The Leading-Order Term in the Asymptotic Expansion of the Scattering Amplitude of a Collection of Finite Number of Dielectric Inhomogeneities of Small Diameter

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ABSTRACT

We rigorously derive the leading-order term in the asymptotic expansion of the scattering amplitude for a collection of a finite number of dielectric inhomogeneities of small diameter. The asymptotic formula derived in this paper provides the basis for the numerical reconstruction of dielectric scatterers of small diameter, as demonstrated by Volkov.

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1. INTRODUCTION

In this paper, we consider the scattering problem for the full Maxwell’s equations from a collection of dielectric inhomogeneities of small diameter. We suppose that there is a finite number of dielectric imperfections in $\mathbb{R}^3$, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^3$ is a bounded, strictly star-shaped smooth domain containing the origin. The total collection of imperfections thus takes the form

$$I_{\alpha} = \bigcup_{j=1}^{m} (z_j + \alpha B_j)$$

The points $z_j \in \mathbb{R}^3$, $j = 1, \ldots, m$, that determine the location of the imperfections are assumed to satisfy

$$0 < d_0 \leq |z_j - z_l| \quad \forall j \neq l$$

We also assume that $\alpha > 0$ and the common order of magnitude of the diameters of the imperfections is small enough, such that the imperfections are disjoint.

Let $\mu^0 > 0$ and $\varepsilon^0 > 0$ denote the permeability and the permittivity of the free space; we shall assume that these are positive constants. Let $\mu^j$ and $\varepsilon^j$ denote the permeability and the permittivity of the $j$th inhomogeneity, $z_j + \alpha B_j$; it is assumed that the $\mu^j$ are positive constants and the $\varepsilon^j$ are either positive constants or constants that are complex and whose real and imaginary parts are positive.

Let

$$\mu_{\alpha}(x) = \begin{cases} 
\mu^0, & x \in \mathbb{R}^3 \setminus I_{\alpha}, \\
\mu^j, & x \in z_j + \alpha B_j, \quad j = 1 \ldots m 
\end{cases}$$

If we allow the degenerate case $\alpha = 0$, then the function $\mu_0(x)$ equals the constant $\mu^0$. The piecewise constant electric permittivity $\varepsilon_{\alpha}(x)$ is defined analogously. We need to introduce some additional notations. Let $\gamma^j, 1 \leq j \leq m$, be a set of positive constants or complex constants whose real and imaginary parts are positive. In effect, $\{\gamma^j\}$ will be either the set $\{\varepsilon^j\}$ or the set $\{\mu^j\}$. For any fixed $1 \leq j_0 \leq m$, let $\gamma$ denote the coefficient given by

$$\gamma(x) = \begin{cases} 
\gamma^0, & x \in \mathbb{R}^3 \setminus \overline{B_{j_0}}, \\
\gamma^{j_0}, & x \in B_{j_0}
\end{cases}$$

By $\phi_l$, $1 \leq l \leq 3$, we denote the solution to

$$\nabla_y \cdot \gamma(y) \nabla_y \phi_l = 0 \quad \text{in} \quad \mathbb{R}^3$$

$$\phi_l - y_l \to 0 \quad \text{as} \quad |y| \to \infty$$

This problem may alternatively be written as

$$\begin{cases} 
\Delta \phi_l = 0 & \text{in} \quad B_{j_0}, \text{ and in} \quad \mathbb{R}^3 \setminus \overline{B_{j_0}} \\
\phi_l & \text{is continuous across} \quad \partial B_{j_0} \\
\frac{\gamma^0}{\gamma^{j_0}} (\partial_{\nu} \phi_l)^+ - (\partial_{\nu} \phi_l)^- = 0 & \text{on} \quad \partial B_{j_0} \\
\phi_l(y) - y_l \to 0 & \text{as} \quad |y| \to \infty
\end{cases}$$

Here, $\nu$ denotes the outward unit normal to $\partial(B_{j_0})$; superscript $+$ and $-$ indicate the limiting values as we approach $\partial(B_{j_0})$ from outside $B_{j_0}$, and from inside $B_{j_0}$. It is obvious that the function $\phi_l$ only depends on the coefficients $\gamma^0$ and $\gamma^{j_0}$ through the ratio $c = \frac{\gamma^0}{\gamma^{j_0}}$. The existence and uniqueness of this $\phi_l$ can be established using single-layer potentials with suitably chosen densities. It is essential here that the constant $c$, by assumption, cannot be 0 or a negative real number.

We now define the polarization tensor $M_{j_0}(c)$ of the inhomogeneity $B_{j_0}$ (with aspect ratio $c$) by

$$M_{j_0}(c) = c^{-1} \int_{B_{j_0}} \partial_{y_k} \phi_l \, dy$$

It is quite easy to see that the tensor $M_{j_0}(c)$ is symmetric; if $c$ is a positive real number, it is furthermore positive definite, see [4, 7].
The scattering amplitude

\[ A_\alpha \left( \frac{x}{|x|}, \eta, \lambda, k \right) = \frac{-ik^3}{4\pi} \alpha^3 \sum_{j=1}^{m} \left[ \left( \mu^0_j - 1 \right) \left( \frac{\mu^0_j}{\mu_j} \right) \eta \times (\eta \times \lambda) \right] \times \frac{x}{|x|} + \left( \frac{\epsilon^0_j}{\epsilon_j} - 1 \right) \left( I_3 - \frac{x x^t}{|x|^2} \right) M^j \left( \frac{\epsilon^0_j}{\epsilon_j} \right) (\eta \times \lambda) \times e^{ik \left( \eta - \frac{\epsilon}{|\epsilon|} \right) z_j} + O(\alpha^4) \]  

for any \( x/|x|, \lambda, \) and \( \eta \in S^2 \), where the remainder \( O(\alpha^4) \) is independent of the set of points \( \{z_j\}_{j=1}^{m} \) provided that (1) holds.

To the best of our knowledge, Eq. (10) is new for both mathematicians and engineers working in the area of electromagnetic theory. Indeed, it is expected to lead to very effective computational algorithms aimed at determining information about the electromagnetic small inhomogeneities from scattering amplitude measurements. Our formula in this paper generalizes that by Ammari et al. [1], where only solutions with transverse electric and transverse magnetic symmetries are considered. In that case, the full Maxwell’s equations are reduced to a scalar Helmholtz equation and the analysis is very much simplified.

Equation (10) proves to be useful for the numerical recovery of dielectric inhomogeneities in space, when they are illuminated by an incoming plane electromagnetic wave, if the scattered amplitude can be measured accurately. Note that in our model, the inhomogeneities have to satisfy a smallness assumption. In contrast, in most models used for radar-type applications, the underlying assumption is rather the opposite, that is, the objects under scrutiny are several times larger in size than a wavelength. The reader might be interested in the implementation of a numerical method based on (10). This method uses measurements of the scattering amplitude from a set of observation points to implement a numerical method based on the inhomogeneity size \( \alpha \) for the scattering amplitude \( A_\alpha \).
angles, for incident plane waves propagating within a chosen set of directions; but all those incident waves have the same wavelength, thus they agree with our smallness assumption. The details of related numerical methods are discussed in [12], where several relevant simulations are presented in Section 3 of that paper.

Finally, we shall mention, in connection with our asymptotic expansion, the works [8–10] and the recent book [2].

2. ASYMPTOTIC FORMULAS FOR THE SOLUTION

In this section, we find and rigorously prove an asymptotic expansion with respect to the inhomogeneity size $\alpha$ for the solution $E_{\alpha}$ in terms of $E_0$. This is an important step in deriving our main asymptotic formula (10).

Throughout this paper, we shall use quite standard $L^2$-based Sobolev spaces to measure regularity. The notation $H(\text{curl}, \Omega)$ is used to denote those functions that along with their rotations are in $L^2(\Omega)$. The space $TH_{\text{div}}^{-1/2}(\partial \Omega)$ denotes the set of tangential vector fields on $\partial \Omega$ that lie in $H^{-1/2}(\partial \Omega)$ and whose surface divergences also lie in $H^{-1/2}(\partial \Omega)$.

Set $\Phi$ to be what is commonly used as a fundamental solution for the Helmholtz equation

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

(11)

The following representation theorem can be found in [5]:

$$(E_{\alpha} - E_0)(x) =$$

$$= -\nabla \times \int_{\partial \Omega} \Phi(x, y) (E_{\alpha} - E_0)(y) \times \nu_y d\sigma_y$$

$$- \nabla \int_{\partial \Omega} \Phi(x, y) (E_{\alpha} - E_0)(y) \cdot \nu_y d\sigma_y$$

$$- \int_{\partial \Omega} \Phi(x, y) [\nabla \times (E_{\alpha} - E_0)(y)] \times \nu_y d\sigma_y$$

for all $x$ in $\mathbb{R}^3 \setminus \Omega$. However since Eq. (5) is valid throughout $\mathbb{R}^3$, the above identity simplifies as

$$E_{\alpha}(x)\times \nu_x = E_0(x)\times \nu_x$$

$$- \left[ \nabla \times \int_{\partial \Omega} \Phi(x, y) (E_{\alpha}(y)\times \nu_y d\sigma_y \right] \times \nu_x$$

$$- \left[ \int_{\partial \Omega} \Phi(x, y) (\nabla \times E_{\alpha}(y)) \times \nu_y d\sigma_y \right] \times \nu_x$$

$$- \int_{\partial \Omega} \nabla \Phi(x, y) \times (\nu_x - \nu_y) (E_{\alpha}(y)) d\sigma_y$$

$$- \int_{\partial \Omega} \Phi(x, y) \text{curl}_{\partial \Omega} (E_{\alpha}(y)) d\sigma_y$$

(12)

In a neighborhood of $\partial \Omega$ in $\mathbb{R}^3 \setminus \Omega$, the normal vector $\nu_x$ is well defined and we infer from the above equality

$$\frac{1}{2} E_{\alpha}(x)\times \nu_x = E_0(x)\times \nu_x$$

$$- (\nabla \times \int_{\partial \Omega} \Phi(x, y) (E_{\alpha}(y)\times \nu_y d\sigma_y \right] \times \nu_x$$

$$- \left[ \int_{\partial \Omega} \Phi(x, y) (\nabla \times E_{\alpha}(y)) \times \nu_y d\sigma_y \right] \times \nu_x$$

$$- \int_{\partial \Omega} \nabla \Phi(x, y) \times (\nu_x - \nu_y) (E_{\alpha}(y)) d\sigma_y$$

$$- \int_{\partial \Omega} \Phi(x, y) \text{curl}_{\partial \Omega} (E_{\alpha}(y)) d\sigma_y$$

(13)
Recall that Maxwell’s equations imply that \( E_\alpha \nu_y \) is related to \((\nabla \times E_\alpha(y)) \times \nu_y\) as follows:

\[
E_\alpha \cdot \nu_y = \frac{1}{k^2} \text{div}_{\partial \Omega} \left[ (\nabla \times E_\alpha(y)) \times \nu_y \right] \quad (14)
\]

where \( \text{div}_{\partial \Omega} \) denotes the surface divergence.

Define the operator on tangential boundary fields on \( \partial \Omega \)

\[ N_\alpha : \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \rightarrow \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \]

\[ N_\alpha f = (\nabla \times v_\alpha)|_{\partial \Omega} \times \nu \]

where \( v_\alpha \) is defined by

\[
\left( \nabla \times \frac{1}{\mu_\alpha} \nabla \times \omega^2 \epsilon_\alpha \right) v_\alpha = 0 \quad \text{in } \Omega \\
v_\alpha \times \nu = f
\]

Define \( N_0 \) to be the analog of \( N_\alpha \) in the absence of inhomogeneities. We also need to define three more operators, \( P, Q, A \) acting on \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \).

For \( \psi \) to be in \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \), let

\[ P\psi(x) = \left( \nabla \times \int_{\partial \Omega} \Phi(x,y) \psi(y) d\sigma_y \right) \times \nu_x \]

\[ Q\psi(x) = \left( \int_{\partial \Omega} \Phi(x,y) \psi(y) d\sigma_y \right) \times \nu_x \]

and

\[ A\psi(x) = \text{div}_{\partial \Omega} \psi(x) \]

It is evident that \( A \) is continuous from \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \) to \( H^{-1/2}(\partial \Omega) \) and it is well known that \( P \) and \( Q \) are continuous mappings of \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \) into itself, see [11]. Actually, \( P \) and \( Q \) are even continuous from \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \) to \( \text{TH}^{1/2}(\partial \Omega) \). Let us now define \( R \), a continuous operator from \( H^s(\partial \Omega) \) to \( \text{TH}^{s+1}(\partial \Omega) \), where \( s \) is any real number. For any \( \phi \) in \( H^s(\partial \Omega) \),

\[
R\phi(x) = \int_{\partial \Omega} \nabla_x \Phi(x,y) \times (\nu_x - \nu_y) \phi(y) d\sigma_y
\]

Finally define \( S \), a continuous operator from \( H^s(\partial \Omega) \) to \( \text{TH}^s(\partial \Omega) \) by the following. For any \( \phi \) in \( H^s(\partial \Omega) \),

\[
S\phi(x) = \int_{\partial \Omega} \Phi(x,y) \overrightarrow{\text{curl}}_{\partial \Omega} \phi(y) d\sigma_y
\]

The reader can find rigorous proofs for the continuity of \( R \) and \( S \) in [11]. It is interesting to notice that the tangential field \( S\phi \) is divergence free. Putting everything together, we observe that \( RA \) and \( SA \) are continuous mappings of \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \) into itself. We can now rewrite (13) as

\[
\frac{I}{2} E_\alpha \times \nu = E_0 \times \nu - P(E_\alpha \times \nu) - \left( Q + \frac{1}{k^2} RA + \frac{1}{k^2} SA \right) N_\alpha
\]

and analogously, set

\[
T_\alpha = \frac{I}{2} + P + \left( Q + \frac{1}{k^2} RA + \frac{1}{k^2} SA \right) N_\alpha
\]

We note the following about \( T_0 \).

**Lemma 2.1** \( T_0 \) is the identity operator on \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \).

**Proof.** For any \( g \) in \( \text{TH}_{\text{div}}^{-1/2}(\partial \Omega) \), there is a unique field \( E \) in \( H(\text{curl}, \Omega) \) such that

\[
(\nabla \times \frac{1}{\mu_0} \nabla \times - \omega^2 \epsilon_0) E = 0 \quad \text{in } \Omega \\
E \times \nu = g \quad \text{on } \partial \Omega
\]
It follows from [5] that for all \( x \) in \( \Omega \)
\[
E(x) = \nabla \times \int_{\partial \Omega} \Phi(x,y) \cdot E(y) \times \nu_y \, d\sigma_y \\
+ \nabla \int_{\partial \Omega} \Phi(x,y) \cdot E(y) \cdot \nu_y \, d\sigma_y \\
+ \int_{\partial \Omega} \Phi(x,y)(\nabla \times E(y)) \times \nu_y \, d\sigma_y
\]

After integration by parts on \( \partial \Omega \) and multiplication by \( \nu_x \), we obtain at the limit as \( x \) approaches \( \partial \Omega \)
\[
\frac{1}{2} E(x) \times \nu_x = \left[ \nabla \times \int_{\partial \Omega} \Phi(x,y) \cdot E(y) \times \nu_y \, d\sigma_y \right] \times \nu_x \\
+ \left[ \int_{\partial \Omega} \Phi(x,y)(\nabla \times E(y)) \times \nu_y \, d\sigma_y \right] \times \nu_x \\
+ \int_{\partial \Omega} \nabla \Phi(x,y)(\nu_x - \nu_y)E_y \, d\sigma_y \\
+ \int_{\partial \Omega} \Phi(x,y) \text{curl}_{\partial \Omega}(E\nu_y) \, d\sigma_y
\]
That is,
\[
\frac{1}{2} g = P g + \left( Q + \frac{1}{k^2} R A + \frac{1}{k^2} S A \right) N_0 g \tag{18}
\]
from which the lemma can be deduced, adding \( \frac{1}{2} g \) on both sides of Eq. (18).

We see from (15) and Lemma 2.1 that
\[
T_\alpha(E_\alpha \times \nu) - T_0(E_0 \times \nu) = 0 \tag{19}
\]
and using the definitions (16) and (17),
\[
T_\alpha(E_\alpha \times \nu - E_0 \times \nu) = T_0(E_0 \times \nu) - T_\alpha(E_0 \times \nu) \\
= \left( Q + \frac{1}{k^2} R A + \frac{1}{k^2} S A \right) \\
\times N_\alpha(E_\alpha \times \nu - E_0 \times \nu) \tag{20}
\]
In order to express the main result in the following proposition, we need to utilize the usual radiating fundamental solution for Maxwell’s equations, namely,
\[
G(x,y) = - \epsilon^0 \left( \Phi(x,y) I_3 + \frac{1}{k^2} D_x^2 \Phi(x,y) \right)
\]
where \( I_3 \) is the \( 3 \times 3 \) identity matrix and \( D_x^2 \) denotes the Hessian.

**Proposition 2.1** Let \( T_\alpha \) and \( T_0 \) be defined by (16) and (17). The following properties hold

(a) \( T_\alpha \) converges to \( T_0 \) pointwise.

(b) \( (T_\alpha - T_0) \) is collectively compact.

(c) There is a constant \( C \) that is independent of \( \alpha \) and the set of points \( \{ z_j \}_{j=1}^m \) such that for any \( g \) in \( TH_{\text{div}}^{-1/2}(\partial \Omega) \), \( T_\alpha^{-1} \) is well defined and
\[
\| T_\alpha^{-1} g \|_{TH_{\text{div}}^{-1/2}(\partial \Omega)} \leq \| g \|_{TH_{\text{div}}^{-1/2}(\partial \Omega)}
\]

(d) The following asymptotic formula holds
\[
(T_0 - T_\alpha)(E_0 \times \nu)(x) \times \nu_x = \alpha^3 \sum_{j=1}^m (\mu^0 - \mu^j) \frac{1}{\epsilon^0 \epsilon^j} \\
\times (\nabla \times G(x,z_j))^j M \left( \frac{\mu^0}{\mu^j} \right) \nabla \times E_0(z_j) + k^2 \left( \frac{1}{\epsilon^j} - \frac{1}{\epsilon^0} \right) \\
\times G(x,z_j) M \left( \frac{\epsilon^0}{\epsilon^j} \right) E_0(z_j) \nu_x + O(\alpha^4) \tag{21}
\]

**Proof.** In an effort to avoid cumbersome notations, we suppose in this proof that there is only a single inhomogeneity. In order to further simplify notation, we assume that the single inhomogeneity has the form \( \alpha B \) (centered at the origin of \( \mathbb{R}^3 \)) and denote the electric permittivity and magnetic permeability inside \( \alpha B \) by \( \epsilon^* \) and \( \mu^* \), respectively.

For any \( g \) in \( TH_{\text{div}}^{-1/2}(\partial \Omega) \), there is a unique field \( \overline{E_0} \) in \( H(\text{curl}, \Omega) \) such that
\[
\left( \nabla \times \frac{1}{\mu_0} \nabla \times - \omega^2 \epsilon_0 \right) \tilde{E}_0 = 0 \text{ in } \Omega
\]

\[
\tilde{E}_0 \times \nu = g \text{ on } \partial \Omega
\]

According to [3], this assumption also ensures well-posedness for the \( \alpha \)-dependent case for \( \alpha \) sufficiently small. So that there is a unique field \( \tilde{E}_\alpha \) in \( H(\text{curl}, \Omega) \) such that

\[
\left( \nabla \times \frac{1}{\mu_\alpha} \nabla \times - \omega^2 \epsilon_\alpha \right) \tilde{E}_\alpha = 0 \text{ in } \Omega
\]

\[
\tilde{E}_\alpha \times \nu = g \text{ on } \partial \Omega
\]

According to Eq. (67) in [3], we know the following expansion to be valid:

\[
(\nabla \times \tilde{E}_\alpha - \nabla \times \tilde{E}_0)(x) \times \nu_x = \alpha^3 \omega^2 (\mu_0 - \mu^*) \frac{\mu_0}{\mu^*} 
\times [G_0(0, x) \times M(\frac{\mu_0}{\mu^*}) \nabla \times \tilde{E}_0(0)] \times \nu_x
\]

\[
+ \alpha^3 k^2 \left( \frac{1}{\epsilon^*} - \frac{1}{\epsilon_0} \right) \nabla \times [G_0(0, x) \times M(\frac{\epsilon_0}{\epsilon^*}) \tilde{E}_0(0)] \times \nu_x
\]

\[
+ O(\alpha^4)
\]  

(22)

where \( G_0 \) is the matrix Green’s function defined by

\[
\left\{ \begin{align*}
\left( \nabla_x \frac{1}{\epsilon_0} \nabla_x - \omega^2 \mu_0 \right) G_0(x, z) &= -\delta_z I_3 \text{ in } \Omega \\
\frac{1}{\epsilon_0} (\nabla_x \times G_0(x, z) \times \nu(x)) &= 0 \text{ on } \partial \Omega
\end{align*} \right.
\]

By elliptic estimates, we know that the difference \( \nabla \times \tilde{E}_\alpha - \nabla \times \tilde{E}_0 \) is smooth in a neighborhood of \( \partial \Omega \) in \( \Omega \), thus the same smoothness holds for the remainder \( O(\alpha^4) \). That remainder was also proven in [3] to be independent of \( \alpha \) and the set of points \( \{z_j\}_{j=1}^m \). Therefore, \( N_\alpha \) is pointwise convergent to \( N_0 \) and \( N_\alpha - N_0 \) is collectively compact because for any \( g \), \( \tilde{E}_0(0) \) and \( \nabla \times \tilde{E}_0(0) \) are bounded by \( \|g\|_{TH_{d_\Omega}^{1/2}(\partial \Omega)} \). Combining this with (16) and (17) and recalling that \( N_\alpha(g) = (\nabla \times \tilde{E}_\alpha) \times \nu \) and \( N_0(g) = (\nabla \times \tilde{E}_0) \times \nu \) yields points (a) and (b). (c) is now a consequence of the theory of collectively compact operators.

To derive (d), we rewrite \((T_0 - T_\alpha)(E_0 \times \nu)\) using the Green function \( G \)

\[
(T_0 - T_\alpha)(E_0 \times \nu)(x) = \left( Q + \frac{1}{k^2} RA + \frac{1}{k^2} SA \right) (N_0 - N_\alpha)(E_0 \times \nu)
\]

\[
= \left( \int_{\partial \Omega} \Phi(x, y)(N_0 - N_\alpha)(E_0 \times \nu) d\sigma_y \right) \times \nu_x
\]

\[
+ \frac{1}{k^2} \int_{\partial \Omega} \nabla_x \Phi(x, y) \times (\nu_x - \nu_y) \text{div}_{\partial \Omega}
\]

\[
+ \frac{1}{k^2} \int_{\partial \Omega} \Phi(x, y) \text{curl}_{\partial \Omega} \text{div}_{\partial \Omega}
\]

\[
= \lim_{x \to \partial \Omega} \left\{ \left( \int_{\partial \Omega} \Phi(x, y)(N_0 - N_\alpha)(E_0 \times \nu) d\sigma_y \right) \times \nu_x
\]

\[
+ \frac{1}{k^2} \int_{\partial \Omega} \nabla_x \Phi(x, y) \times (\nu_x - \nu_y) \text{div}_{\partial \Omega}
\]

\[
+ \frac{1}{k^2} \int_{\partial \Omega} \Phi(x, y) \text{curl}_{\partial \Omega} \text{div}_{\partial \Omega}
\]

\[
= \lim_{x \to \partial \Omega} \left( - \int_{\partial \Omega} \frac{1}{\epsilon_0} G(x, y)(N_0 - N_\alpha)(E_0 \times \nu) d\sigma_y \right) \times \nu_x
\]

(23)

Note that it is possible in the above equalities to pass to the limit because the principal parts of the integration kernels are homogeneous of degree -1. But according to Eq. (22), we know the following expansion to be valid
\((N_0 - N_\alpha)(E_0 \times \nu) (z) \times \nu_z = \)
\[-\alpha^3 \omega^2 (\mu^0 - \mu^\ast) \nu^0 \times G_0(0, z)^t M \left( \frac{\mu^0}{\mu^\ast} \right) \nabla \times E_0(0) \times \nu_z \]
\[-\alpha^3 \omega^2 \left( \frac{1}{\epsilon^s} - \frac{1}{\epsilon^\ast} \right) \times (\nabla \times G_0(0, z))^t M \left( \frac{\epsilon^0}{\epsilon^\ast} \right) E_0(0) \times \nu_z + O(\alpha^4) \quad (24)\]

If we just plug Eq. (24) into Eq. (23), we get a fairly complicated formula. However, it can be significantly simplified. There appears a term in the form

\[
\int_{\partial \Omega} G(x, y) \left[ G_0(0, y)^t M \left( \frac{\mu^0}{\mu^\ast} \right) \nabla \times E_0(0) \right] \times \nu_y d\sigma_y
\]

To make things clearer, denote \(R_1(x, y)^t\) the first row of \(G(x, y)\). \(R_1(x, y)^t\) is then a column vector. \(\nu\) will denote the usual scalar product in \(\mathbb{R}^3\).

\[
\begin{align*}
R_1(x, y)^t & \left[ G_0(0, y)^t M \left( \frac{\mu^0}{\mu^\ast} \right) \nabla \times E_0(0) \right] \times \nu_y \\
& = - \left[ G_0(0, y)^t M \left( \frac{\mu^0}{\mu^\ast} \right) \nabla \times E_0(0) \right] [R_1(x, y)^t \times \nu_y] \\
& = - \left[ M \left( \frac{\mu^0}{\mu^\ast} \right) \nabla \times E_0(0) \right] [G_0(0, y) R_1(x, y)^t \times \nu_y]
\end{align*}
\]

It then suffices to use Lemma 11 from [3], which asserts that for any smooth vector field \(w\) satisfying

\[
(\nabla \times \frac{1}{\epsilon^0} \nabla \times - \omega^2 \nu^0) w = 0
\]

in a neighborhood of \(\Omega\), the following identity holds:

\[
- \int_{\partial \Omega} G_0(0, y) w(y) \times \nu_y d\sigma_y = \frac{1}{\omega^2 \mu^0} \nabla_y \times w(0)
\]

to readily get

\[
- \int_{\partial \Omega} G_0(0, y) R_1(x, y)^t \times \nu_y d\sigma_y = \frac{1}{\omega^2 \mu^0} \nabla_y \times R_1(x, 0)^t
\]

Since \(G(x, y)\) is a symmetric matrix and \(R_1(x, 0)^t\) is equal to the first column of \(G(x, 0)\), we conclude that

\[
\frac{1}{\omega^2 \mu^0} \int_{\partial \Omega} G(x, y) \left[ G_0(0, y)^t M \left( \frac{\mu^0}{\mu^\ast} \right) \nabla \times E_0(0) \right] \times \nu_y d\sigma_y = \frac{1}{\omega^2 \mu^0} \int_{\partial \Omega} \left( \nabla_y \times G(x, 0) \right)^t M \left( \frac{\mu^0}{\mu^\ast} \right) \nabla \times E_0(0)
\]

We also want to simplify a term in the form

\[
\begin{align*}
\frac{1}{\omega^2 \mu^0} \int_{\partial \Omega} G(x, y) (\nabla \times G_0(0, y))^t M \left( \frac{\epsilon^0}{\epsilon^\ast} \right) E_0(0) \times \nu_y d\sigma_y & \\
= \frac{1}{\omega^2 \mu^0} \int_{\partial \Omega} \left[ (\nabla \times G_0(0, y))^t M \left( \frac{\epsilon^0}{\epsilon^\ast} \right) E_0(0) \times \nu_y d\sigma_y \right] \times \nu_y & \\
= - \left[ (\nabla \times G_0(0, y))^t M \left( \frac{\epsilon^0}{\epsilon^\ast} \right) E_0(0) \times \nu_y d\sigma_y \right] \times \nu_y & \\
= - \left[ M \left( \frac{\epsilon^0}{\epsilon^\ast} \right) E_0(0) \times \nu_y d\sigma_y \right] \times \nu_y & \\
= \int_{\partial \Omega} \nabla_y \times G_0(0, y) R_1(x, y)^t \times \nu_y
\end{align*}
\]

It then suffices to use Lemma 11 from [3] to get

\[
- \int_{\partial \Omega} \nabla \times G_0(0, y) R_1(x, y)^t \times \nu_y d\sigma_y = \epsilon^0 R_1(x, 0)
\]

We finally get the formula announced in (d) by letting \(x\) tend to \(\partial \Omega\).

Denote

\[
E^{(1)}(x) = \sum_{j=1}^{m} \left( \frac{\mu^0 - \mu^\ast}{\epsilon^0 \mu^\ast} \right) \frac{1}{\omega^2 \mu^0} \nabla \times E_0(z_j)
\]

\[
\times (\nabla \times G(x, z_j))^t M \left( \frac{\mu^0}{\mu^\ast} \right) \times \nabla \times E_0(z_j)
\]

\[
+ k^2 \left( \frac{1}{\epsilon^s} - \frac{1}{\epsilon^\ast} \right) G(x, z_j) M \left( \frac{\epsilon^0}{\epsilon^\ast} \right) E_0(z_j)
\]
\( E^{(1)} \) is the first term in the asymptotic expansion for \( E_\alpha - E_0 \) as stated in the following lemma.

**Lemma 2.2** The following estimate holds

\[
\| E_\alpha \times \nu - E_0 \times \nu - \alpha^3 E^{(1)} \times \nu \|_{T H^{-1/2}_0(\Omega)} = O(\alpha^4)
\]

where \( O(\alpha^4) \) is a term of order 4 independent of the set of points \( \{ z_j \}_{j=1}^m \).

**Proof.** This lemma is a straightforward consequence of (19), Lemma 2.1, and Proposition 2.1. Indeed,

\[
\begin{align*}
T_\alpha (E_\alpha \times \nu - E_0 \times \nu - \alpha^3 E^{(1)} \times \nu) &= (T_\alpha - T_0)(E_0 \times \nu) - \alpha^3 T_\alpha (E^{(1)} \times \nu) \\
&= \alpha^3 E^{(1)} \times \nu - \alpha^3 T_\alpha (E^{(1)} \times \nu) + O(\alpha^4) \\
&= \alpha^3 (T_0 - T_\alpha)(E^{(1)} \times \nu) + O(\alpha^4) \\
&= O(\alpha^4)
\end{align*}
\]

Using the preceding lemma, it is now possible to derive an expansion for \( E_\alpha \) in the open set \( \mathbb{R}^3 \setminus \Omega \).

**Theorem 2.1** Let \( E_\alpha \) solve (7) and satisfy the radiation condition (8) and let \( M^j(\mu^0/\mu^j) \) and \( M^j(\varepsilon^0/\varepsilon^j) \) be the polarization tensors for the shapes \( B_j \) defined by (4). Then for \( x \in \mathbb{R}^3 \setminus \Omega \) bounded away from \( \partial \Omega \), the following pointwise expansion holds:

\[
E_\alpha(x) = E_0(x) + \alpha^3 \sum_{j=1}^m \left( \frac{\mu^0 - \mu^j}{\varepsilon^0 \mu^j} \right) (\nabla z \times G(x, z_j))^t \\
\times M^j \left( \frac{\mu^j}{\mu^0} \right) \nabla \times E_0(z_j) + k^2 \left( \frac{1}{\varepsilon^j} - \frac{1}{\varepsilon^0} \right) \\
\times G(x, z_j) M^j \left( \frac{\varepsilon^0}{\varepsilon^j} \right) E_0(z_j) + O\left( \frac{\alpha^4}{|x|} \right) \quad (26)
\]

where the remainder \( O(\alpha^4/|x|) \) is independent of the set of points \( \{ z_j \}_{j=1}^m \) provided that (1) holds.

**Proof.** We first introduce the matrix valued Green function \( G_1 \) defined in \( \mathbb{R}^3 \setminus \Omega \) by

\[
\begin{align*}
\left\{ \begin{array}{l}
\nabla \times \nabla \times \omega^2 \mu^0 G_1(x, y) = -\delta_x I_3 \text{ in } \mathbb{R}^3 \setminus \Omega, \\
G_1(x, y) \times \nu(x) = 0 \text{ on } \partial \Omega, \\
\left| \nabla \epsilon^0 \right| G_1(x, y) - \frac{1}{i \omega \sqrt{\mu^0}} \left( \nabla \times G_1(x, y) \right) \times \frac{y}{|y|} = O\left( \frac{1}{|y|^2} \right)
\end{array} \right.
\]

We then use \( G_1 \) for a representation of \( E_\alpha - E_0 \) in \( \mathbb{R}^3 \setminus \Omega \) to get

\[
(E_\alpha(x) - E_0(x)) = -\int_{\partial \Omega} \frac{1}{\varepsilon^0} \left[ (\nabla y \times G_1(x, y)) \times \nu_y \right]^t \\
\times (E_\alpha - E_0) d\sigma_y
\]

Applying Lemma 2.2 to the above we obtain, for those points \( x \) bounded away from \( \partial \Omega \), that

\[
(E_\alpha(x) - E_0(x)) = -\alpha^3 \int_{\partial \Omega} \frac{1}{\varepsilon^0} \left[ (\nabla y \times G_1(x, y)) \times \nu_y \right]^t \\
\times E^{(1)}(y) d\sigma_y + O\left( \frac{\alpha^4}{|x|} \right) \quad (27)
\]

The \( O(\frac{\alpha^4}{|x|}) \) is independent of \( \{ z_j \}_{j=1}^m \). It is also independent of \( \frac{x}{|x|} \) because of the behavior at infinity of \( G_1 \). We plug the value of \( E^{(1)} \) in Eq. (27). Now, notice that the functions

\[
y \to (\nabla z \times G(y, z_j))^t
\]

and

\[
y \to G(y, z_j)^t
\]

are smooth for \( y \) in \( \mathbb{R}^3 \setminus \Omega \). They also satisfy the same radiation condition as \( G_1 \). Thus,

\[
- \int_{\partial \Omega} \frac{1}{\varepsilon^0} \left[ (\nabla y \times G_1(x, y)) \times \nu_y \right]^t (\nabla z \times G(y, z_j))^t \\
= (\nabla z \times G(x, z_j))^t
\]

and
for $x$ in $\mathbb{R}^3 \setminus \Omega$. Equation (26) ensues.

3. ASYMPTOTIC FORMULA FOR THE SCATTERING AMPLITUDE

In this section, we find and rigorously prove a formula, asymptotic with respect to the inhomogeneity size $\alpha$, for the scattering amplitude $A_{\alpha}$.

To find the leading-order term in the asymptotic expansion of $A_{\alpha}$, we first expand $G(x, y)$ for a large $x$ and a fixed $y$

$$G(x, y) = -e^{0}(\Phi(x, y)I_{3} + \frac{1}{\kappa^2} D^{2}_{x} \Phi(x, y))$$

$$= -e^{0} \left[ \frac{e^{i|x-y|}}{4\pi|x-y|} I_{3} - \frac{(x-y)(x-y)^{t}}{4\pi|x-y|^{3}} e^{i|x-y|} \right]$$

$$+ O \left( \frac{1}{|x|^{2}} \right) = -e^{0} \left( I_{3} - \frac{x^{t}}{|x|^{2}} \frac{e^{-i\frac{x}{|x|}y}}{4\pi} \right)$$

$$+ O \left( \frac{1}{|x|^{2}} \right)$$

(28)

We perform a similar expansion for $(\nabla_{y} \times G(x, y))^{t}$

$$(\nabla_{y} \times G(x, y))^{t} = -e^{0} ik \frac{e^{i|x-y|}}{4\pi|x-y|^{2}}$$

$$\times \left[ (y-x) \times e_{1}, (y-x) \times e_{2}, (y-x) \times e_{3} \right]^{t}$$

$$+ O \left( \frac{1}{|x|^{2}} \right) = -e^{0} ik \frac{e^{-i\frac{x}{|x|}y}}{4\pi|x|}$$

$$\times \left( -x \times e_{1}, -x \times e_{2}, -x \times e_{3} \right) + O \left( \frac{1}{|x|^{2}} \right)$$

(29)

where $\{e_{1}, e_{2}, e_{3}\}$ is the natural basis for $\mathbb{R}^3$. Note that for a fixed vector $u$ in $\mathbb{R}^3$

$$(x \times e_{1}, x \times e_{2}, x \times e_{3})^{t} u = u \times x$$

From (5) we infer

$$E_{0}(z_{j}) = ik\eta \times \lambda e^{ik\eta \cdot z_{j}}$$

and

$$\nabla \times E_{0}(z_{j}) = -k^{2} \eta \times (\eta \times \lambda) e^{ik\eta \cdot z_{j}}$$

Finally, putting everything together we find that the asymptotic formula (10) for the scattering amplitude holds.

REFERENCES


