varepsilon \rho l_\rho) = 0 \), and satisfying
\[
l_\rho(x) \times \nu = - \left( e^* \left( \frac{x}{\rho} \right) + \frac{1}{\rho} \nabla q^* \left( \frac{x}{\rho} \right) \right) \times \nu \quad \text{on} \ \partial \Omega,
\]
and
\[
\|l_\rho\|_{L^2(\Omega)} + \|\nabla \times l_\rho\|_{L^2(\Omega)} \leq C \rho^{3/2}.
\] (37)
We can then set the following definition for \((e_\rho^* - e^*)\): find \((e_\rho^* - e^* - l_\rho)\) in \( Z(\Omega) \) such that
\[
\int_{\Omega} \frac{1}{\mu_\rho} \nabla \times (e_\rho^* - e^* - l_\rho) \cdot \nabla \times w = - \int_{\Omega} \nabla \times l_\rho \cdot \nabla \times w.
\]
for all \( w \) in \( Z_\rho(\Omega) \). It can easily be verified that \( \nabla \times (\frac{1}{\rho} \nabla \times (e^*_\rho - e^*)) = 0 \) inside \( \Omega \) and that

\[
(e^*_\rho - e^*) \times \nu = -\left( e^* + \frac{1}{\rho} \nabla q^*_\rho \left( \frac{x}{\rho} \right) \right) \times \nu \quad \text{on} \ \partial \Omega.
\]

From Lemma 2.3 and (37) we infer

\[
\| e^*_\rho - e^* \|_{L^2(\Omega)} + \| \nabla_x \times (e^*_\rho - e^*) \|_{L^2(\Omega)} \leq C \rho^{3/2},
\]

which is what Lemma 3.3 is claiming. \( \square \)

We now proceed to construct an asymptotic expansion of \( u^i_\rho \). Set

\[
R_\rho(x) = u^i_\rho(x) - \frac{1}{\omega_0 e^0} \left( E^i_0(x) + \rho e^*_\rho \left( \frac{x}{\rho} \right) + (\nabla_y q^*_\rho) \left( \frac{x}{\rho} \right) \right).
\]

Notice that \( R_\rho \) lies in the space \( Z_\rho(\Omega) \). For any \( v \) in \( Z_\rho(\Omega) \), we can perform the following integration by parts:

\[
\int_{\Omega} \frac{1}{\mu_\rho} (\nabla \times R_\rho) \cdot (\nabla \times v) - \int_{\partial \Omega_B} \frac{1}{\mu B} \left( \nabla \times \left( u^i_\rho(x) - \frac{1}{\omega_0 e^0} \left( E^i_0(x) + \rho e^*_\rho \left( \frac{x}{\rho} \right) \times \nu \right) \right) \right) \cdot v
\]

\[
+ \int_{\partial \Omega_B} \frac{1}{\mu^0} \left( \nabla \times \left( u^i_\rho(x) - \frac{1}{\omega_0 e^0} \left( E^i_0(x) + \rho e^*_\rho \left( \frac{x}{\rho} \right) \times \nu \right) \right) \cdot \nu \right) \cdot v
\]

\[
+ \int_{\partial \Omega_B} \frac{1}{\omega_0 e^0} (1 - \frac{\mu_0}{\mu^1}) E^i_0 \cdot v
\]

\[
= \frac{1}{\omega_0 e^0} \int_{\partial \Omega_B} \left( \frac{1}{\mu^0} - \frac{1}{\mu^1} \right) (\nabla \times E^i_0(x) - \nabla \times E^i_0(0)) \cdot \nabla \times v. \tag{39}
\]

Since \( E^i_0 \) is a smooth function in \( \Omega \), \( (\nabla \times E^i_0(x) - \nabla \times E^i_0(0)) \) is of order \( \rho \) for \( x \) in \( \rho B \) and therefore the last term in (39) is of order \( \rho^{3/2} \). Due to Lemma 2.3 we derive,

\[
\| R_\rho \|_{H(\text{curl}, \Omega)} \leq C \rho^{5/2}.
\]

Recalling (38) and (32), we obtain

\[
\left\| u^i_\rho(x) - \frac{1}{\omega_0 e^0} \left( E^i_0(x) + \rho e^*_\rho \left( \frac{x}{\rho} \right) + (\nabla_y q^*_\rho) \left( \frac{x}{\rho} \right) \right) \right\|_{L^2(\Omega)} \leq C \rho^{5/2}, \tag{40}
\]

and

\[
\left\| \nabla_x \times \left( u^i_\rho(x) - \frac{1}{\omega_0 e^0} \left( E^i_0(x) + \rho e^*_\rho \left( \frac{x}{\rho} \right) \right) \right) \right\|_{L^2(\Omega)} \leq C \rho^{5/2}. \tag{41}
\]

Recalling (33), (40) can be rewritten as follows.
Lemma 3.4. There exists a constant $C$ that is independent of $\rho$ and the set of points $(z_j)_{j=1}^m$ such that

$$
\left\| u^i_{\rho}(x) - \frac{1}{\omega_0^2} \left( E^i_0(x) + \left( \nabla q^* \left( \frac{x}{\rho} \right) \right) \right) \right\|_{L^2(\Omega)} \leq C \rho^{5/2}.
$$

(42)

Approximating the two following integrals in the lemma below will prove to be useful in presenting our final result in a form utilizing polarization tensors rather than integrals involving $u^i_{\rho}$.

Lemma 3.5. Let $u^i_{\rho}$ be the unique solution of (28), (29). The following asymptotic formulas hold:

$$
\int_{\Omega} u^i_{\rho} \cdot E^i_0 = \rho^3 \omega_0 \int_{\Omega} \nabla \cdot \left( \frac{\varepsilon_0^0}{\varepsilon^j} M^j \left( \frac{\varepsilon_0^0}{\varepsilon^j} \right) E^i_0(z_j) \right) \cdot E^i_0(z_j) + O(\rho^4),
$$

and

$$
\int_{\Omega} \nabla \times u^i_{\rho} \cdot \nabla \times E^i_0 = \rho^3 \omega_0 \int_{\Omega} \nabla \times \left( \frac{\varepsilon_0^0}{\varepsilon^j} M^j \left( \frac{\varepsilon_0^0}{\varepsilon^j} \right) \nabla \times E^i_0(z_j) \right) \cdot \nabla \times E^i_0(z_j) + O(\rho^4).
$$

Here, the terms $O(\rho^4)$ are bounded by $C \rho^4$, where the constant $C$ is independent of $\rho$ and the set of points $(z_j)_{j=1}^m$.

Proof. Upon insertion of (42) we get

$$
\int_{B} u^i_{\rho} \cdot E^i_0 = \int_{B} u^i_{\rho}(\rho y) \cdot E^i_0(\rho y) \, dy = \rho^3 \int_{B} \frac{1}{\omega_0^2} \left( E^i_0(\rho y) + \left( \nabla q^* \right)(y) \right) \cdot E^i_0(0) + O(\rho^4)
$$

$$
= \rho^3 \frac{1}{\omega_0^2} \left( \int_{B} E^i_0(0) \cdot E^i_0(0) + \int_{B} \nabla q^* \cdot E^i_0(0) \, dy \right) + O(\rho^4).
$$

From the definition of the function $q^*$ it follows immediately that

$$
\psi(y) = q^*(y) + E^i_0(0) \cdot y
$$

is the unique solution to

$$
\begin{cases}
\Delta \psi = 0 & \text{in } B, \text{ and in } \mathbb{R}^3 \setminus B, \\
\psi \text{ is continuous across } \partial B, \\
\frac{\varepsilon_0}{\varepsilon^j} (\partial_\nu \psi)^+ - (\partial_\nu \psi)^- = 0 & \text{on } \partial B, \\
\psi(y) - E^i_0(0) \cdot y \to 0 & \text{as } |y| \to +\infty,
\end{cases}
$$

and therefore

$$
\int_{B} \nabla q^* \cdot E^i_0(0) + \int_{B} \nabla \psi \cdot E^i_0(0) = \int_{B} \nabla \psi \cdot E^i_0(0) = \frac{\varepsilon_0}{\varepsilon^j} \left[ M \left( \frac{\varepsilon_0^0}{\varepsilon^j} \right) E^i_0(0) \right] \cdot E^i_0(0).
$$
Here $M(\varepsilon^0/\varepsilon^1)$ is the polarization tensor defined by (9). Similarly, we obtain

$$\int_{\rho B} \nabla \times u^i \cdot \nabla \times E^i_0 = \rho^3 \int_B \nabla \times u^i(\rho y) \cdot \nabla \times E^i_0(\rho y) \, dy$$

$$= \rho^3 \int_B \frac{1}{\omega_0^2 \varepsilon^0} (\nabla \times E^i_0(\rho y) + \nabla \times (\varepsilon^*)(y)) \, dy \cdot \nabla \times E^i_0(0) + O(\rho^4)$$

$$= \rho^3 \int_B \frac{1}{\omega_0^2 \varepsilon^0} \left( |B| \nabla \times E^i_0(0) \cdot \nabla \times E^i_0(0) + \int_B \nabla \times \varepsilon^*(y) \, dy \cdot \nabla \times E^i_0(0) \right) + O(\rho^4)$$

$$= \rho^3 \int_B \frac{1}{\omega_0^2 \varepsilon^0} \left[ M \left( \frac{\mu_0}{\mu^i} \right) \nabla \times E^i_0(0) \right] \cdot \nabla \times E^i_0(0) + O(\rho^4).$$

The reader is referred to Section 7 in [3] for more details on these calculations.

3.2. Proof of main result

**Lemma 3.6.** Let $P^i_\rho$, $1 \leq i \leq n$, denote the orthogonal projection in $X_\rho(\Omega)$ on the eigenspace of $T_\rho$ corresponding to the eigenvalue $1/(\omega_i^2)^2$, let $P_0$ denote the orthogonal projection in $X_0(\Omega)$ on the eigenspace of $T_0$ corresponding to the eigenvalue $1/\omega_0^2$, and denote $P_\rho = \sum_{i=1}^n P^i_\rho$. We then have the following estimate relating $P_\rho$ to $P_0$

$$\| P_\rho R_\rho(f) - P_0 R_0(f) \|_{L^2(\Omega)} \leq C \rho^{3/2} \| f \|_{X_\rho(\Omega)}$$

(45)

for all $f$ in $X_\rho(\Omega)$.

**Proof.** Let $\Gamma$ be the circle in the complex plane centered at $\omega_0^2$ of radius $\delta$ and $\delta$ is small enough so that the Maxwell equations (5)–(7) have no other eigenvalue inside $\Gamma$. For $z \in \Gamma$, let $w_\rho$ be the unique solution in $Z_\rho(\Omega)$ to the following equations:

$$\begin{cases}
\nabla \times \left( \frac{1}{\mu_\rho} \nabla \times w_\rho \right) - z \varepsilon_\rho w_\rho = f & \text{in } \Omega, \\
\nabla \cdot (\varepsilon_\rho w_\rho) = 0 & \text{in } \Omega, \\
w_\rho \times \nu = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $f \in X_\rho(\Omega)$. Note that

$$(I - z T_\rho)w_\rho = T_\rho \left( \frac{f}{\varepsilon_\rho} \right).$$

Analogously, we define for any $f \in X_0(\Omega)$ and $z \in \Gamma$, $w_0$ as the unique solution in $Z_0(\Omega)$ to the following equations:

$$\begin{cases}
\nabla \times \left( \frac{1}{\mu_0} \nabla \times w_0 \right) - z \varepsilon_0 w_0 = f & \text{in } \Omega, \\
\nabla \cdot (\varepsilon_0 w_0) = 0 & \text{in } \Omega, \\
w_0 \times \nu = 0 & \text{on } \partial \Omega.
\end{cases}$$
The following estimate is derived in the appendix
\[ \|w_\rho - w_0\|_{H^1(\Gamma, \Omega)} \leq C \rho^{3/2} \|f\|_{L^2(\Omega)}, \] (46)
uniformly in \( z \in \Gamma \). The following calculation can be justified using classical results on symmetric compact operators as developed in [9] for example. Provided that
\[ 0 < \delta < \omega_0^2 \]
the image of \( \Gamma \) under the conformal mapping of the complex plane \( z \rightarrow 1/z \) is the circle \( \Gamma' \) centered at \( \omega_0^2/\omega_0^2 - \delta^2 \) and of radius \( \delta/(\omega_0^2 - \delta^2) \). Note that \( 1/\omega_0^2 \) is on the inside of \( \Gamma' \), and that this mapping from \( \Gamma \) to \( \Gamma' \) reverses the orientation of parameterization. Setting \( \zeta = 1/z \), for \( \rho \) small enough,
\[
\frac{1}{2i\pi} \int_\Gamma w_\rho \, \text{d}z = \frac{1}{2i\pi} \int_\Gamma (I - zT_\rho)^{-1}T_\rho \left( \frac{f}{\epsilon_\rho} \right) \, \text{d}z = -\frac{1}{2i\pi} \int_\Gamma \frac{1}{\zeta} (\zeta I - T_\rho)^{-1}T_\rho \left( \frac{f}{\epsilon_\rho} \right) \, \text{d}\zeta \\
= \sum_{i=1}^n (\omega_\rho^i)^2 P^i_\rho \left( \frac{f}{\epsilon_\rho} \right) = \sum_{i=1}^n (\omega_\rho^i)^2 T_\rho P^i_\rho \left( \frac{f}{\epsilon_\rho} \right) = P_\rho \left( \frac{f}{\epsilon_\rho} \right). \tag{47}
\]
We have used the fact that \( T_\rho \) and \( P^i_\rho \) commute. Similarly
\[
\frac{1}{2i\pi} \int_\Gamma w_0 \, \text{d}z = \omega_0^2 P_0 T_0 \left( \frac{f}{\epsilon_0} \right) = P_0 \left( \frac{f}{\epsilon_0} \right). \tag{48}
\]
Combining (46)–(48) we arrive at (45). \( \square \)

The usual way to get to the average of the \( 1/(\omega_\rho^i)^2 \) is to introduce linear operators on a finite dimensional space and use the fact that the trace is independent of the choice of a basis. Such a method is put into practice in [11] for example. We can claim in view of (45) that for \( \rho \) small enough \( P_\rho R_\rho \) is one to one from \( \mathbf{R}(P_\rho) \) into \( \mathbf{R}(P_\rho) \), where \( \mathbf{R} \) denotes the range. But these two finite dimensional spaces have the same dimension, thus \( P_\rho R_\rho \) is an isomorphism from \( \mathbf{R}(P_\rho) \) onto \( \mathbf{R}(P_\rho) \). Set
\[
\tilde{T}_\rho = (P_\rho R_\rho)^{-1}T_\rho P_\rho R_\rho \quad \text{and} \quad \tilde{T} = T_0|_{\mathbf{R}(P_\rho)}
\]
The trace of a finite dimensional linear operator is independent of the choice of a basis. Consequently
\[
\frac{1}{\omega_0^2} - \frac{1}{n} \sum_{i=1}^n \frac{1}{(\omega_\rho^i)^2} = \frac{1}{n} \text{trace}(\tilde{T}_0 - \tilde{T}_\rho) = \frac{1}{n} \sum_{i=1}^n \left\langle (\tilde{T}_0 - \tilde{T}_\rho) E_0^i, E_0^i \right\rangle_{X_0(\Omega)} \\
= \frac{1}{n} \sum_{i=1}^n \left\langle \frac{1}{\omega_0^2} E_0^i, E_0^i \right\rangle_{X_0(\Omega)} - \left\langle \tilde{T}_\rho E_0^i, E_0^i \right\rangle_{X_0(\Omega)} \tag{49}
\]
We now want to introduce \( u_\rho^i \) in \( (\tilde{T}_\rho E_0^i, E_0^i)_{X_0(\Omega)} \). We first split the latter in two terms, the second of these two terms will be proved to be of lower order. For ease of notation we only treat the case of a single
Finally we want to estimate the order of

\[ \langle \tilde{T}_\mu E^0_\rho, E^0_\rho \rangle_{X_\rho(\Omega)} = \int_\Omega \epsilon^0 (P_\rho R_\rho)^{-1} T_\rho P_\rho R_\rho E^i_\rho \cdot E^i_0 \]

\[ = \int_\Omega \epsilon^0 R_\rho^{-1} T_\rho R_\rho E^i_\rho \cdot E^i_0 + \int_\Omega \epsilon^0 ((P_\rho R_\rho)^{-1} T_\rho P_\rho R_\rho - R_\rho^{-1} T_\rho R_\rho) E^i_0 \cdot E^i_0. \]

(50)

Recalling that \( T_\rho R_\rho E^i_0 = u^i_\rho \) we have

\[ \int_\Omega R_\rho^{-1} T_\rho R_\rho E^i_0 \cdot E^i_0 = \int_\Omega \epsilon_\rho u^i_\rho \cdot E^i_0 = \epsilon^0 \int_\Omega u^i_\rho \cdot E^i_0 + (\epsilon^1 - \epsilon^0) \int_{\rho B} u^i_\rho \cdot E^i_0 \]

\[ = \frac{1}{\omega_0^2 \mu_0} \int_\Omega \frac{1}{\omega_0^2} \nabla \times (\nabla \times E^i_0) + (\epsilon^1 - \epsilon^0) \int_{\rho B} u^i_\rho \cdot E^i_0 \]

\[ = \frac{1}{\omega_0^2} \int_\Omega \frac{1}{\omega_0^2} \nabla \times u^i_\rho \cdot \nabla \times E^i_0 + \frac{1}{\omega_0^2} \mu_0 \cdot (\frac{1}{\mu_0} - \frac{1}{\mu_1}) \int_{\rho B} \nabla \times u^i_\rho \cdot \nabla \times E^i_0 + (\epsilon^1 - \epsilon^0) \int_{\rho B} u^i_\rho \cdot E^i_0 \]

\[ = \frac{1}{\omega_0^2} \int_\Omega E^i_0 \cdot E^i_0 + \frac{1}{\omega_0^2} \mu_0 \cdot (\frac{1}{\mu_0} - \frac{1}{\mu_1}) \int_{\rho B} \nabla \times u^i_\rho \cdot \nabla \times E^i_0 + (\epsilon^1 - \epsilon^0) \int_{\rho B} u^i_\rho \cdot E^i_0. \]

(51)

Finally we want to estimate the order of

\[ A_\rho = \int_\Omega \epsilon^0 ((P_\rho R_\rho)^{-1} T_\rho P_\rho R_\rho - R_\rho^{-1} T_\rho R_\rho) E^i_0 \cdot E^i_0. \]

Since \( T_\rho \) and \( P_\rho \) commute as linear operators of \( X_\rho(\Omega) \), this term is equal to

\[ \int_\Omega \epsilon^0 ((P_\rho R_\rho)^{-1} P_\rho - R_\rho^{-1}) u^i_\rho \cdot E^i_0. \]

Since \( u^i_\rho = R_\rho \epsilon_\rho u^i_\rho \), \( A_\rho \) is again equal to

\[ \int_\Omega \epsilon^0 ((P_\rho R_\rho)^{-1} P_\rho R_\rho - I) \epsilon_\rho u^i_\rho \cdot E^i_0. \]

But since \( P_0 \) is an orthogonal projection and \( E^i_0 \) lies in the range of \( P_0 \), the following equality holds

\[ \int_\Omega \epsilon^0 (P_0 - I) \epsilon_\rho u^i_\rho \cdot E^i_0 = 0. \]

Therefore

\[ A_\rho = \int_\Omega \epsilon^0 ((P_\rho R_\rho)^{-1} P_\rho R_\rho - P_0) \epsilon_\rho u^i_\rho \cdot E^i_0. \]

We use again the fact that \( P_0 \) is an orthogonal projection to write

\[ \int_\Omega \epsilon^0 (I - P_0)(P_\rho R_\rho)^{-1} P_\rho R_\rho \epsilon_\rho u^i_\rho \cdot (P_\rho R_\rho)^{-1} P_\rho R_\rho \epsilon_\rho u^i_\rho = 0. \]
Since $P_0 = P_0(P_\rho R_\rho)^{-1} P_\rho R_\rho$ we rewrite $A_\rho$ as

$$
\int_{\Omega} e^0((P_\rho R_\rho)^{-1} P_\rho R_\rho - P_0) \epsilon_\rho u^i_{\rho} \cdot (E^i_0 - \omega^2_0(P_\rho R_\rho)^{-1} P_\rho R_\rho \epsilon_\rho u^i_{\rho}).
$$

Finally, remarking that $((P_\rho R_\rho)^{-1} P_\rho R_\rho - P_0)E^i_0 = 0$, $A_\rho$ is equal to

$$
\int_{\Omega} e^0((P_\rho R_\rho)^{-1} P_\rho R_\rho - P_0) \left( \epsilon_\rho u^i_{\rho} - \frac{E^i_0}{\omega^2_0} \right) \cdot (E^i_0 - \omega^2_0(P_\rho R_\rho)^{-1} P_\rho R_\rho \epsilon_\rho u^i_{\rho}).
$$

We notice that for all $f$ in $X_0(\Omega)$

$$
\left\| (R_\rho - R_0)P_0(f) \right\|_{L^2(\Omega)} = \left( \int_{\rho_B} \left( \frac{1}{\epsilon_1} - \frac{1}{\epsilon_0} \right)^2 \left| P_0(f) \right|^2 \right)^{1/2}.
$$

But all the elements in the space $R(P_0)$ are smooth and any of their norm is bounded by the $L^2$ norm. We infer that

$$
\left\| (R_\rho - R_0)P_0(f) \right\|_{L^2(\Omega)} \leq C \rho^{3/2} \left\| P_0(f) \right\|_{L^2(\Omega)} \leq C \rho^{3/2} \left\| f \right\|_{L^2(\Omega)}.
$$

Now as a consequence of (45) we may write for all $f$ in $X_0(\Omega)$

$$
\left\| P_\rho R_\rho(f) - R_\rho P_0(f) \right\|_{L^2(\Omega)} \leq C \rho^{3/2} \left\| f \right\|_{X_0(\Omega)}. \tag{53}
$$

Since the operators $P_\rho$ are uniformly bounded in $X_\rho(\Omega)$ and $P_\rho^2 = P_\rho$, it follows that

$$
\left\| P_\rho R_\rho(f) - P_\rho R_\rho P_0(f) \right\|_{L^2(\Omega)} \leq C \rho^{3/2} \left\| f \right\|_{X_0(\Omega)}. \tag{54}
$$

Finally by definition of $(P_\rho R_\rho)^{-1}$ we can write the following estimate

$$
\left\| (P_\rho R_\rho)^{-1} P_\rho R_\rho(f) - P_0(f) \right\|_{L^2(\Omega)} \leq C \rho^{3/2} \left\| f \right\|_{X_0(\Omega)}. \tag{55}
$$

We are now ready to estimate the order of the term $A_\rho$. First remember that we have proved

$$
\left\| \epsilon_\rho u^i_{\rho} - \frac{E^i_0}{\omega^2_0} \right\|_{X_0(\Omega)} \leq C \rho^{3/2}. \tag{56}
$$

Combining (55) and (56), we find

$$
\left\| (P_\rho R_\rho)^{-1} P_\rho R_\rho - P_0 \right\|_{L^2(\Omega)} \left( \epsilon_\rho u^i_{\rho} - \frac{E^i_0}{\omega^2_0} \right) \leq C \rho^3. \tag{57}
$$
Next

\[
\| E_0^i - \omega_0^2 (P_\rho R_\rho)^{-1} P_\rho R_\rho \epsilon_\rho u_\rho^i \|_{L^2(\Omega)}
\leq \| P_0 (E_0^i - \omega_0^2 \epsilon_\rho u_\rho^i) \|_{L^2(\Omega)} + \| ((P_\rho R_\rho)^{-1} P_\rho R_\rho - P_0) \omega_0^2 \epsilon_\rho u_\rho^i \|_{L^2(\Omega)}
\leq C \| E_0^i - \omega_0^2 \epsilon_\rho u_\rho^i \|_{L^2(\Omega)} + C \rho^{3/2} \| \epsilon_\rho u_\rho^i \|_{L^2(\Omega)} \leq C \rho^{3/2}.
\] (58)

Combining (57) and (58) we find the estimate

\[
| A_\rho | \leq C \rho^{3/2}.
\]

We conclude using (49)–(51) that

\[
\frac{1}{\omega_0^2} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(\omega_\rho^i)^2} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\epsilon_\rho^0}{\omega_\rho^0 - \omega_\rho^i} - \frac{1}{\omega_\rho^0} \right) \int_{\rho B} \nabla \times u_\rho^i \cdot \nabla \times E_0^i + (\epsilon_\rho^0 - \epsilon_\rho^1) \epsilon_\rho^0 \int_{\rho B} u_\rho^i \cdot E_0^i + O(\rho^{9/2}).
\] (59)

Finally, Lemma 3.5 applied to (59) yields our main result.

**Appendix**

We begin by deriving two estimates from [3]. In this paragraph we will make references to [3] and use the same notations, that is in this paragraph only, the notations from [3] will overrule the current notations of this paper.

Note that the function \( q^* \) lies in the space \( \mathbb{R} \). It follows from the definition of \( q^* \) and an integration by parts that

\[
\int_{\mathbb{R}^3 \setminus B} \mu^0 |\nabla q^*|^2 + \int_B \mu^1 |\nabla q^*|^2 = \int_B (\mu^0 - \mu^1) \nabla q^* \cdot H_0(0)
\]

consequently

\[
\| \nabla q^* \|_{L^2(\mathbb{R}^3)} \leq C |H_0(0)|
\]

and using the first estimate in Proposition 1 in [3]

\[
\| H_\rho - H_0 \|_{L^2(\Omega)} \leq C |H_0(0)| \rho^{3/2},
\]

where \( C \) does not depend on the boundary data \( g \).

The results from Section 3 of [3] can be carried out to our case although we now have a tangential zero boundary condition instead of the normal zero boundary condition in [3]. For ease of notation, we assume that there only is a single inhomogeneity centered at 0. Supposing that \( f \) is smooth, which makes \( w_0 \) smooth, we can construct two auxiliary fields akin to those from Section 5 of [3]. We then arrive at the equivalent of the two estimates in Proposition 1 in [3] and just as shown above, we derive

\[
\| w_\rho - w_0 \|_{L^2(\Omega)} \leq C \rho^{3/2} |w_0(0)|,
\] (60)
The $H^2(\Omega)$ norm of $w_0$ depends only on $\|f\|_{L^2(\Omega)}$. Recall that $H^2(\Omega)$ is continuously embedded in $C^0(\overline{\Omega})$ since $\partial \Omega$ was chosen to be smooth. We derive from (60)

$$\|w_\rho - w_0\|_{L^2(\Omega)} \leq C\rho^{3/2}\|f\|_{L^2(\Omega)}.$$  

(61)

Now estimate (61) can be inferred by density if $f$ has only an $X_0(\Omega)$ regularity.

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References