Scattering by a perfect conductor in a waveguide: energy preserving schemes for integral equations

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ABSTRACT

The scattering matrix for a perfectly conducting electrical cylinder, (or a sound hard obstacle), in a waveguide is unitary. This is a well known result which is a consequence of the conservation of power. When a numerical method is employed to approximate the reflection and transmission coefficients of the cylinder, an approximate scattering matrix can be constructed. An integral equation of the second kind for an unknown density can be solved, and the density can then be used for computing the entries of the approximate scattering matrix. We show that this approximate matrix is unitary for cylinders of symmetric cross section, regardless of the order of the approximation. In the non symmetric case, the approximate scattering matrix still satisfies a conservation of energy condition, albeit in an unfamiliar form. As the order of the approximation is increased, conservation of energy is also satisfied in the more familiar form to machine accuracy.

1 Introduction

The scattering of an incident mode by a perfectly conducting cylinder is a fundamental problem in microwave and acoustic technology. It has a number of applications in devices such as tuners, resonant cavities, and as a basic component of a photonic structure [11, 22].
Accordingly, the study of this scattering problem has received considerable attention over the last fifty years.

If the radius of the cylinder $R_0$ is very small compared to the width $W$ of the waveguide, i.e., $R_0/W << 1$, then the solution of the problem can be approximated by variational, heuristic, or asymptotic methods [11, 16, 7]. All give approximations to the reflection and transmission coefficients. If on the other hand $R_0/W$ is of order 1, then numerical methods are employed to yield approximations to these coefficients. In either limit the reflection and transmission coefficients can be combined to construct an approximation to the scattering matrix, $P$, for the cylinder.

Assuming that the cylinder is a perfect electrical conductor (or acoustically hard), the scattering matrix $P$ for the target is unitary [7]. This is just the mathematical statement of the conservation of power. We are interested in knowing which numerical methods, for $R_0/W$ of order 1, yield approximate scattering matrices which are unitary. This is not only a fundamentally interesting question, but is important in applications where the cylinder is used in a more complicated circuit.

The ability of certain numerical methods to preserve conservation and reciprocity laws has been considered by a number of authors in the context of scattering problems [1, 2, 9]. The results we present here add to this body of work. They all possess an interesting common feature; if the numerical method is carefully chosen, then the approximations obtained satisfy exactly these laws. These remarkable facts are true, regardless of the method’s accuracy. Thus, the adherence of a numerical method to these laws does not guarantee accuracy.

In this paper we first reformulate the scattering problem as an integral equation of the second kind for an unknown density $\mu$. Our approach closely follows the well known technique employed for compact scatterers in an infinite homogeneous environment [5, 6]. Integral equations of the second kind to be solved for unknown densities have also been used for elastic waves in waveguides, in particular for determining resonant frequencies, see [13]. We show that under suitable non resonance conditions, the integral equation we set up for $\mu$ is well posed. We use Nystrom’s method, or more precisely a quadrature method adapted to singularities of logarithmic type, to discretize it. Using the computed approximate density $\mu$, we proceed to find approximations to the reflection and transmission coefficients. These are then used to construct our approximate scattering matrix, that we still denote $P$ for ease of notations. For scatterers that are symmetric with respect to a line perpendicular to the walls of the waveguides, we prove that $P$ is symmetric. This is true, regardless of the order of interpolation or the number of collocation points in Nystrom’s method. This result is similar in spirit to those presented in references [2, 7, 9]. We also show that our approximate matrix $P$ is no longer unitary when the cylinder is no longer symmetric, however an equivalent, and more natural, conservation identity is preserved in that case. Of course, as the order of our approximation is increased the numerical approximation to the continuous scattering matrix becomes unitary to machine accuracy.

We shall now briefly outline the remainder of this paper. In section 2, we state our scattering problem and recall the definition of transmission and reflection coefficients. We then prove that the scattering matrix is unitary, and the less known fact that, in the case of one propagating mode, it is also symmetric. Symmetry of the scattering matrix is related to complex conjugation, which reverses incoming waves. Along those lines, we relate in identity (18) a field produced by a left incoming wave to the field produced by a right incoming
wave, using reflection and transmission coefficients. Identity (18) can be regarded to be a reciprocity identity for guided waves impinging a non symmetric perfect conductor. Next we derive our integral equation of the second kind for the unknown density $\mu$, and indicate how reflection and transmission coefficients are calculated from $\mu$. As identity (18) is crucial throughout this paper, we derive its analog (36) for the density $\mu$, directly from the integral equation satisfied by $\mu$. In section 3, we show how we discretized the integral equation for $\mu$, first using a general projection operator $\Pi$, and then making that projection operator explicit for the numerical scheme we followed. We then present numerical results for the calculation of reflection and transmission coefficients, and we verify the discrete conservation of energy property. For comparison, we present another numerical simulation based on the finite element method. Finally we derive identity (36) in the discrete case assuming that the projection operator $\Pi$ preserves complex conjugation. Discrete conservation laws follow immediately. Section 4 is concerned with scatterers that are symmetric about a line perpendicular to the walls of the waveguide. If the projector operator $\Pi$ satisfies an additional property expressing conservation of symmetries, then the symmetry for the transmission coefficients is preserved at the discrete level, and conservation of energy identities appear at the discrete level in their more familiar form. Section 5 is a discussion of different generalizations of previously stated results: first to an alternate formulation of our integral equation method, then to the multiple propagating mode case, finally to problems with periodic wall conditions.

2 The continuous case

2.1 Conservation of energy identities and properties of the scattering matrix

We are concerned with the scattering properties of a hard object immersed in a two dimensional hard waveguide. This models the cross section of a cylinder that is a perfect conductor in between two plane walls that are also perfect conductors, and it is assumed that a Transverse Magnetic (TM) field propagates in this structure. This leads to Neumann conditions on the boundary of the two dimensional object and on the walls. Transverse Electric (TE) fields lead to Dirichlet boundary conditions: this case can be handled in a similar fashion, but we will not treat it in this paper. Finally, we assume single mode propagation in most of this paper. We indicate in the end how our results can be generalized to multiple mode propagation.

We denote $\tilde{x}$ and $\tilde{y}$ the two dimensional spatial variables for this problem. Assume the rectangular waveguide is unbounded in the $\tilde{x}$ direction and has width $w$ in the $\tilde{y}$ direction. We choose the lower edge of the waveguide to be the line $\tilde{y} = 0$. We non dimensionalize our coordinate system by setting $x = \frac{\tilde{x}}{w}$, $y = \frac{\tilde{y}}{w}$.

The dimensionless wave number for our problem is now defined by $k = \frac{\omega w}{c_0}$. We denote the scatterer by $S$ and assume its boundary $\partial S$ is a smooth closed curve. This object is situated
between the lines $x = a$ and $x = b$, with $a < b$, and is bounded away from the waveguide walls, $\partial W$. we refer the reader who is unfamiliar with waveguide theory to [7], chapter 5, for the definition of propagating modes. Since we have assumed single mode propagation, this requires $0 < k < \pi$. Suppose an incident time harmonic wave $A e^{ikx}$ strikes the object $S$ from the left and an incident wave $B e^{-ikx}$ strikes the object $S$ from the right. Note that we have discarded the time dependency part, which can be restored by multiplying by $e^{-i\omega t}$. Denote $u$ the resulting total field. $u$ satisfies,

$$\begin{align*}
(\Delta + k^2)u &= 0 \text{ in } W \setminus S \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial W \text{ and on } \partial S \\
u(x, y) &\sim A e^{ikx} + Ce^{-ikx} \text{ for } z << a \quad (3) \\
u(x, y) &\sim Be^{-ikx} + De^{ikx} \text{ for } z >> b, \quad (4)
\end{align*}$$

for a pair of unknown complex numbers $C$ and $D$, and where the decay in the asymptotic estimates is exponential in $|x|$. Note that the perfect conductor condition in the TM case appears in equation (2). The underlying assumption for the rest of the paper is that problem (1-4) has a unique solution. This is known to be true, except possibly, for a discrete set of resonant wavenumbers, see [4]. The existence of resonant wavenumbers is still an area under investigation. Linton and Evans [13] brought strong numerical evidence of existence of those resonant wavenumbers for a wide class of shapes. Note that, if the Neumann conditions were replaced by Dirichlet conditions, and under some geometric assumption on $S$, Vainberg and Mazya proved that (1-4) would indeed have a unique solution, see [20, 21]. More recently, Shipman and Volkov proved an analogous result for dielectrics in inverse periodic structures in the case of one propagating mode, see [18] (note that inverse structures refer to structures in which embedded dielectrics have a lower wavenumber than that of the surrounding medium).

For sake of completion, we derive the energy conservation identity for $u$ satisfying (1-4).

Applying Green’s theorem in

$$D = \{(x, y)|(x, y) \in W \setminus S, a \leq x \leq b\}$$

we find

$$\int_{x=a} u^*(x, y) \frac{\partial}{\partial x} u(x, y) dy - \int_{x=b} u^*(x, y) \frac{\partial}{\partial x} u(x, y) dy = \int_D (k^2 |u|^2 - |\nabla u|^2) dx dy,$$

It follows that

$$\text{Im} \left\{ \int_{x=a} u^*(x, y) \frac{\partial}{\partial x} u(x, y) dy - \int_{x=b} u^*(x, y) \frac{\partial}{\partial x} u(x, y) dy \right\} = 0. \quad (5)$$

A calculation shows that (5) combined with (3, 4) yields, if we let $a \to -\infty$ and $b \to \infty$, the energy conservation identity

$$|A|^2 + |B|^2 = |C|^2 + |D|^2. \quad (6)$$

Reflection and transmission coefficients for the scatterer $S$ are defined through the application of incoming unit waves. More precisely, if the only incoming wave is $e^{ikx}$ ($A = 1, B = 0$),
then we define $R$ and $T$ by
\[
\begin{align*}
u(x, y) &\sim e^{ikx} + Re^{-ikx} \text{ for } z << a \\
u(x, y) &\sim Te^{ikx} \text{ for } z >> b,
\end{align*}
\]
and if the only incoming wave is $e^{-ikx}$ ($A = 0, B = 1$), then we define the reflexion coefficient $R'$ and the transmission coefficient $T'$ by
\[
\begin{align*}
u(x, y) &\sim T'e^{-ikx} \text{ for } z << a \\
u(x, y) &\sim e^{-ikx} + R'e^{ikx} \text{ for } z >> b.
\end{align*}
\]
Define $P$ the scattering matrix,
\[P = \begin{pmatrix} R & T' \\ T & R' \end{pmatrix}.
\]
We recall that the scattering metric acts by multiplication on incident fields vectors to obtain scattered fields. In addition, it is well known that this matrix is unitary. However, it is less commonly known that $P$ is also symmetric, and thus we derive these two properties below.

**Proposition 2.1** The scattering matrix $P$ is unitary and symmetric.

**proof:** Going back to $u$ verifying (1-4), we have, by linearity
\[
P \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C \\ D \end{pmatrix}.
\]
(7)
Since (6) and (7) hold for any complex numbers $A$ and $B$ we can claim that the matrix $P$ is unitary. To show that $P$ is also symmetric, denote $v$ the solution to (1-4) for the choice $A = 1, B = 0$. Then $C = R, D = T$. We notice that $v^*$ is the solution to (1-4) for the choice $A = R^*, B = T^*$ and
\[
P \begin{pmatrix} R^* \\ T^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
(8)
From where, using the fact that $P$ is unitary
\[
\begin{pmatrix} R^* \\ T^* \end{pmatrix} = (P^T)^* \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
(9)
which yields $T = T'$. ◦
Note that saying that $P$ is unitary and symmetric is equivalent to the following set of identities,
\[
T = T' \quad (10)
\]
\[|R| = |R'| \quad (11)
\]
\[|R|^2 + |T'|^2 = 1 \quad (12)
\]
\[R^*T + R'T^* = 0 \quad (13)
\]
or to the set

\begin{align*}
T &= T' \\
|R| &= |R'| \\
|R|^2 + T^*T' &= 1 \\
R^*T + R'T^* &= 0
\end{align*}

The goal of this paper is to construct a numerical method that will calculate very accurately all transmission and reflection coefficients, in such a way that the energy identities (16) and (17) are exact at the discrete level. Moreover, it will become apparent why the more unfamiliar identity expressing energy conservation (16,17) is in our case more natural to consider than the traditional form (12,13).

### 2.2 A relation between left and right incoming fields

Conservation of energy implies an interesting relation between fields produced by a unit left and a unit right incoming field, respectively. In the remainder of this paper we will denote by \( u \) the field resulting by a left incoming wave \( e^{ikx} \). \( u \) solves (1-4) for \( A = 1, B = 0, \) and \( C = R, D = T \) have to be determined. Similarly denote by \( u' \) the field resulting by a left incoming wave \( e^{-ikx} \). \( u' \) solves (1-4) for \( A = 0, B = 1, \) and \( C = T', D = R' \) have to be determined. Since there is a very simple relation between the incoming wave producing \( u \) and the incoming wave producing \( u' \), a relation between the total fields \( u \) and \( u' \) is to be expected.

**Proposition 2.2** The fields \( u \) and \( u' \) are related by the following identity

\[ u^* = R^*u + T^*u'. \]

**proof:** From the asymptotic estimates

\begin{align*}
u(x, y) &\sim e^{ikx} + Re^{-ikx} \quad \text{for } z << a \\
u(x, y) &\sim Te^{ikx} \quad \text{for } z >> b,
\end{align*}

and

\begin{align*}
u'(x, y) &\sim T'e^{-ikx} \quad \text{for } z << a \\
u'(x, y) &\sim e^{-ikx} + R'e^{ikx} \quad \text{for } z >> b,
\end{align*}

it follows that

\begin{align*}
T^*u'(x, y) + R^*u(x, y) &\sim (T'T^* + RR^*) e^{-ikx} + R^* e^{ikx} \quad \text{for } z << a \\
T^*u'(x, y) + R^*u(x, y) &\sim T^* e^{-ikx} + (R'T^* + R^*T) e^{ikx} \quad \text{for } z >> b,
\end{align*}

In view of (16) and (17), \( T^*u' + R^*u \) has the same asymptotics as \( u^* \). As (1-4) was assumed to have no trivial solution, we conclude that \( u^* = R^*u + T^*u' \). \( \diamond \)
2.3 An integral equation for the scattered field

The finite element method is a popular means of evaluating \( u \) and inferring \( R \) and \( T \). Some packages are commercially available, and we will give later in this paper an example of how FEMLAB can be put into use. However, we show in this paper that boundary integral equation methods yield more accurate results while using linear systems of much smaller size, and under some natural conditions on how they are discretized, they preserve the energy identities (16-17).

To form an integral equation for \( u \), we need the suitable Green’s function for this problem. It can be represented in the following series form,

\[
G(x, y, x_0, y_0) = \sum_{n=0}^{\infty} -\frac{1}{2ik_n} e^{ik_n|x-x_0|} \psi_n(y) \psi_n(y_0),
\]

where

\[
k_n = \begin{cases} k & \text{if } n = 0 \\ i\sqrt{n^2\pi^2 - k^2} & \text{if } n \geq 1 \end{cases},
\]

and

\[
\psi_n(y) = \begin{cases} 1 & \text{if } n = 0 \\ \sqrt{2} \cos(\pi ny) & \text{if } n \geq 1 \end{cases}.
\]

Note that

\[\Delta_{x,y} G = -\delta_{x_0,y_0},\]

and \( G \) satisfies the Neumann conditions for \((x_0, y_0)\) on \(\partial W\) and \((x, y)\) in \(W\). Applying Green’s theorem, \( u \) can be represented using its value on \(\partial S\), and on the portions of \(W\) where \(x = a\) and where \(x = b\),

\[
u(x_0, y_0) = \int_{\partial S} \frac{\partial G}{\partial \nu}(x, y, x_0, y_0)u(x, y)d\sigma(x, y)
+ \int_{\{x=a\}} \frac{\partial G}{\partial x}(x, y, x_0, y_0)u(x, y)dy - \int_{\{x=a\}} G(x, y, x_0, y_0)\frac{\partial u}{\partial x}(x, y)dy
- \int_{\{x=b\}} \frac{\partial G}{\partial x}(x, y, x_0, y_0)u(x, y)dy + \int_{\{x=b\}} G(x, y, x_0, y_0)\frac{\partial u}{\partial x}(x, y)dy,
\]

for \((x_0, y_0)\) in \(W \setminus S\) such that \(a < x_0 < b\), where \(d\sigma\) is the length element on \(\partial S\), and where \(\frac{\partial}{\partial \nu}\) is the normal derivative \(\nabla_{x,y} \cdot \nu_{x,y}, \nu_{x,y}\) being the exterior normal derivative to \(\partial S\) at the point \((x, y)\). Note that this latter equation uses the fact that \( u \) has zero normal derivative on \(\partial S\). Using again our single propagating mode assumption we observe that as \(x \to -\infty\),

\[
G \sim -\frac{1}{2ik} e^{-ik(x-x_0)}, \quad \frac{\partial G}{\partial x} \sim \frac{1}{2} e^{-ik(x-x_0)}
\]

\[
u \sim e^{ikx} + Re^{-ikx}, \quad \frac{\partial u}{\partial x} \sim ike^{ikx} - ikRe^{-ikx}
\]

and as \(x \to \infty\),

\[
G \sim -\frac{1}{2ik} e^{ik(x-x_0)}, \quad \frac{\partial G}{\partial x} \sim -\frac{1}{2} e^{ik(x-x_0)}
\]

\[
u \sim Te^{ikx}, \quad \frac{\partial u}{\partial x} \sim ikTe^{ikx}.
\]
Combining (20-22), as \( a \to -\infty \) and \( b \to \infty \), we obtain

\[
u(x_0, y_0) = e^{ikx_0} + \int_{\partial S} \frac{\partial G}{\partial \nu}(x, y, x_0, y_0)u(x, y)\,d\sigma(x, y), \tag{23}
\]

for \((x_0, y_0)\) in \( W \setminus \bar{S} \). As \((x_0, y_0)\) approaches \( \partial S \), using the classical jump formula for double layer potential, we obtain

\[rac{1}{2}u(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial \nu}(x, y, x_0, y_0)u(x, y)\,d\sigma(x, y) = e^{ikx_0}, \tag{24}
\]

for \((x_0, y_0)\) on \( \partial S \). Identity (24), may serve as an integral equation for solving for \( u \) on \( \partial S \). Plugging the solution into (23), \( u \) can be evaluated everywhere and the reflection coefficient \( R \) and the transmission coefficient \( T \) may be calculated.

There is an alternative way of representing the solution \( u \) to (1-4), using an integral involving some layer density \( \mu \) on \( \partial S \). If we set

\[
\tilde{u}(x_0, y_0) = e^{ikx_0} + \int_{\partial S} G(x, y, x_0, y_0)\mu(x, y)\,d\sigma(x, y), \tag{25}
\]

for \((x_0, y_0)\) in \( W \setminus \bar{S}, \tilde{u} \) clearly satisfies (1) and

\[
\begin{align*}
\tilde{u}(x, y) & \sim e^{ikx} + \tilde{R}e^{-ikx} \text{ for } z << a\\
\tilde{u}(x, y) & \sim \tilde{T}e^{ikx} \text{ for } z >> b,
\end{align*}
\]

for some complex numbers \( \tilde{R} \) and \( \tilde{T} \). As \((x_0, y_0)\) approaches \( \partial S \), if we require

\[
\frac{1}{2}\mu(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial \nu}(x, y, x_0, y_0)\mu(x, y)\,d\sigma(x, y) = ike^{ikx_0}\nu_{x_0}, \tag{26}
\]

where \( \frac{\partial}{\partial \nu_0} \) designates the normal derivative \( \nabla_{x_0,y_0} \cdot \nu_{x_0,y_0} \) (\( \nu_{x_0,y_0} \) being the exterior normal derivative to \( \partial S \) at the point \((x_0, y_0)\)), due to the classical jump formula for double layer potential, we can claim that \( \tilde{u} \) satisfies (2). Due to the assumed uniqueness for problem (1-4), we can claim that \( \tilde{u} \) is equal to \( u \).

In our numerical work, equation (26) is the actual integral equation we solved for \( \mu \) on \( \partial S \). We then plugged the numerical solution into (25), and the reflection coefficient \( R \) and the transmission coefficient \( T \) could be inferred. In effect,

\[
R = -\frac{1}{2ik} \int_{\partial S} e^{ikx}\mu\,d\sigma, \tag{27}
\]

\[
T = 1 - \frac{1}{2ik} \int_{\partial S} e^{-ikx}\mu\,d\sigma. \tag{28}
\]

Notice that we chose formulation (25,26) over (23,24). There is a slight advantage in making that choice insofar that formula (25) can be regarded as simpler than (23). This unknown density formulation is not novel. It was successfully put into use by Linton and Evans, [13]. A theoretical argument proves that the integral equations (24) and (26) are well posed if \( k \) is not a resonant wavenumber, in a sense specified in the following proposition. This result was proved by Ursell [19]. We provide a proof in this paper for sake of completion.
Proposition 2.3 The integral equations (24) and (26) have each a unique solution if the problem (1-4) has a unique solution, and if $k$ is not a resonant wavenumber for the Dirichlet problem in $S$.

proof: Suppose there is a continuous density $\mu$ such that

$$\frac{1}{2} \mu(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial \nu_0}(x, y, x_0, y_0) \mu(x, y) d\sigma(x, y) = 0,$$

for $(x_0, y_0)$ on $\partial S$. Set

$$v(x_0, y_0) = \int_{\partial S} G(x, y, x_0, y_0) \mu(x, y) d\sigma(x, y)$$

for $(x_0, y_0)$ in $W$. $v$ satisfies $\frac{\partial v}{\partial \nu}$ = 0 on the outside of $\partial S$, therefore $v$ satisfies problem (1-4) with $A = B = 0$. As problem (1-4) is well posed, $v$ is uniformly zero on $W \setminus S$. Note that $v$ is continuous across $\partial S$. Then as $k$ is not a resonant wavenumber for the Dirichlet problem in $S$, $v$ is uniformly zero in $S$. It follows that

$$\frac{\partial v}{\partial \nu}|^+ - \frac{\partial v}{\partial \nu}|^- = 0 = \mu,$$

This proves uniqueness for equation (26). Existence follows by a classical Fredholm argument. Finally, equations (26) and (24) are dual in a sense explained in [5], ensuring that equation (24) is well posed too. ♦

In the remainder of this paper we will assume that the hypothesis of Proposition 2.3 are satisfied, and thus integral equation (26) is well posed.

Remark: If $k$ is a resonant wavenumber for the Dirichlet problem in $S$, then integral equations (24) and (26) are ill posed.

2.4 An analog to identity (18) for surface densities

For the remainder of this paper we will denote $\mu$ and $\mu'$ the densities related to $u$ and $u'$, that is,

$$u(x_0, y_0) = e^{ikx_0} + \int_{\partial S} G(x, y, x_0, y_0) \mu(x, y) d\sigma(x, y),$$

and

$$u'(x_0, y_0) = e^{-ikx_0} + \int_{\partial S} G(x, y, x_0, y_0) \mu'(x, y) d\sigma(x, y).$$

Accordingly, $\mu$ and $\mu'$ satisfy the integral equations,

$$\frac{1}{2} \mu(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial \nu_0}(x, y, x_0, y_0) \mu(x, y) d\sigma(x, y) = ike^{ikx_0} \nu_{x_0},$$

$$\frac{1}{2} \mu'(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial \nu_0}(x, y, x_0, y_0) \mu'(x, y) d\sigma(x, y) = -ike^{-ikx_0} \nu_{x_0},$$

The densities $\mu$ and $\mu'$ are related by the following identity.
Proposition 2.4

\[ \mu^* = R^* \mu + T^* \mu'. \] (36)

Identity (36) will prove to be essential in deriving the discrete versions of the conservation of energy formulas (16,17). Therefore, we will provide two derivations of (36). The first one hinges on identity (18), and is the more straightforward. The second one is only based on equations (34,35): the second derivation will be suitable for the derivation of the discrete analog of (36).

first derivation of (36):

From (18,32,33), we derive,

\[ R^* e^{ikz_0} + T^* e^{-ikz_0} - e^{-ikz_0} + \int_{\partial S} G(R^* \mu + T^* \mu') - G^* \mu^* d\sigma = 0. \]

But,

\[ \int_{\partial S} G^* \mu^* d\sigma = \int_{\partial S} G\mu^* d\sigma - \frac{1}{ik} \int_{\partial S} \cos(k(x - x_0)) \mu^*(x,y) d\sigma(x,y) \]
\[ = \int_{\partial S} G\mu^* d\sigma - \frac{1}{2ik} \int_{\partial S} (e^{ik(x-x_0)} + e^{-ik(x-x_0)}) \mu^*(x,y) d\sigma(x,y) \]
\[ = \int_{\partial S} G\mu^* d\sigma - \frac{1}{2ik} \int_{\partial S} (e^{ik(x-x_0)} + e^{-ik(x-x_0)}) \mu^*(x,y) d\sigma(x,y) \]
\[ = \int_{\partial S} G\mu^* d\sigma - (T^* - 1)e^{-ikx_0} - R^* e^{ikx_0}. \]

This proves that

\[ \int_{\partial S} G(R^* \mu + T^* \mu' - \mu^*) = 0, \] (37)

for all \((x_0, y_0)\) in \(W \setminus \overline{S}\). We may then take the normal derivative of (37) and take the limit as \((x_0, y_0)\) approaches \(\partial S\). Identity (36) follows from the well posedness assumption for equation (26).

second derivation of (36):

From (34), we derive,

\[ \frac{1}{2} \mu^*(x_0, y_0) = \int_{\partial S} \frac{\partial G^*}{\partial v_0}(x,y,x_0,y_0) \mu^*(x,y) d\sigma(x,y) = -iek^{-ikx_0} \nu_{x_0}. \]

In this second derivation too, we will take advantage of the fact that the imaginary part of the Green's function G is regular.

\[ \int_{\partial S} \frac{\partial G^*}{\partial v_0} \mu^* d\sigma = \int_{\partial S} \frac{\partial G}{\partial v_0} \mu^* d\sigma - \frac{1}{ik} \int_{\partial S} \frac{\partial}{\partial v_0} \cos(k(x - x_0)) \mu^*(x,y) d\sigma(x,y) \]
\[ = \int_{\partial S} \frac{\partial G}{\partial v_0} \mu^* d\sigma - \frac{1}{2ik} \int_{\partial S} \frac{\partial}{\partial v_0} (e^{ik(x-x_0)} + e^{-ik(x-x_0)}) \mu^*(x,y) d\sigma(x,y) \]
\[ = \int_{\partial S} \frac{\partial G}{\partial v_0} \mu^* d\sigma - \frac{1}{2} \nu_{x_0} \int_{\partial S} (-e^{ik(x-x_0)} + e^{-ik(x-x_0)}) \mu^*(x,y) d\sigma(x,y) \]
\[ = \int_{\partial S} \frac{\partial G}{\partial v_0} \mu^* d\sigma + ik(T^* - 1)e^{-ikx_0} \nu_{x_0} - ikR^* e^{ikx_0} \nu_{x_0}. \]
We conclude that \( \mu^* \) satisfies the equation

\[
\frac{1}{2} \mu^*(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial \nu_0}(x, y, x_0, y_0) \mu^*(x, y) d\sigma(x, y) = -i k T^* e^{-i k x_0} \nu_{x_0} + i k R^* e^{i k x_0} \nu_{x_0},
\]

which is the same equation as that satisfied by \( T^* \mu' + R^* \mu \). In view of Proposition 2.3, we must have \( \mu^* = T^* \mu' + R^* \mu \).

\[\Box\]

**Remark 2.4:** The first derivation of identity (36) is based on (18), its analog for total fields. In turn (18) was derived from the conservation of energy identities. In contrast the second derivation of (36) makes only use of the definition of the reflection and transmission coefficients \( R \) and \( T \). We observe that the conservation of energy identities (16-17) can be then deduced from (36). Recalling the expressions of transmission and reflection coefficients expression in terms of the densities \( \mu \) and \( \mu' \),

\[
\begin{align*}
R & = -\frac{1}{2ik} \int_{\partial S} e^{ikx} \mu d\sigma \\
T & = 1 - \frac{1}{2ik} \int_{\partial S} e^{-ikx} \mu d\sigma \\
R' & = -\frac{1}{2ik} \int_{\partial S} e^{-ikx} \mu' d\sigma \\
T' & = 1 - \frac{1}{2ik} \int_{\partial S} e^{ikx} \mu' d\sigma
\end{align*}
\]

To obtain (16) we just multiply (36) by \( -\frac{1}{2ik} e^{ikx} \), and we then integrate over \( \partial S \), leading to

\[-(T^* - 1) = R^* R + T^* (T' - 1).\]

To obtain (17) we just multiply (36) by \( -\frac{1}{2ik} e^{-ikx} \), and we then integrate over \( \partial S \), leading to

\[-R^* = R^* (T - 1) + T^* R'.\]

We will derive the discrete equivalent of conservation laws (16-17) in the same fashion.

### 3 The discrete conservation laws

#### 3.1 Discretizing integral equation (34)

We denote by

\[ t \rightarrow (x(t), y(t)), \quad 0 \leq t < 2\pi, \]

a parametrization of the curve \( \partial S \). \( x \) and \( y \) are assumed to be \( 2\pi \) periodic. An efficient numerical method for solving integral equation (34) must take into account the logarithmic singularities appearing in the Green’s function \( G \). It is possible to split \( G \) in the sum of the usual radiating fundamental solution of the Helmholtz operator in free space and an analytic function. Consequently we may write

\[
\frac{\partial G}{\partial \nu_{x_0,y_0}}(x(v), y(v), x_0(t), y_0(t)) d\sigma(x(v), y(v)) = \ln(4 \sin^2(\frac{t-v}{2})) K_1(t, v) + K_2(t, v),
\]

(42)
where \((x(v), y(v))\) and \((x_0(t), y_0(t))\) are two points on \(\partial S\) and \(K_1\) and \(K_2\) are two functions that are \(2\pi\) periodic in each of their arguments. An explicit formula for \(K_1\) is for \(t \neq v\) is given by

\[
K_1(t, v) = -\frac{1}{4\pi} J_0'(k)|x(v) - x(t), y(v) - y(t)| \frac{(x(t) - x(v)) \cdot \nu_x(t) + (y(t) - y(v)) \cdot \nu_y(t)}{|(x(v) - x(t), y(v) - y(t))|} \sigma(v),
\]

where \(J_0'\) is the derivative of the zero-th order Bessel function of first kind, \(\sigma(v) = \sqrt{x'(v) + y'(v)}\), and \(K_1(t, t) = 0\). This decomposition for \(G\) assumes that the contour \(\partial S\) is smooth enough so that an exterior normal can be defined. Having one or a finite number of corner points is still acceptable, but usually requires changing variables for the parametrization of \(\partial S\).

Integral equation (34) can be now rewritten as

\[
\frac{1}{2} \mu(t) - \int_0^{2\pi} \ln(4 \sin^2(\frac{t - v}{2})) K_1(t, v) \mu(v) dv - \int_0^{2\pi} K_2(t, v) \mu(v) dv = ike^{ikx(t)} \nu_x(t). \tag{44}
\]

The unknown in equation (44) can be viewed as living in a Hölder space of continuous periodic functions over \([0, 2\pi]\), or a Sobolev space, see [6], which we denote \(E\) in either case. A numerical method will solve (44) in a finite dimensional subspace of \(E\), denoted \(F\). We fix a linear projection operator \(\Pi\) from \(E\) onto \(F\). The projection of equation onto \(F\) is

\[
\frac{1}{2} \mu(t) - \Pi\{\int_0^{2\pi} \ln(4 \sin^2(\frac{t - v}{2})) K_1(t, v) \mu(v) dv\} - \Pi\{\int_0^{2\pi} K_2(t, v) \mu(v) dv\} = \Pi\{ike^{ikx(t)} \nu_x(t)\},
\]

where the unknown \(\mu\) is in \(F\). It would be clumsy to try to evaluate the integrals in the equation above to machine precision, as the overall precision of the numerical method depends also on the error done in approximating a function in \(E\) by its projection on \(F\). Accordingly, we propose to solve the equation

\[
\frac{1}{2} \mu(t) - \Pi\{\int_0^{2\pi} \ln(4 \sin^2(\frac{t - v}{2})) \Pi_v\{K_1(t, v) \mu(v)\} dv\} - \Pi\{\int_0^{2\pi} \Pi_v\{K_2(t, v) \mu(v)\} dv\} = \Pi\{ike^{ikx(t)} \nu_x(t)\}, \tag{45}
\]

### 3.2 Our numerical scheme for solving equation (45)

We chose to solve equation (45) by using Nystrom’s method, that is we discretized integrals by quadrature, while being careful about logarithmic singularities, see [3, 8]. Let \(L_j\) be the Lagrange trigonometric polynomial of order \(n\),

\[
L_j(t) = \frac{1}{2n} (1 + \sum_{l=1}^{n-1} \cos l(t - \frac{j}{n}) + \cos n(t - \frac{j}{n})). \tag{46}
\]

The projection operator in the method we followed is defined by

\[
\Pi f(t) = \sum_{j=0}^{2n-1} f(\frac{j}{2n})L_j(t).
\]
Using Lagrange trigonometric polynomials gives an advantage when it comes to evaluating the singular part involved in our integral equation. Indeed, Kress derived in [8] the exact formula

$$\int_0^{2\pi} \ln(4 \sin^2(\frac{t-v}{2})) L_j(v) \, dv = \frac{2\pi}{n} \left( \sum_{m=1}^{n-1} \frac{1}{m} \cos m(t - \frac{j}{m}) + \frac{1}{2n} \cos n(t - \frac{j}{m}) \right),$$

(47)

and showed that if a function $f$ is $2\pi$ periodic and real analytic, the error in approximating

$$\int_0^{2\pi} \ln(4 \sin^2(\frac{t-v}{2})) f(v) \, dv,$$

by

$$\int_0^{2\pi} \ln(4 \sin^2(\frac{t-v}{2})) \Pi f(v) \, dv,$$

is of order $e^{-Cn}$, for some positive constant $C$. Kress has studied the numerical solution to a similar well posed linear integral equation of the second kind by quadrature. Theoretical results [3, 8] guarantee that the finite dimensional linear equation (45) has a unique solution, and converges to the true solution as $n$ grows large. If in our case we assume that the parametric equations defining the contour $\partial S$ are real analytic, that convergence will occur at exponential rate, provided the continuous equation is well posed. That rate is verified in our numerical simulations presented later in this paper.

### 3.3 On the evaluation of the Green’s function

Numerical methods for evaluating the Green’s function $G$ in a very accurate and efficient manner are by now well known. For values of $|x - x_0|$ that are large enough, a truncated version of formula (19) is adequate. Otherwise Ewald’s method, which uses completely different series, will be effective, see [12]. Notice also that the function defined by the difference

$$G(x, y, x_0, y_0) - \frac{i}{4} H_0^1(k |(x - x_0, y - y_0)|),$$

(48)

and whose derivative is involved in calculating $K_2$, is smooth for $(x, y, x_0, y_0)$ in $W \times W$. For small values of $|(x - x_0, y - y_0)|$, each of the two terms in (48) is close to being singular. There is a way of computing (48) with the same accuracy and same order of number of operations no matter how small $|(x - x_0, y - y_0)|$ is. A detailed account of such a computation for another Green’s function can be found in [23].

### 3.4 Numerical results

In this section we verify the discrete energy conservation laws on two different scatterers. The first one is shaped as an ellipse and the second one has a non convex geometry. In each case the value for the wavenumber $k$ was set equal to 1.7, ensuring single mode propagation in the empty waveguide. Equation (45) was solved in each case as described in previous sections. Then transmission and reflection coefficients were computed following formulas
(38-41). In the first experiment the scatterer is an ellipse whose surface \( \partial S \) is given by the parametric equations

\[
\begin{pmatrix}
    x(t) \\
    y(t)
\end{pmatrix} = \begin{pmatrix}
    \cos \frac{\pi}{6} - \sin \frac{\pi}{6} \\
    \sin \frac{\pi}{6} \cos \frac{\pi}{6}
\end{pmatrix} \begin{pmatrix}
    .4 \cos(t) \\
    .2 \sin(t)
\end{pmatrix} + \begin{pmatrix}
    .6 \\
    0
\end{pmatrix}
\]

(49)

and is shown in Figure 1. In table 1 we give numerical values for reflection and transmission coefficients calculated through our method. The number of quadrature point is \( 2n \). In the last two rows of table 1, the announced conservation of energy laws are verified. Note however that the identity \( T = T' \) is only verified as \( n \) increases.

\[ \begin{array}{c|c|c|c}
   n & 2 & 4 & 8 \\
   \hline
   R & 1.9401e-01 - 5.9983e-01i & 8.6759e-02 - 5.2440e-01i & 8.7080e-02 - 5.2482e-01i \\
   R' & 2.0097e-01 - 5.9753e-01i & 1.2209e-01 - 5.1732e-01i & 1.2232e-01 - 5.1774e-01i \\
   T & 7.5819e-01 + 2.5011e-01i & 8.3086e-01 + 1.6658e-01i & 8.3017e-01 + 1.6674e-01i \\
   T' & 7.1675e-01 + 2.3644e-01i & 8.3017e-01 + 1.6644e-01i & 8.3017e-01 + 1.6674e-01i \\
   |R|^2 + T^*T' - 1 & 2.2204e-16 - 2.7756e-17i & 0 - 8.3267e-17i & -4.4409e-16 - 1.1102e-16i \\
   R^*T + T^*R' & -1.1102e-16 & -2.6336e-16 + 5.5112e-17i & -1.9429e-16 - 1.6653e-16i \\
\end{array} \]

Figure 1: the scatterer \( S \) defined by equations (49) immersed in the waveguide \( W \).
Table 1: Reflection and transmission coefficients for the perfect conductor sketched in Figure 1.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>16</td>
</tr>
<tr>
<td>$R$</td>
<td>$8.7079e-02 - 5.2482e-01i$</td>
</tr>
<tr>
<td>$R'$</td>
<td>$1.2232e-01 - 5.1774e-01i$</td>
</tr>
<tr>
<td>$T$</td>
<td>$8.3017e-01 + 1.6674e-01i$</td>
</tr>
<tr>
<td>$T'$</td>
<td>$8.3017e-01 + 1.6674e-01i$</td>
</tr>
<tr>
<td>$</td>
<td>R</td>
</tr>
<tr>
<td>$R^*T + T^*R'$</td>
<td>$3.1919e-16 - 5.5511e-17i$</td>
</tr>
</tbody>
</table>

In a second example, the scatterer $S$, plotted in Figure 2, has a non convex shape whose surface $\partial S$ is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} .2 \cos(t) \sqrt{1.1 + \cos(t)} \\ .2 \sin(t) \sqrt{1.1 + \cos(t)} \end{pmatrix} + \begin{pmatrix} .5 \\ 0 \end{pmatrix}$$

(50)

The same observations made in the first example apply here too. However convergence is slower this time, possibly because of the cusp like indentation of the target: more points are necessary to accurately approximate this shape.

Figure 2: the scatterer $S$ defined by equations (50) immersed in the waveguide $W$
<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>$-6.7566e-01 + 7.6906e-01i$</td>
<td>$1.7748e-01 - 3.3240e-01i$</td>
<td>$1.6164e-01 - 3.2857e-01i$</td>
</tr>
<tr>
<td>$R'$</td>
<td>$9.7260e-01 - 3.1941e-01i$</td>
<td>$-6.4143e-02 - 3.7131e-01i$</td>
<td>$-5.5772e-02 - 3.6191e-01i$</td>
</tr>
<tr>
<td>$T$</td>
<td>$-1.8401e-02 + 1.2151e-02i$</td>
<td>$9.2750e-01 + 1.4938e-01i$</td>
<td>$9.2010e-01 + 1.4108e-01i$</td>
</tr>
<tr>
<td>$T'$</td>
<td>$1.8156e+00 - 1.1990e+00i$</td>
<td>$9.0169e-01 + 1.4523e-01i$</td>
<td>$9.1949e-01 + 1.4098e-01i$</td>
</tr>
<tr>
<td>$</td>
<td>R</td>
<td>^2 + T^*T' - 1$</td>
<td>$2.2204e-16 - 3.1225e-17i$</td>
</tr>
<tr>
<td>$R^*T + T^*R'$</td>
<td>$0 - 2.9490e-17i$</td>
<td>$2.7756e-17 - 5.5511e-17i$</td>
<td>$8.3267e-17 + 5.5511e-17i$</td>
</tr>
</tbody>
</table>

Table 2: Reflection and transmission coefficients for the perfect conductor sketched in Figure 2.

**Remark:** It might be surprising to read that the coefficient $T'$ for $n = 2$ in Table 2 is greater than 1, which certainly violates identity (12). It is shown later in the paper that the numerical method preserves identity (12) for scatterers $S$ that are symmetric about the y-axis. In contrast, the sole application of identities (16-17) does not guarantee $|T'| \leq 1$. Notice also that $T$ and $T'$ are quite different from each other for the run $n = 2$.

### 3.5 Counter example using FEMLAB

In this section we go back to the example involving the tilted ellipse sketched in Figure 1. We propose to compute reflection and transmission coefficients using a finite element package. We conducted our calculation with Femlab 3.1. The waveguide was truncated: it was modeled to be a long rectangle of length of length $2L$, with $L = 10$. Artificial boundary conditions were imposed on the two narrow edges:

\[
\begin{align*}
\frac{\partial u}{\partial x} + iku &= 2i e^{ikx}, \text{ for } x = -L \\
\frac{\partial u}{\partial x} - iku &= 0, \text{ for } x = L,
\end{align*}
\]

for the scalar field $u$ produced by a the left incoming wave $e^{ikx}$. These boundary conditions are asymptotically correct in the case of one propagating mode. We do not address the question of convergence of the truncated problem in this paper. In the case of multiple propagating modes, higher order truncations of the Dirichlet to Neumann maps have to be used to ensure high accuracy. We refer to [10] for a practical approach to this problem. Meshes were generated automatically. We reproduced a picture with a mesh made up of 641 elements below.
Figure 3: a 641 element mesh for the truncated wave guide containing the perfect conductor shaped as a tilted ellipse.

The finite element space was selected by default to be quadratic Lagrange finite elements. After also solving for \( u' \), the coefficients \( R, T, R', T' \) were then estimated by averaging the value of the fields over the artificial walls \( x = -L \) and \( x = L \). Numerical results appear in Table 3, where \( n' \) is the number of mesh elements for each run.

<table>
<thead>
<tr>
<th>( n' )</th>
<th>100</th>
<th>641</th>
<th>2564</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>1.5115e-003 -5.2468e-001i</td>
<td>4.1900e-002 -5.3286e-001i</td>
<td>8.3542e-002 -5.2541e-001i</td>
</tr>
<tr>
<td>( R' )</td>
<td>1.9394e-002 -5.2431e-001i</td>
<td>6.9362e-002 -5.2998e-001i</td>
<td>1.1827e-001 -5.1870e-001i</td>
</tr>
<tr>
<td>( T )</td>
<td>8.5112e-001 +1.6950e-002i</td>
<td>8.4055e-001 +8.7967e-002i</td>
<td>8.3135e-001 +1.6069e-001i</td>
</tr>
<tr>
<td>( T' )</td>
<td>8.5112e-001 +1.6950e-002i</td>
<td>8.4055e-001 +8.7967e-002i</td>
<td>8.3135e-001 +1.6069e-001i</td>
</tr>
<tr>
<td>(</td>
<td>R</td>
<td>^2 + T^*T' - 1 )</td>
<td>-2.4591e-005</td>
</tr>
<tr>
<td>( R^*T + T^*R' )</td>
<td>1.2184e-005 +9.8508e-006i</td>
<td>2.5863e-005 +1.3888e-006i</td>
<td>-4.8375e-007 +5.9944e-007i</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>10256</th>
<th>41024</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>8.6838e-002 -5.2486e-001i</td>
<td>8.7063e-002 -5.2482e-001i</td>
</tr>
<tr>
<td>( R' )</td>
<td>1.2205e-001 -5.1780e-001i</td>
<td>1.2230e-001 -5.1774e-001i</td>
</tr>
<tr>
<td>( T )</td>
<td>8.3025e-001 +1.6633e-001i</td>
<td>8.3018e-001 +1.6671e-001i</td>
</tr>
<tr>
<td>( T' )</td>
<td>8.3025e-001 +1.6633e-001i</td>
<td>8.3018e-001 +1.6671e-001i</td>
</tr>
<tr>
<td>(</td>
<td>R</td>
<td>^2 + T^*T' - 1 )</td>
</tr>
<tr>
<td>( R^*T + T^*R' )</td>
<td>1.4411e-007 +1.3980e-009i</td>
<td>7.2156e-007 -4.1418e-008i</td>
</tr>
</tbody>
</table>

Table 3: Reflection and transmission coefficients for the perfect conductor sketched in Figure 1, calculated by the finite element method

This numerical table indicates some progress of the quantities \(|R|^2 + T^*T' - 1\) and \( R^*T + T^*R' \) toward 0, as \( n \) grows. Comparing Table 2 and Table 3, we note that the integral equation method is still more accurate in the 16 quadrature points \( (n = 8) \) run, than the finite element method in the \( n' = 41024 \) mesh elements run.

### 3.6 Derivation of the discrete energy conservation laws

We state and derive in this section the main result of this paper. Let \( \mu \) and \( \mu' \) be in the finite dimensional space \( F \) and solve, respectively

\[
\frac{1}{2} \mu(t) - \Pi \{ \int_0^{2\pi} \ln(4\sin^2(t/2)) \Pi_v \{ K_1(t,v)u(v) \} dv \} - \Pi \{ \int_0^{2\pi} \Pi_v \{ K_2(t,v)v(u) \} dv \} = \Pi \{ ike^{ikx(t)}v_x(t) \} \tag{51}
\]

\[
\frac{1}{2} \mu'(t) - \Pi \{ \int_0^{2\pi} \ln(4\sin^2(t/2)) \Pi_v \{ K_1(t,v)\mu'(v) \} dv \} - \Pi \{ \int_0^{2\pi} \Pi_v \{ K_2(t,v)\mu'(v) \} dv \} = \Pi \{ -ike^{-ikx(t)}v_x(t) \} \tag{52}
\]
We assume that these two linear equations are uniquely solvable. We have to assume that the projection operator $\Pi$ preserves complex conjugation. More precisely, for any $f$ in $E$,

$$\Pi(f^*) = \Pi(f)^*$$  \hfill (53)

In order to avoid cumbersome notations, we still denote reflection and transmission coefficients for the discrete case in the same fashion as in the continuous case. More precisely,

$$R = -\frac{1}{2ik} \int_0^{2\pi} \Pi(e^{ikx(\nu)} \mu(\nu)\sigma(\nu))d\nu$$  \hfill (54)

$$T = 1 - \frac{1}{2ik} \int_0^{2\pi} \Pi(e^{-ikx(\nu)} \mu(\nu)\sigma(\nu))d\nu$$  \hfill (55)

$$R' = -\frac{1}{2ik} \int_0^{2\pi} \Pi(e^{-ikx(\nu)}' \mu'(\nu)\sigma(\nu))d\nu$$  \hfill (56)

$$T' = 1 - \frac{1}{2ik} \int_0^{2\pi} \Pi(e^{ikx(\nu)}' \mu'(\nu)\sigma(\nu))d\nu$$  \hfill (57)

**Proposition 3.1** The discrete energy conservation laws hold. More precisely,

$$|R|^2 + T^*T' = 1$$

$$R^*T + R'T^* = 0,$$

where $R, T, R', T'$ are given by (54-57), and $\mu$ and $\mu'$ satisfy (51,52).

The proof pretty much follows the second derivation of (36), followed by the derivation of the energy estimates suggested in Remark 2.4. Therefore, we first prove the analog of (36) for discrete densities, as made explicit in the following lemma.

**Lemma 3.1** The discrete densities $\mu$ and $\mu'$ defined by equations (51) and (52) satisfy

$$\mu^* = R^*\mu + T^*\mu',$$  \hfill (58)

where $R$ and $T$ are defined by (54) and (55).

**proof of lemma (3.1):** From (51), we derive using (53) and the fact that $K_1$ is real valued,

$$\frac{1}{2} \mu^*(t) - \Pi\left\{ \int_0^{2\pi} \ln(4\sin^2(\frac{t - v}{2})) \Pi_v \{ K_1(t, v)\mu^*(v) \} dv \right\} - \Pi\left\{ \int_0^{2\pi} \Pi_v \{ K_2(t, v)\mu^*(v) \} dv \right\} =$$

$$\Pi\left\{-i(\ln^2)_{\nu_x(t)}\right\}$$

But,

$$K_2^*(t, v) = K_2(t, v) - \frac{1}{ik} \frac{\partial}{\partial \nu_x(t)} \cos(k(x(v) - x(t)))\mu^*(v)\sigma(v)$$

$$= K_2(t, v) - \nu_x(t)\frac{1}{2}(e^{-ikx(t)}e^{-ikx(v)} + e^{ikx(t)}e^{-ikx(v)})\mu^*(v)\sigma(v),$$

thus,

$$\Pi\left\{ \int_0^{2\pi} \Pi_v \{ K_2(t, v)\mu^*(v) \} dv \right\} =$$

$$\Pi\left\{ \int_0^{2\pi} \Pi_v \{ K_2(t, v)\mu^*(v) \} dv \right\} + (T^* - 1)\Pi(ik\nu_x(t)) - R^*\Pi(ik\nu_x(t)),$$
where we have used the discrete definitions (54,55) for $R$ and $T$. We thus proved

$$\frac{1}{2} \mu^*(t) - \Pi\{ \int_0^{2\pi} \ln(4 \sin^2(\frac{t - v}{2})) \Pi_v \{ K_1(t, v) \mu^*(v) \} dv \} - \Pi\{ \int_0^{2\pi} \Pi_v \{ K_2(t, v) \mu^*(v) \} dv \} = \Pi\{-ikT^* e^{-ik\sigma(t)} + ikR^* e^{ik\sigma(t)} \nu_x(t) \}.$$

Comparing (59) to (51,52), we see that $\mu^*$ and $R^* \mu + T^* \mu'$ satisfy the same linear equation, which was assumed to be regular. ⋄

**proof of proposition (3.1):** We just have to multiply identity (58) by $-\frac{1}{2} ike^{-ik(t)} \sigma(t)$, apply the projection operator $\Pi$, and integrate over $[0, 2\pi]$, to obtain the first identity in proposition 3.1. To obtain the second, multiply identity (58) by $-\frac{1}{2} ike^{-ik(t)} \sigma(t)$, apply the projection operator $\Pi$, and integrate over $[0, 2\pi]$. ⋄

4 The symmetric case

As noted earlier, the scattering matrix $P$ is symmetric, regardless of the scatterer $S$. In this section we assume that $S$ is symmetric about the $y$ axis. If $x(t)$ and $y(t)$ are parametric equations for the contour $\partial S$, it proves convenient to assume that $x(t)$ and $y(t)$ are defined over $[-\pi, \pi]$ and that $x(-t) = -x(t)$ and $y(-t) = y(t)$, for all $t$ in $[-\pi, \pi]$.

4.1 Additional relations for some continuous quantities

A change of variables shows that the symmetry assumption implies the additional identity

$$R = R',$$  \hspace{1cm} (60)

for the reflection coefficients. According to the definition of our Green’s function (19)

$$G(-x, y, x_0, y_0) = G(x, y, -x_0, y_0)$$  \hspace{1cm} (61)

If the point $(x, y)$ is on the contour $\partial S$ then the point $(-x, y)$ is also on $\partial S$ and the respective normal vectors at those two points have opposite $x$ component and same $y$ component. Consequently,

$$\left( \frac{\partial G}{\partial \nu_0} \right)(-x, y, x_0, y_0) = \left( \frac{\partial G}{\partial \nu_0} \right)(x, y, -x_0, y_0),$$  \hspace{1cm} (62)

for $(x_0, y_0)$ on $\partial S$. The identity

$$\mu(-t) = \mu'(t)$$  \hspace{1cm} (63)

for the densities defined in (32) and (33) follows easily.
4.2 Numerical results for a symmetric scatterer

They were also conducted following Nystrom’s quadrature method described in section 3.3. In this last example the scatterer $S$ is an ellipse whose contour $\partial S$ is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} .2 \cos(t) \\ .4 \sin(t) \end{pmatrix} + \begin{pmatrix} .6 \\ 0 \end{pmatrix}$$

This scatterer is identical in shape to the one in the first but the rotation angle is different: it is now symmetric about a line parallel to the $y$-axis. The scatterer is plotted in figure 3.

![Figure 4: the symmetric scatterer $S$ defined by equations (64) immersed in the waveguide $W$](image)

We present computed numerical values for reflection and transmission coefficients in table 4. Discrete conservation laws hold here too. In addition, identities $R = R'$ and $T = T'$ are satisfied at the discrete level, as shown in the next section. Consequently, the energy conservation laws $(16,17)$ are observed at any level in the more familiar form $|R|^2 + |T|^2 = 1$ and $\text{Re}(RT^*) = 0$.

<table>
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<th>$n$</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
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<tr>
<td>$R$</td>
<td>1.9426e-02 - 3.6094e-01i</td>
<td>3.0880e-02 - 4.1159e-01i</td>
<td>3.0999e-02 - 4.1196e-01i</td>
</tr>
<tr>
<td>$R'$</td>
<td>1.9426e-02 - 3.6094e-01i</td>
<td>3.0880e-02 - 4.1159e-01i</td>
<td>3.0999e-02 - 4.1196e-01i</td>
</tr>
<tr>
<td>$T$</td>
<td>9.3104e-01 + 5.0108e-02i</td>
<td>9.0829e-01 + 6.8146e-02i</td>
<td>9.0811e-01 + 6.8333e-02i</td>
</tr>
<tr>
<td>$T'$</td>
<td>9.3104e-01 + 5.0108e-02i</td>
<td>9.0829e-01 + 6.8146e-02i</td>
<td>9.0811e-01 + 6.8333e-02i</td>
</tr>
<tr>
<td>$</td>
<td>R</td>
<td>^2 + T^*T' - 1$</td>
<td>0 + 1.3878e-17i</td>
</tr>
<tr>
<td>$R^*T + T^*R'$</td>
<td>4.1633e-17 + 5.5511e-17i</td>
<td>-6.9389e-18</td>
<td>5.2042e-17 - 5.5511e-17i</td>
</tr>
<tr>
<td>$n$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
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<td></td>
</tr>
<tr>
<td>$R$</td>
<td>3.0999e-02 - 4.1196e-01i</td>
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<tr>
<td>$T$</td>
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<td>$T'$</td>
<td>9.0811e-01 + 6.8333e-02i</td>
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<tr>
<td>$</td>
<td>R</td>
<td>^2 + T^*T' - 1$</td>
<td>2.2204e-16 - 1.3878e-17i</td>
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<tr>
<td>$R^*T + T^*R'$</td>
<td>0 + 1.1102e-16i</td>
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<td></td>
</tr>
</tbody>
</table>

Table 4: Reflection and transmission coefficients for the symmetric perfect conductor sketched in Figure 4.

4.3 A proof for the discrete energy conservation laws in the symmetric case

We have to make the following assumptions on the functional space $F$: if a function $f$ is in the finite dimensional space $F$ then the function $g$ defined by $g(t) = f(-t)$, is also in $F$. Note also that this implies that the linear projector $\Pi$ maps even functions onto even functions, and odd functions onto odd functions. Note that, in our numerical method, we chose $F$ to be the vector space spanned by Lagrange trigonometric interpolants, $L_j$, $j = -n \cdots n$, of order $2n$, as made explicit in equation (46). The functions $L_j$, $j = -n \cdots n$ satisfy

$$L_j(-t) = L_{-j}(t),$$

which is in agreement with the symmetry assumption for the space $F$. Finally, notice that the symmetry properties for the Green’s function $G$ (62), and symmetry properties for the parametric equations $x(t), y(t)$ imply symmetries for the integration kernels $K_1$ and $K_2$,

$$K_1(-t, v) = K_1(t, -v), \quad K_2(-t, v) = K_2(t, -v) \quad (64)$$

**Proposition 4.1** Under the symmetry properties for the space $F$ and for the integration kernels $K_1$ and $K_2$, and assuming that the projection operator preserves complex conjugation (53), the discrete identities $R = R'$ and $T = T'$ hold, where $R, T, R', T'$ are given by (54-57), and $\mu$ and $\mu'$ satisfy (51,52).

**proof:** We write the equation defining $\mu$, this time integrating over $[-\pi, \pi]$,

$$\frac{1}{2} \mu(t) - \Pi \left\{ \int_{-\pi}^{\pi} \ln(4\sin^2\left(\frac{t-v}{2}\right)) \Pi_v \{ K_1(t, v)\mu(v) \} dv \right\} - \Pi \left\{ \int_{-\pi}^{\pi} \Pi_v \{ K_2(t, v)\mu(v) \} dv \right\} = \Pi \{ ike^{ikx(t)}\nu_{x(t)} \}. $$

Equivalently,

$$\frac{1}{2} \mu(-t) - \Pi \left\{ \int_{-\pi}^{\pi} \ln(4\sin^2\left(\frac{-t-v}{2}\right)) \Pi_v \{ K_1(-t, v)\mu(v) \} dv \right\} - \Pi \left\{ \int_{-\pi}^{\pi} \Pi_v \{ K_2(-t, v)\mu(v) \} dv \right\} = \Pi \{ ike^{ikx(-t)}\nu_{x(-t)} \}. $$

After applying symmetry properties,

$$\frac{1}{2} \mu(-t) - \Pi \left\{ \int_{-\pi}^{\pi} \ln(4\sin^2\left(\frac{-t-v}{2}\right)) \Pi_v \{ K_1(t, -v)\mu(v) \} dv \right\} - \Pi \left\{ \int_{-\pi}^{\pi} \Pi_v \{ K_2(t, -v)\mu(v) \} dv \right\} = \Pi \{ -ike^{-ikx(t)}\nu_{x(t)} \}. $$
Finally after a change of variables in each of the two integrals,
\[ \frac{1}{2} \mu(-t) - \Pi\{ \int_{-\pi}^{\pi} \ln(4 \sin^2(\frac{-t + v}{2})) \Pi \{ K_1(t, v) \mu(-v) \} dv \} - \Pi\{ \int_{-\pi}^{\pi} \Pi \{ K_2(t, v) \mu(-v) \} dv \} = \Pi\{ -ike^{-ix(t)} x(t) \}, \]

which is the same equation as the one satisfied by \( \mu' \). We thus proved that \( \mu(-t) = \mu'(t) \), for all \( t \) in \([ -\pi, \pi ]\). From there, a simple change of variables in the integrals defining \( R, T, R', T' \) in formulas (54-57) shows that \( R = R' \) and \( T = T' \).

**Remark** Combining proposition 3.1 and proposition 4.1 we obtain in the symmetric case the discrete conservation laws \( |R|^2 + |T|^2 = 1 \) and \( \text{Re}(RT^*) = 0 \).

## 5 Generalization

### 5.1 The other form: integral equation (24)

The analogs of propositions 3.1 and 4.1 hold for the integral equation of the second kind (24) resulting from the representation formula (23). Following the same argument, we could derive the discrete analog of formula (18). Discrete formulas for reflection and transmission coefficients would then be given by

\[ R = -\frac{1}{2ik} \int_{0}^{2\pi} \Pi(ike^{ikx(v)} u(v) \nu_x(v) \sigma(v)) dv, \]

\[ T = 1 - \frac{1}{2ik} \int_{0}^{2\pi} \Pi(-ike^{-ikx(v)} u(v) \nu_x(v) \sigma(v)) dv, \]

\[ R' = -\frac{1}{2ik} \int_{0}^{2\pi} \Pi(-ike^{-ikx(v)} u'(v) \nu_x(v) \sigma(v)) dv, \]

\[ T' = 1 - \frac{1}{2ik} \int_{0}^{2\pi} \Pi(ike^{ikx(v)} u'(v) \nu_x(v) \sigma(v)) dv. \]

### 5.2 The multiple propagating mode case

For ease of exposition, we will only consider the case of two propagating modes in this section. This implies that the wave number is in the range \(( \pi, 2\pi )\). Adding more propagating modes is then a straightforward generalization. The two propagating modes 0 and 1 are

\[ m_0(x, y) = e^{ikx}, \quad m'_0(x, y) = e^{-ikx}, \]

and

\[ m_1(x, y) = e^{i\sqrt{k^2 - \pi^2} x} \sqrt{2} \cos(\pi y), \quad m'_1(x, y) = e^{-i\sqrt{k^2 - \pi^2} x} \sqrt{2} \cos(\pi y). \]

The scattering matrix \( P \) is of dimension 4 \( \times \) 4. We write in the block form

\[ P = \begin{pmatrix} R & T' \\ T & R' \end{pmatrix}. \]

The \( R \) block is the matrix

\[ R = \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{pmatrix}. \]
where $\sum_{j=0}^{1} R_{ij} m_j'$ is the reflexion produced by the left incoming wave of mode $m_i$. The other three blocks in $P$ are defined along the same lines. It is well known that the matrix $P$ is block unitary and block symmetric. Define
\[ u_i(x_0, y_0) = m_i(x, y) + \int_{\partial S} G(x, y, x_0, y_0) \mu_i(x, y) d\sigma(x, y), \] (69)
and
\[ u_i'(x_0, y_0) = m_i'(x, y) + \int_{\partial S} G(x, y, x_0, y_0) \mu_i'(x, y) d\sigma(x, y), \] (70)
for $i = 0, 1$ for $(x_0, y_0)$ in $W \setminus S$, where $\mu$ is an unknown density. As $(x_0, y_0)$ approaches $\partial S$, taking the normal derivative of (69) and of (70) and using the classical jump formula for double layer potential, we obtain
\[ \frac{1}{2} \mu_i(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial v_0}(x, y, x_0, y_0) \mu_i(x, y) d\sigma(x, y) = \frac{\partial m_i}{\partial v}(x_0, y_0), \] (71)
\[ \frac{1}{2} \mu_i'(x_0, y_0) - \int_{\partial S} \frac{\partial G}{\partial v_0}(x, y, x_0, y_0) \mu_i'(x, y) d\sigma(x, y) = \frac{\partial m_i'}{\partial v}(x_0, y_0), \] (72)
for $(x_0, y_0)$ on $\partial S$. In the two propagating mode case, the imaginary part of the Green’s has two terms
\[ -\frac{1}{2ik} \cos k(x - x_0) - \frac{1}{2i\sqrt{k^2 - \pi^2}} \cos \sqrt{k^2 - \pi^2} (x - x_0) \cos(\pi(y - y_0)). \]
The analog of propositions 2.4 and 3.1 is
\[ \left( \begin{array}{c} u_0' \\ u_1' \end{array} \right) = R^* \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) + T^* \left( \begin{array}{c} u_0' \\ u_1' \end{array} \right), \] (73)
for the total fields and
\[ \left( \begin{array}{c} \mu_0' \\ \mu_1' \end{array} \right) = R^* \left( \begin{array}{c} \mu_0 \\ \mu_1 \end{array} \right) + T^* \left( \begin{array}{c} \mu_0' \\ \mu_1' \end{array} \right), \] (74)
for the associated densities. Solving equations (71,72) through the use of Nystrom’s method, if the projection operator still satisfies the properties required in the preceding section, we are lead to
\[ R^* R + T^* T' = I_2 \]
\[ R^* R + T^* T' = 0, \] (75)
where $I_2$ is the 2 by 2 identity matrix. Furthermore, if the scatterer $S$ is symmetric about the $y$ axis, the matrix identities $R = R'$ and $T = T'$ are observed at the discrete level. The numerical codes that were developed for producing the examples attached to Tables 1, 2, and 4, can be used again in the case of two propagating modes: one only needs to change the input value for $k$. The conservation laws (75) can then be observed with the same order of precision as that from the previously exposed one propagating mode case. We omit numerical values in the present case for brevity.
5.3 Periodic boundary conditions

It is common to model periodic electromagnetic structures or photonic gratings using a modified waveguide. Instead of Dirichlet or Neumann conditions on the walls, periodic or pseudo periodic conditions are imposed, see [15]. We adjusted the Green’s function to this new framework, we computed fields, then reflection and transmission coefficients, still following Nystrom’s quadrature method. We observed discrete energy conservation laws, and symmetry properties, when applicable. The incidence angle for the incoming wave $\gamma$ adds a new parameter to this problem which contributes to a richer algebraic structure.

6 Conclusion

In this paper we have studied the numerical approximation to the scattering matrix of a perfect electrical conductor target in a waveguide. We have reformulated the scattering problem as an integral equation of the second kind and have employed Nystrom’s quadrature method for weakly singular integral equations to obtain approximations to the scattering matrix. We have proved and numerically verified under certain symmetry restrictions that the approximation is unitary, regardless of the order of the numerical method! When these symmetries are broken, the unitary character of the approximate only becomes evident as the number of collocation points increase. We also showed that two energy conservation identities are preserved by the approximate scattering matrix, even in the non symmetric case. Analogous results were observed for an alternative formulation of the integral equation approach for this problem, and we have extended this method to study photonic structures with perfectly conducting cylinders. This gives rise to waveguides with periodic boundary conditions. Similar results were obtained if perfectly conductive cylinders were replaced by lossless dielectrics. These results are in essence similar and will be published at a later date.

References


