An inverse problem for the recovery of active faults from surface observations

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Abstract

We discuss in this paper the possibility of detecting slow slip events (such as silent earthquakes, or earthquake nucleation phases) in the vicinity of geological faults, and the possible localization of those faults from GPS observations. A nonlinear eigenvalue problem (of Steklov type), modeling the slow evolution of the slip is stated as a direct problem. The recovery of an active fault from surface observations is formulated as the related inverse problem.

We perform an asymptotic analysis of the solution with respect to the depth of the fault. We start from an integral formulation for the direct problem. We prove that the differences between the eigenvalues and eigenfunctions attached to the half space problem and those attached to the free space problem, is of the order of $d^{-2}$, where $d$ is a depth parameter. An asymptotic formula for the observed surface displacement, with a remainder of same order is then derived. From that formula, we infer two inversion techniques for the recovery of faults from surface observations. The first one involves a least square minimization method; the second one uses the momentum method. The recovered information contains only the depth of the fault and the "normalized seismic moment", which is related to the fault shape. We test the two inversion methods for line segment faults in numerical simulations. We are lead to conclude that the momentum method gives a very good initial guess for the the least square minimization method, which turns out to be sharp, robust and computationally inexpensive. Unexpectedly, the latter method is also very efficient for faults that are close to the observation surface, despite the fact that our asymptotic approximation may not be valid in that region. Finally we assess how our method for detecting active faults is affected by the sensitivity of the observation apparatus and the stepsize for the grid of surface observation points. The maximum permissible stepsize for such a grid is computed for different values of fault depth and orientation.

Keywords: inverse problems, domains with cracks, integral equations, spectral analysis, slip-dependent friction, earthquake initiation, silent earthquakes

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1 Introduction

Understanding the dynamics of slow slip events (for a classification of such phenomena, see [12]) on geological faults can be vitally important in seismology. By slow event, we mean important slip taking place on an intermediate time scale (i.e. minutes to months). This is much longer than seismic time scales (seconds) but much shorter than geological time scales (hundreds of years). Since slow slip events are aseismic (i.e. there is no associated seismic wave), their detection is possible only by means of modern GPS techniques which can resolve ”less-than-cm” surface displacements. Two types of phenomena can be related to slow slip events: silent earthquakes and nucleation (or initiation) phases for (ordinary) earthquakes. Either phenomenon can be modeled using the same physics (slip-weakening of friction force) in association to the same mathematics which involve eigenvalue analysis.

Accounts of silent earthquakes in subduction zones near Japan [18], New Zealand, Alaska and Mexico [14, 13] were recently reported in the literature. Silent earthquakes are rather large (6. \leq M_w \leq 8.) and produce surface displacements (range about 2. – 6.cm) that can be picked up by GPS techniques.

The earthquake nucleation (or initiation) phase, which precedes dynamic rupture, was uncovered by detailed seismological observations [11, 6] and recognized in laboratory experiments [5, 16]. Important physical properties of the nucleation phase (characteristic time, critical fault length, etc.) were obtained in [2, 7, 4, 1, 19] through simple mathematical properties of unstable evolution. Early detection of nucleation phase from surface displacements has the potential to play a key role in short time prediction of large earthquakes.

The aim of this paper is to study the possible recovery of faults from surface displacements due to slow slip events. Our two fold goal is to detect active faults and to localize them using GPS measurements. We will not attempt to recover the shape of a detected fault: this could be a very difficult or even an ill posed problem. We restrict ourselves to deriving an inversion technique capable of detecting the presence of active faults and of producing limited qualitative information on the fault position and seismic moment.

We now give an outline of this paper. The nonlinear eigenvalue problem describing the slow evolution of the slip is stated in section 2. We discuss both the direct and the inverse problems. A simplified (linearized) model is also presented, and in many situations, it leads to the same solution. In section 3 we introduce a physical model for the the anti-plane configuration, we recall some mathematical properties of the related elastic energy and we derive the eigenvalue problem for the spectral stability analysis of the slip. In the following section we give an integral formulation of the direct problem in the half plane and we indicate analogs of those results in the free space case.

In section 5, we assume that the first eigenspace is one dimensional and we perform an asymptotic analysis for the corresponding first eigenvector with respect to fault depth. We prove that eigenvalues and eigenfunctions for the half space differ from those in free space by a quantity of order \(d^{-2}\), where \(d\) is a depth parameter. We derive an asymptotic formula for the observed surface displacement, valid within the same order.
That formula then serves as the starting point for devising an efficient recovery method for faults. We illustrate the previous asymptotic analysis by direct computations of eigenfunctions.

In section 6 we deduce two inversion techniques for recovering faults from surface observations based on our asymptotic formula. The first one involves a least square minimization method and the second one uses the momentum method. We only recover the depth of the fault and the "normalized seismic moment" associated to the fault shape.

In section 7 we present numerical examples of reconstruction of line segment faults. We compare the two inversion methods and conclude that the momentum method gives a very good initial guess for the least square minimization method. We demonstrate that this combined approach proves to be sharp, robust and computationally inexpensive. Unexpectedly, this method also performs rather well for faults that are close to the observation surface, despite the fact that our asymptotic formula may not to be valid in that region.

Finally, in section 8, we assess how our method for detecting active faults is affected by the sensitivity of the observation apparatus and the stepsize for the grid of surface observation points. The maximum permissible stepsize for such a grid is computed for different values of fault depth and orientation.

Two appendices appear in the end of this paper. In the first one we perform an asymptotic analysis of the direct problem that is specific to line segment faults. We prove that if such faults are far enough from the surface, then the first eigenspace for the displacement eigenvalue problem is one dimensional. In the second appendix, we specify the discretization scheme that was employed for the numerical solution to the direct problem.

2 Problem Statement

We denote \( \mathcal{D} \) a fixed two dimensional domain in the non dimensional coordinate system \( Ox_1 x_2 \). This domain is not necessarily bounded and contains no cuts, i.e. \( \mathcal{D} = \mathcal{D}^{\circ} \). Its boundary, called the exterior boundary and denoted by \( \Gamma_{\text{ext}} := \partial \mathcal{D} \), contains two parts: the Dirichlet boundary \( \Gamma_0 \) (which may be empty) and the Neumann boundary \( \Gamma_{\text{obs}} \), called the "surface observation" boundary. Let \( \Gamma \) be a bounded connected arc, called cut, crack or fault, included in \( \mathcal{D} \). We denote by \( \Omega = \Omega(\Gamma) \) the domain \( \mathcal{D} \) without the fault, \( \Omega := \mathcal{D} \setminus \Gamma \) which has \( \Gamma \) as an internal boundary (see figure 1). Notice that our problem will be formulated in a non dimensional coordinates system, which means that we chose a characteristic length \( L \). A natural choice for \( L \) is provided by relating it to the physical length of the fault. In our coordinate system we decide to fix the length of the fault, by imposing \( |\Gamma| = 2 \).
2.1 Direct problem

Let us start by defining the direct problem. We consider the following eigenproblem involving the Laplace operator and Signorini-type boundary conditions:

\[ \begin{align*}
\text{div}(\nabla \Phi) &= 0 \quad \text{in } \Omega, \\
\Phi &= 0 \quad \text{on } \Gamma_0, \\
\partial_n \Phi &= 0 \quad \text{on } \Gamma_{\text{obs}}, \\
[\partial_n \Phi] &= 0, \\
[\Phi] &\geq 0, \\
\partial_n \Phi &\geq \beta [\Phi], \\
[\Phi](\partial_n \Phi - \beta [\Phi]) &= 0 \quad \text{on } \Gamma,
\end{align*} \]

where we have denoted by \([\ ]\) the jump across \(\Gamma\) (i.e. \([w] = w^+ - w^-\)), and by \(\partial_n = \nabla \cdot n\) the corresponding normal derivative, with the unit normal \(n\) pointing toward the positive side.

Let us now give the variational formulation for the above nonlinear eigenvalue problem. We introduce, as in [15, 10], the space \(V\) of functions of finite elastic energy. Let \(V\) be the following subspace of \(H^1(\Omega)\):

\[ V = \{ v \in H^1(\Omega); v = 0 \quad \text{on } \Gamma_d, \quad \text{there exists } R > 0 \text{ such that } v(x) = 0 \text{ if } |x| > R \} \]

endowed with the norm \(\| \cdot \|_V\) defined by the following dot product:

\[ (u,v)_V = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad \|u\|_V^2 = (u,u)_V, \quad \forall u,v \in V. \]

We define \(V\) as the closure of \(V\) in the norm \(\| \cdot \|_V\). The dot product \((u,v)_V\) in \(V\) is still defined by \(\int_{\Omega} \nabla u \cdot \nabla v dx\). The space \(V\) is continuously embedded in \(H^1(\Omega_R)\) for all \(R > 0\), with \(\Omega_R := \{ x \in \Omega / |x| < R \}\). If \(\Omega\) is not bounded, \(V\) is not a subspace of \(H^1(\Omega)\). Indeed, if \(v \in V\) then \(v(x)\) is not vanishing for \(|x| \to +\infty\).

If we denote

\[ V_+ := \{ v \in V / [v] \geq 0 \quad \text{on } \Gamma \}, \]
then the nonlinear eigenvalue problem (1-2) can be written as

\[
\begin{cases}
\text{find } \Phi \in V_+, \Phi \neq 0 \text{ and } \beta \in \mathbb{R}_+ \text{ such that } \\
\int_\Omega \nabla \Phi \cdot \nabla (v - \Phi) \, dx \geq \beta \int_\Gamma |[\Phi]| [v - \Phi] \, d\sigma, \ \forall \ v \in V_+.
\end{cases}
\]  

(4)

The above eigenvalue problem was studied in [9]. For bounded domains, it was proved in [9] that the spectrum of (1-2) consists of a nondecreasing and unbounded positive sequence of eigenvalues \( \beta \). We can associate the Rayleigh quotient \( \beta_0 \) to the eigenvalue variational inequality (4). \( \beta_0 \) is the smallest eigenvalue \( \beta \) for the problem (1-2). More precisely we have the following result (see [10]):

**Theorem 2.1.** There exists \((\Phi_0, \beta_0)\) a solution to the nonlinear eigenvalue problem (4) such that

\[
\beta_0 = \frac{\int_\Omega |\nabla \Phi_0|^2 \, dx}{\int_\Gamma [\Phi_0]^2 \, d\sigma} = \min_{v \in V_+} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Gamma [v]^2 \, d\sigma}.
\]  

(5)

Moreover, if \((\Phi, \beta)\) is another solution of (4) then \( \beta \geq \beta_0 \).

\( \Phi_0 \) can be normalized in different ways. We will focus on two possible normalizations, one through the \( L^2(\Gamma) \) norm,

\[
\int_\Gamma [\Phi_0]^2 \, d\sigma = 1,
\]

and the other one relative to the maximum slip

\[
\max_{x \in \Gamma} [\Phi_0](x) = 1.
\]

We can now formulate our direct problem: it consists of finding the first eigenfunction \( \Phi_0 = \Phi_0(\Gamma) \) for a given fault \( \Gamma \subset \mathcal{D} \). We then define the observable part of the first eigenfunction denoted by \( \psi(\Gamma) := \Phi_0|_{\Gamma_{\text{obs}}} \in H^{1/2}(\Gamma_{\text{obs}}) \), i.e. the restriction of \( \Phi_0 \) on \( \Gamma_{\text{obs}} \). In conclusion the direct problem maps \( \Gamma \subset \mathcal{D} \) to the observable part \( \psi(\Gamma) \in H^{1/2}(\Gamma_{\text{obs}}) \) of the first eigenfunction.

### 2.2 Inverse problem

We now introduce the related inverse problem. To any distribution of surface observation \( \omega \in H^{1/2}(\Gamma_{\text{obs}}) \), there corresponds a dislocation of amplitude \( m \) along the fault \( \Gamma \) associated to the first eigenvalue \( \beta_0 \), i.e. \( \omega = m \psi(\Gamma) \) on \( \Gamma_{\text{obs}} \).

The inverse problem consists of identifying the set of surface observations

\[
H_{\text{obs}} := \{ m \psi(\Gamma) ; \ \Gamma \subset \mathcal{D} \} \subset H^{1/2}(\Gamma_{\text{obs}})
\]

corresponding to existing active faults. If the observation \( \omega \) belongs to \( H_{\text{obs}} \), we will seek to recover the location of the corresponding fault \( \Gamma \).

A full fledged inverse problem, that is, the attempt of recovering exactly the cut \( \Gamma \), could be very difficult to tackle, it could even be ill posed. The inverse problem is
considerably simplified if we forgo the attempt to recover the shape of the fault. Note
that in practice, detecting the presence of active faults and obtaining some additional
qualitative information such as location and seismic moment, is already a worthy ac-
complishment.

It is convenient to assume that the first eigencone for the problem (1-2), is one
dimensional, i.e. the eigenfunction Φ₀(Γ) is unique. This assumption does not hold
in general, but if Γ is a line segment which is not too close to the surface observation
Γₖₜ this dimension assumption turns out to be correct (see section 9.2). Although the
geometry of the geological fault may be non trivial, in practice, it is often close to
being a line segment. Solving the inverse problem is much simpler under that dimension
assumption.

2.3 Linear problem

The above direct and inverse problems are associated to the nonlinear eigenvalue
problem (1-2). In most cases this nonlinear problem can be replaced by a linear problem,
that is, equation (1) can be replaced by the boundary condition

$$ [\partial_n \Phi] = 0, \quad \partial_n \Phi = \beta [\Phi] \text{ on } \Gamma. $$

Equivalently, the linearized problem can be stated in its variational form,

$$ \begin{cases} 
\text{find } \Phi \in V, \Phi \neq 0 \text{ and } \beta \in \mathbb{R}_+ \text{ such that} \\
\int_\Omega \nabla \Phi \cdot \nabla v dx = \beta \int_\Gamma [\Phi] [v] d\sigma, \quad \forall v \in V.
\end{cases} $$

(7)

The linearized problem (1),(6) was analyzed in [8] in the case of bounded domains and
in [3] in the case of unbounded domains. In either case the spectrum consists of a
nondecreasing and unbounded positive sequence of eigenvalues β. The first eigenvalue
is given by the Rayleigh quotient,

$$ \beta_0 = \frac{\int_\Omega |\nabla \Phi_0|^2 dx}{\int_\Gamma [\Phi_0]^2 d\sigma} = \min_{v \in V} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Gamma [v]^2 d\sigma}. $$

(8)

Let us remark that, if Φ is a solution to (1), (6) and [Φ] ≥ 0 on Γ, then Φ is also a
solution for (1), (2). If Γ is comprised of a set of co-linear faults or if the fault has a small
curvature, the first eigenfunction Φ₀ corresponding to the linear eigenvalue was found
in numerical computations to have a positive slip on Γ (see [3, 4]), thus the equations
for the linear case lead to a satisfactory model for the initiation of instabilities.

In the linear case the direct and inverse problems have the same formulation as
previously. We just need to replace the nonlinear eigenvalue problem (5) with the linear
problem (8).

3 Physical motivation

Consider, as in [3, 4, 19], the anti-plane shearing on a fault (or a system of fi-
nite faults) under a slip-dependent friction in a linear elastic domain \( \Omega \times \mathbb{R} \), in non-
Fig. 2 – The piecewise linear slip weakening friction law (solid line). Without constraints on the sign of the slip and shear stress, the linearization can lead to solutions lying on the dashed line.

dimensional coordinates $Ox_1x_2x_3$, for which a characteristic length $L$ was chosen. We suppose that the displacement field $u = (u_1, u_2, u_3)$ is zero in the $Ox_1$ and $Ox_2$ directions and that $u_3$ does not depend on $x_3$. The displacement is therefore simply denoted by $w = w(t, x_1, x_2)$. Assume that the elastic medium has shear rigidity $G$, density $\rho$ and shear velocity $c = \sqrt{G/\rho}$. The non-vanishing shear stress components are $\sigma_{31} = \tau_1^\infty + \frac{G}{\tau} \partial_1 w$, $\sigma_{32} = \tau_2^\infty + \frac{G}{\tau} \partial_2 w$, and $\sigma_{11} = \sigma_{22} = -S$, where $\tau^\infty$ is the pre-stress and $S > 0$ is the normal stress on the faults. We assume that $S, \tau_1^\infty, \tau_2^\infty$ are continuous in $\Omega$.

The boundary $\Gamma_{obs} \times \mathbb{R}$ corresponds to the surface of the earth where a stress free condition is imposed. Thus the pre-stress and the over stress have to vanish on that boundary, which is expressed by $\tau^\infty \cdot n = 0$ in conjunction with $G \partial_n w = 0$.

On the interface $\Gamma$, the shear stress has no jumps $[G \partial_n w] = 0$. We now introduce a friction type constitutive law. The friction force depends on the slip $[w]$ through a friction coefficient $\mu = \mu([w])$ which is multiplied by the normal stress $S$. We assume that the friction coefficient is a Lipschitz continuous function, with respect to the slip. Let $H$ be the antiderivative

$$H(x, u) := S(x) \int_0^u \mu(x, s) \, ds.$$  

We suppose that there exist some constants $l, a, \alpha \geq 0$, such that

$$|\mu(x, s_1) - \mu(x, s_2)| \leq l|s_1 - s_2|, \quad H(x, s) - S(x)\mu(x, 0)s + \alpha s^2 / 2 + as^3 \geq 0, \quad (9)$$

for almost all $x \in \Gamma$, and for all $s, s_1, s_2 \in \mathbb{R}_+$. It is quite common to use piecewise linear friction laws, (see [17]). These are in the
following form:

\[
\{ \begin{array}{ll}
\mu(x, u) = \mu_s(x) - \frac{\mu_s(x) - \mu_d(x)}{2D_c(x)} u & \text{if } u \leq 2D_c(x) \\
\mu(x, u) = \mu_d(x) & \text{if } u > 2D_c(x),
\end{array} \}
\] (10)

where \( u \) is the relative slip, \( \mu_s \) and \( \mu_d \) (\( \mu_s > \mu_d \)) are the static and dynamic friction coefficients, and \( D_c \) is the critical slip. This piecewise linear model for the friction coefficient is a reasonable approximation to the experimental observations reported in [16]. If we put

\[
\alpha := \text{ess sup}_{x \in \Gamma} \frac{(\mu_s(x) - \mu_d(x))S(x)}{2D_c(x)},
\]

then (9) holds.

Let us now describe the static (or quasi-static) problem associated to this friction law. These processes correspond to "slow" slip events which characterize earthquakes developing on intermediate time scales (days, month). Compared to geological time scales, these phenomena are sufficiently rapid to be considered as earthquakes. The time scale governing usual earthquakes is of the order of seconds: the process is then fully dynamic. Even if the formulation is quite different in that case, the same approach is valid during the first part of the initiation (or nucleation) phase. The dynamical process is then quite slow and the same eigenvalue analysis is applicable, see [1, 2, 4, 19].

The slip dependent friction law on \( \Gamma \) in the static case is described by:

\[
\frac{G}{L} \partial_n w + q = -\mu([w(t)]) S \text{sign}([w]), \quad \text{if } [w] \neq 0,
\]

(11)

\[
|\frac{G}{L} \partial_n w + q| \leq \mu([w])S \quad \text{if } [w] = 0,
\]

(12)

where \( q := \tau^\infty \cdot n \) is the tangential pre-stress acting on the fault. The above equations assert that the tangential (frictional) stress is bounded by the normal stress \( S \) multiplied by the value of the friction coefficient \( \mu \). If such a limit is not attained, sliding does not occur. Otherwise the frictional stress is opposed to the slip \([w]\) and its absolute value depends on the slip through \( \mu \).

We suppose that we can choose the orientation of the unit normal of the fault (cut) \( \Gamma \) such that \( q(x) = \tau^\infty(x) \cdot n(x) \leq q_0 < 0 \) almost everywhere in \( \Gamma \). This choice is possible in many concrete applications, where the pre-stress \( \tau^\infty \) gives a dominant direction of slip. Since we are looking for equilibrium positions in the neighborhood of \( w \equiv 0 \), and since the direction of slip is given by \( \tau^\infty \), we can restrict the above friction law to the case of nonnegative slip \(([w] \geq 0)\). This is a common assumption in earthquake source geophysics.

From the equilibrium equation and the boundary conditions, we derive the following static problem (SP): find \( w : \Omega \to \mathbb{R} \) such that

\[
\text{div} \left( \frac{G}{L} \nabla w \right) = 0 \quad \text{in } \Omega,
\]

(13)
\[ w = 0 \quad \text{on } \Gamma_d, \quad [\partial_n w] = 0 \quad \text{on } \Gamma \] (14)

\[ [w] \geq 0, \quad \frac{G_L}{L} \partial_n w + q + S \mu([w]) \geq 0, \quad [w](\frac{G_L}{L} \partial_n w + q + S \mu([w])) = 0 \quad \text{on } \Gamma \] (15)

The following quasi-variational inequality represents the variational formulation for problem (SP): find \( w \in V_+ \) such that

\[
\int_{\Omega} \frac{G}{L} \nabla w \cdot \nabla (v - w) \, dx + \int_{\Gamma} S \mu([w])([v] - [w]) \, d\sigma + \int_{\Gamma} q([v] - [w]) \, d\sigma \geq 0,
\]

for all \( v \in V_+ \). If we consider \( W : V \to \mathbb{R} \) the energy functional:

\[
W(v) := \frac{1}{2} \int_{\Omega} \frac{G}{L} |\nabla v|^2 \, dx + \int_{\Gamma} H([v]) + q[v] \, d\sigma,
\]

and if \( w \in V_+ \) is a local extremum for \( W \), then \( w \) is a solution of (16) (see [8, 10]). Moreover, there exists at least a global minimum for \( W \) on \( V_+ \).

Let us now analyze the stability of the equilibrium \( w \equiv 0 \). To this end, we will suppose that \( q(x) + S(x) \mu(x,0) \leq 0 \), for almost all \( x \) in \( \Gamma \). This is true if and only if \( w \equiv 0 \) is a solution of (16). The following theorem, to be found in [10], asserts that the stability of the equilibrium is related to the first eigenvalue of the nonlinear problem (1-2).

**Theorem 3.1.** Let \( \alpha \) be as in (9) and let \( \beta_0 \) be given by Theorem 2.1. If

\[
\frac{\alpha L}{G} < \beta_0
\]

then \( w \equiv 0 \) is an isolated local minimum for \( W \), i.e. there exists \( \delta > 0 \) such that \( W(0) < W(v), \quad \forall v \in V_+, \, v \neq 0, \|v\|_V < \delta. \)

The linearized eigenvalue problem, that is equations (1)(6), can also be useful for the stability analysis (see [8]). For co-linear faults, or for a fault with small curvature, the first eigenfunction \( \Phi_0 \) was found in numerical computations to have a positive slip on \( \Gamma \) (see [3, 4]), thus the linearized equations provide a satisfactory model for the onset of instabilities. If the faults are not coplanar, then the unilateral condition is no longer satisfied, that is the first eigenfunction of the tangent (linear) problem has no physical significance. Thus, in the non coplanar case, a nonlinear (unilateral) eigenvalue problem has to be considered for modeling initiation of friction instabilities. This difficulty arises with the effect of stress shadowing. This effect does not exist for coplanar fault segments. This can be understood by looking at the linearization of the friction law around the equilibrium position \( w \equiv 0 \) (see Fig. 2): an unconstrained linearization can lead to solutions lying on the dashed line of the friction law, whereas the constrained formulation ensures that the solutions lie on the solid line, so that the corresponding slip is necessarily of constant sign.

It follows from theorem 3.1 that the solution to the static eigenvalue problem is the stability limit of the solution to the dynamical problem. Indeed, if for some reason
the stability condition $\alpha L/G < \beta_0$ is no longer valid, then the part of the solution associated with the first (positive) eigenvalue will have an exponential growth in time. Thus, after some time, this part will become dominant, while the other modes will undergo a wave-type evolution. The propagative terms are rapidly negligible and the shape of the slip distribution is fairly well approximated by the first eigenfunction $\Phi_0$ during all of the nucleation phase of an earthquake. The accuracy of the approximation of the dominant part (i.e. the first unstable eigenfunction) was illustrated by many numerical comparisons. The dominant part was compared in [2, 4, 20] with the full solution computed by a finite difference method. In each case the difference was found to be of the order of the initial perturbation, which is negligible with respect to the final amplitude of the solution at the end of the initiation phase. In conclusion, the distribution of the displacement on the earth surface (i.e. $w(t, x)$ on $\Gamma_{obs}$) is fairly well approximated by $x \rightarrow \exp(\lambda t)\Phi_{\lambda_0}(x)$ during a "long" period of time $t \in [0, T_c]$, called the nucleation phase. During that nucleation phase the slip $[w(t, x)]$ is less than the critical slip $D_c$ everywhere on the fault $\Gamma$. At the beginning of the initiation phase the exponential growth exponent $\lambda$ is small enough and the eigenfunction $\Phi_{\lambda_0}$ can be approximated by the static eigenfunction $\Phi_0$ given by (5) or (8).

4 Integral formulation in the half plane

Starting from this section we assume that $D$ is the half plane defined by $x_2 < 0$ and that the part of the boundary $\Gamma_0$ where Dirichlet conditions are applied, is empty. On $\Gamma_{obs}$, here defined by $x_2 = 0$, Neumann conditions are applied. In that case, an adapted Green's function is readily available. Denoting $G_0$ the free space Green's function for the Laplacian,

$$G_0(x_1, x_2, y_1, y_2) = \frac{1}{4\pi} \log \frac{1}{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

the half space Green's function $G$ with zero normal derivative at the line $x_2 = 0$ is

$$G(x_1, x_2, y_1, y_2) = G_0(x_1, x_2, y_1, y_2) + G_0(x_1, x_2, y_1, -y_2).$$

Furthermore we assume that the fault line $\Gamma$ is an oriented curve of class $C^1$, or piecewise $C^1$, that $\Gamma$ has no double points, and that

$$(x_1(v), x_2(v)), v \in [-1, 1],$$

are parametric equations for $\Gamma$. The parameter $v$ is supposed to be the arc length, that means that we implicitly assumed that $\Gamma$ is scaled to have the length 2. We take the unit normal $n$ to be indirectly perpendicular to the tangent vector.

We can now reformulate the characterization of the first eigenvalue $\beta_0$ defined by the Rayleigh quotient (8), in association with the linear eigenvalue problem (7), using integral operators on the curve $\Gamma$. We first review basic properties of double layer potentials, and normal derivatives of double layer potentials. The latter have to be understood in Hadamard’s finite part sense for hypersingular integrals. We will throughout this paper
use the work by Wendland et al. [21] to refer to regularity properties for hypersingular integrals.

We define \( \tilde{H}^{1/2}(\Gamma) \) as in [21]: let \( \tilde{\Gamma} \) be a simple smooth closed curve in \( \mathbb{R}^2 \) such that \( \Gamma \subset \tilde{\Gamma} \). Then,

\[
\tilde{H}^{1/2}(\Gamma) = \{ u \in H^{1/2}(\tilde{\Gamma}) | \text{supp}(u) \subset \Gamma \}.
\]

The norm on \( \tilde{H}^{1/2} \) is defined by

\[
\| u \|_{\tilde{H}^{1/2}(\Gamma)} = \| u \|_{H^{1/2}(\tilde{\Gamma})}.
\]

**Lemma 4.1.** Let \( \varphi \) in \( \tilde{H}^{1/2}(\Gamma) \) be non zero. Set

\[
u(y_1, y_2) = -\int_{\Gamma} \partial_n G \varphi(v) dv.
\]

Then \( u \) satisfies

\[
\Delta u = 0 \quad \text{in} \quad \Omega \setminus \tilde{\Gamma},
\]

\[
\partial_n u = 0 \quad \text{along} \quad \Gamma_{\text{obs}},
\]

\[
[u] = \varphi \quad \text{across} \quad \Gamma,
\]

\[
[\partial_n u] = 0 \quad \text{across} \quad \Gamma,
\]

\( \nabla u \) is in \( L^2(\Omega) \) and finally, if \( \varphi \neq 0 \),

\[
\int_{\Gamma} \frac{\partial u}{\partial n} \varphi dv > 0.
\]

**Proof.** Identities (18-19) are obvious. To derive the other identities, we first extend the \( \Gamma \) to \( \tilde{\Gamma} \), the boundary of a smooth domain \( U \) whose closure is included in \( \Omega \). We can do it in a such a way that \( n \) be the interior normal on the \( \Gamma \) part of \( \tilde{\Gamma} \), and that the orientation defined by parameterizing \( \Gamma \) as \( v \) increases be positive. \( \varphi \) is extended to \( \tilde{\Gamma} \) by 0. Classical potential theory indicates that properties (20-21) hold. We now apply Green’s theorem:

\[
\int_{\Gamma} \partial_n u \varphi dv = \int_{\Gamma} \partial_n u(u^+ - u^-) dv = \int_{U} |\nabla u|^2 + \int_{\Omega \setminus U} |\nabla u|^2,
\]

where we have used that \( \partial_n u = 0 \) on \( \Gamma_S \), and the fact that \( u(y) \) decays as \( y \) approaches infinity.

Now, if \( \int_{U} |\nabla u|^2 + \int_{\Omega \setminus U} |\nabla u|^2 = 0 \), then \( u \) is a constant in \( U \) and is zero in \( \Omega \setminus U \). By making another choice for \( U \), we can argue that \( u \) is zero everywhere in \( \Omega \setminus \Gamma \). Recalling \([u] = -\varphi\), this implies that \( \varphi \) is equal to zero.

We are now ready to reformulate the characterization of the first eigenvalue \( \beta_0 \) defined by the Rayleigh quotient (8), in association with the linear eigenvalue problem (7), using integral operators on the curve \( \Gamma \). This is done in the following proposition.

**Proposition 4.1.** The first eigenvalue \( \beta_0 \) defined by the Rayleigh quotient (8), associated to the linear eigenvalue problem (1),(6), can also be defined by this other quotient,

\[
\beta_0 = \inf_{\varphi \in \tilde{H}^{1/2}(\Gamma), \varphi \neq 0} \frac{-\int_{\Gamma} \partial_n \int_{\Gamma} \partial_n G \varphi(v) d\sigma(v) \varphi(u) d\sigma(u)}{\int_{\Gamma} \varphi^2 d\sigma},
\]

where \( G \) stands for \( G(x_1(v), x_2(v), y_1(u), y_2(u)) \).
Proof. Set
\[
\beta'_0 = \inf_{\varphi \in \tilde{H}^2_1(G), \varphi \neq 0} - \int_G \partial_{n_u} \int_G \partial_{n_x} G\varphi(v) \, d\sigma(v) \, \varphi(u) \, d\sigma(u) \int_G \varphi^2 \, d\sigma,
\]  
(24)
and
\[
\beta_0 = \min_{v \in V_+, v \neq 0} \int_\Omega |\nabla v|^2 \, dx \int_G [v]^2 \, d\sigma,
\]  
(25)
where \(V_+\) was introduced in Section 2. We want to prove that \(\beta'_0 = \beta_0\). Let us denote \(v_0\) a function achieving the minimum in (25). \(v_0\) is guaranteed to exist in virtue of theorem 2.1. It is also known from [8] that \(v_0\) satisfies
\[
\Delta v_0 = 0 \text{ in } \Omega \setminus \overline{\Gamma}, \\
\partial_n v_0 = 0 \text{ along } \Gamma_{obs}, \\
\beta_0[v_0] = \partial_n v_0 \text{ across } \Gamma, \\
[,\partial_n v_0] = 0 \text{ across } \Gamma.
\]
It follows that
\[
v_0(y_1, y_2) = -\int_G (\partial_{n_x} G)[v_0(x_1(v), y_1(v))] \, d\sigma(v).
\]
Therefore \(\beta_0 \geq \beta'_0\).

Arguing by contradiction, assume that \(\beta_0 > \beta'_0\). Then, for some positive \(\epsilon\) there exists \(\varphi \in \tilde{H}^1_1(\Gamma)\) such that \(\|\varphi\|_{L^2(\Gamma)} = 1\) and
\[
-\int_G \partial_{n_x} \int_G \partial_{n_x} G\varphi(v) \, d\sigma(v) \, \varphi(u) \, d\sigma(u) \leq \beta_0 - \epsilon.
\]
Set
\[
u(y_1, y_2) = -\int_G \partial_{n_x} G\varphi(v) \, dv.
\]
Then
\[
\int_G \partial_n u^+(u^- - u^-) \leq \beta_0 - \epsilon \\
\Delta u = 0 \text{ in } H^1(\Pi^- \setminus \Gamma) \\
\partial_n u = 0 \text{ on } \Gamma_{obs} \\
[,\partial_n u] = 0 \text{ across } \Gamma.
\]
But then,
\[
\int_\Omega |\nabla u|^2 \leq \beta_0 - \epsilon \text{ and } \int_G [u]^2 = 1,
\]
which contradicts the definition of \(\beta_0\). \(\blacksquare\)
Remark. The proof of Proposition 4.1 also showed that the infimum of the Rayleigh quotient (23) is achieved.

Proposition 4.1 and the remark that follows have analogs in the free space case, which is the case where $\mathcal{D} = \mathbb{R}^2$. In that case the functional space $V$ is simply the closure of smooth and compactly supported in $\mathbb{R}^2 \setminus \Gamma$ functions for the norm

$$\|v\| = \sqrt{\int_{\mathbb{R}^2} |\nabla v|^2 dx}.$$  

Accordingly, we introduce the following notations for the first eigenvalue for the linear problem (1),(6) in free space given by (8),

$$\beta_0^\infty = \frac{\int_{\mathbb{R}^2} |\nabla \Phi_0^\infty|^2 dx}{\int_{\Gamma} [\Phi_0^\infty]^2 d\sigma} = \min_{v \in V} \frac{\int_{\mathbb{R}^2} |\nabla v|^2 dx}{\int_{\Gamma} [v]^2 d\sigma}. \quad (26)$$

The analog of proposition 4.1 uses the free space Green’s function $G_0$ and states

$$\beta_0^\infty = \min_{\varphi \in H^2(\Gamma)} \frac{-\int_{\Gamma} \partial_n \varphi \int_{\Gamma} \partial_r G_0 \varphi(v) d\sigma(v) \varphi(u) d\sigma(u)}{\int_{\Gamma} \varphi^2 d\sigma}. \quad (27)$$

5 Fault depth asymptotic analysis

Let us start by giving some specific notations useful in this section, where we will derive an asymptotic formula relating the first eigenvector for the linear eigenvalue problem (1),(6), to the first eigenvector for the equivalent problem in free space. The asymptotic parameter will be a measure of depth for the fault $\Gamma$. More precisely, we fix an oriented curve of class $C^1$, or piecewise $C^1$, $\Gamma$ has no double points, and we fix parametric equations for $\Gamma$ $(x_1(v), x_2(v))$, $v \in [-1, 1]$, which defines a unit tangent vector to $\Gamma$. As commonly assumed, we take the unit normal vector $n$ to be indirectly perpendicular to the tangent vector, as in Figure 1. Next we, assume that $(x_1(0), x_2(0)) = (0, 0)$ and we define $\Gamma_d$ to be the curve obtained from $\Gamma$ by translation of vector $(0, -d)$ (see figure 3). We will assume that $d$ is large enough to ensure that $\Gamma_d$ is included in the half plane $x_2 < 0$.

We denote $\beta_0^d$ the corresponding first eigenvalue given by equation (8), where $\Gamma$ is replaced by $\Gamma_d$. Let $\Phi_0^d$ be a function satisfying (1),(6) with $\beta = \beta_0^d$ and such that $\int_{\Gamma_d} [\Phi_0^d]^2 = 1$. In this section we assume that the first eigenspace for the linear problem (1),(6), in the half plane ruptured by the fault $\Gamma_d$ is one dimensional. Similarly, we assume that the first eigenspace for the linear problem (1),(6), in the whole plane ruptured by the fault $\Gamma$ is one dimensional. We already denoted $\beta^\infty_0$ the corresponding eigenvalue. Let $\Phi_0^\infty$ be a function satisfying (1),(6) where we set $\beta = \beta_0^\infty$ and such that $\int_{\Gamma} [\Phi_0^\infty]^2 = 1$. 

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We first analyze regularity properties for the jumps of \( \Phi_d^0 \) and \( \Phi_0^\infty \) across the fault lines. We denote these jumps by
\[
\varphi_d := [\Phi_d^0], \quad \varphi_\infty := [\Phi_0^\infty].
\]
After a linear change of variables, \( \varphi_d \) can also be regarded as a function in the space \( \tilde{H}^{\frac{1}{2}}(\Gamma) \). We also assume that \( \varphi_d \geq 0 \) and \( \varphi_\infty \geq 0 \), which, as explained earlier, gives physical significance to the linear problem. This ensures that we are only considering solutions related to the the nonlinear eigenproblem (5).

The goal of this section is first to prove the convergence \( \beta_d^0 \rightarrow \beta_0^\infty \), and then assuming that the associated eigenspaces are one dimensional, prove convergence of scaled eigenvectors \( \varphi_d = [\Phi_d^0] \) associated to \( \beta_d^0 \), to a scaled eigenvector \( \varphi_\infty = [\Phi_0^\infty] \) associated to \( \beta_0^\infty \). As pointed out earlier, the dimension assumption on those eigenspaces does not hold in general. Nevertheless, we will prove in section 9.2 that if \( \Gamma \) is a line segment and \( d \) is large enough, this dimension assumption holds.

From the previous section
\[
\beta_0^d = \min_{\varphi \in \tilde{H}^{\frac{1}{2}}(\Gamma_d)} \frac{-\int_{\Gamma_d} \partial_{ny} \int_{\Gamma_d} \partial_{nx} G_d \varphi(v) d\sigma(v) \varphi(u) d\sigma(u)}{\int_{\Gamma_d} \varphi^2 d\sigma}. \tag{28}
\]
A simple change of variables allows us to deal only with integral operators on the fixed curve \( \Gamma \). The change of variables induces an integration kernel \( G_d \). With that change of variables identity (28) becomes
\[
\beta_0^d = \min_{\varphi \in \tilde{H}^{\frac{1}{2}}(\Gamma)} \frac{-\int_{\Gamma} \partial_{ny} \int_{\Gamma} \partial_{nx} G_d \varphi(v) d\sigma(v) \varphi(u) d\sigma(u)}{\int_{\Gamma} \varphi^2 d\sigma}. \tag{29}
\]
It will also prove convenient to adopt a simpler notation for the hypersingular operators of interest acting on \( \tilde{H}^{\frac{1}{2}}(\Gamma) \). We denote,
\[
G^{hyp}_d \varphi := -\partial_{ny} \int_{\Gamma} \partial_{nx} G_d \varphi(v) d\sigma(v)
\]
\[
G^{hyp}_\infty \varphi := -\partial_{ny} \int_{\Gamma} \partial_{nx} G \varphi(v) d\sigma(v)
\]
5.1 Asymptotic behavior of the first eigenvalue

Proposition 5.1. Let $\beta_0^d$ be the first eigenvalue for the linear problem (1),(6) in the half plane ruptured by the fault $\Gamma_d$, and $\beta_0^\infty$ the first eigenvalue for the linear problem (1),(6) in the whole plane ruptured by the fault $\Gamma$. Then there exists a constant $C$ depending only on the fixed curve $\Gamma$ such that

$$|\beta_0^d - \beta_0^\infty| \leq \frac{C}{d^2}. \quad (30)$$

Proof. Calculations show that $G_{hyp}^d - G_{hyp}^\infty$ is smooth and that, for all $t, v$ in $[-1,1]$, $G_{hyp}^d(x_1(v), x_2(v), y_1(t), y_2(t)) - G_{hyp}^\infty(x_1(v), x_2(v), y_1(t), y_2(t)) \leq \frac{C_2}{d^2}$. \( (31) \)

Let $\varphi_d$ achieve the minimum for defining $\beta_0^d$, that is $\|\varphi_d\|_{L^2(\Gamma)} = 1$ and

$$\beta_0^d = \int_\Gamma (G_{hyp}^d \varphi_d)(v) \varphi_d(v) d\sigma(v) = \min_{\|\varphi\|_{L^2(\Gamma)} = 1} \int_\Gamma (G_{hyp}^d \varphi)(v) \varphi(v) d\sigma(v). \quad (32)$$

Similarly, define $\varphi_\infty$ in $\dot H^{\frac{1}{2}}(\Gamma)$ such that $\|\varphi_\infty\|_{L^2(\Gamma)} = 1$ and

$$\beta_0^\infty = \int_\Gamma (G_{hyp}^\infty \varphi_\infty)(v) \varphi_\infty(v) d\sigma(v) = \min_{\|\varphi\|_{L^2(\Gamma)} = 1} \int_\Gamma (G_{hyp}^\infty \varphi)(v) \varphi(v) d\sigma(v). \quad (33)$$

According to estimate (31),

$$\beta_0^\infty + \frac{C_2}{d^2} |\Gamma| \geq \int_\Gamma (G_{hyp}^\infty \varphi_\infty)(v) \varphi_\infty(v) d\sigma(v) \geq \beta_0^d,$$

where $|\Gamma|$ is the arc length of $\Gamma$, and similarly

$$\beta_0^d + \frac{C_2}{d^2} |\Gamma| \geq \int_\Gamma (G_{hyp}^d \varphi_d)(v) \varphi_d(v) d\sigma(v) \geq \beta_0^\infty.$$

The last two estimates lead to the estimate for the first eigenvalues, (30).

We now present two numerical runs which illustrate the derived asymptotic behavior. Each of these two runs involve faults that are line segments of length 2. In the first run, the line segment is parallel to the observation surface, or in other words, the inclination angle $\theta$ is 0. In the second run, the inclination angle $\theta$ is $\pi/3$. The first eigenvalue $\beta_0^d$ was computed following the numerical scheme presented in appendix 9.1 for different values of the depth $d$. In Figure 4 we have plotted the remainder $|\beta_0^d - \beta_0^\infty|$ versus the depth $d$ in a $\log_{10}$ scale. The announced decay of order 2 given in (30) is clearly observed in each case. It is noteworthy that for $d = 1$, the numerical values within 3 decimals are $\beta_0^d = 1.627$ for the first fault ($\theta = 0$.) and $\beta_0^d = 1.567$ for the second fault ($\theta = \pi/3$) which are already somewhat close to $\beta_0^\infty = 1.158\ldots$. 


5.2 Asymptotic behavior of the first eigenfunction

We first analyze regularity properties for the functions $\varphi_d$ and $\varphi_\infty$. As

\[
G_{d}^{\text{hyp}} \varphi_d = \beta_d^0 \varphi_d,
\]

and

\[
G_{\infty}^{\text{hyp}} \varphi_\infty = \beta_0^\infty \varphi_\infty,
\]

the a priori estimates stated in Theorem 1.8 by Wendland et al. [21], ensure that $\varphi_d$ and $\varphi_\infty$ are in $C^1_2(\Gamma)$. Furthermore, the singularities of $\varphi_d$ and $\varphi_\infty$ at the tips of $\Gamma$ are exactly of square root type.

**Proposition 5.2.** There exists a constant $C$ depending only on the curve $\Gamma$ such that

\[
\max_{v \in [-1,1]} |\varphi_d(v) - \varphi_\infty(v)| \leq \frac{C}{d^2}. \tag{34}
\]

**Proof.** Using (31) and (30), we may write

\[
G_{\infty}^{\text{hyp}} \varphi_d = \beta_0^d \varphi_d + O\left(\frac{1}{d^2}\right) = \beta_0^\infty \varphi_d + O\left(\frac{1}{d^2}\right).
\]

Let $P$ the orthogonal projection onto the nullspace of $G_{\infty}^{\text{hyp}} - \beta_0^\infty I$. We have the following estimate

\[
(G_{\infty}^{\text{hyp}} - \beta_0^\infty I)(I - P)\varphi_d = O\left(\frac{1}{d^2}\right).
\]

Noticing that $(G_{\infty}^{\text{hyp}} - \beta_0^\infty I)^{-1}$ is continuous from the range of $(I - P)$ into $H^\frac{1}{2}(\Gamma)$, we derive

\[
(I - P)\varphi_d = O\left(\frac{1}{d^2}\right),
\]
in the $H^{\frac{1}{2}}(\Gamma)$ norm. Equivalently,

$$\varphi_d - \varphi_d, \varphi_\infty > \varphi_\infty = O\left(\frac{1}{d^2}\right),$$

(35)

thus, taking the dot product by $\varphi_d$,

$$< \varphi_d, \varphi_\infty >^2 = 1 + O\left(\frac{1}{d^2}\right)$$

As we chose $\varphi_d$ and $\varphi_\infty$ to be non negative, we infer,

$$< \varphi_d, \varphi_\infty > = 1 + O\left(\frac{1}{d^2}\right)$$

and plugging back into (35),

$$\varphi_d - \varphi_\infty = O\left(\frac{1}{d^2}\right),$$

(36)

in the $H^{\frac{1}{2}}(\Gamma)$ norm. As $H^{\frac{1}{2}}(\Gamma)$ is not included in $L^\infty(\Gamma)$, we need to do more work. We will use once again the a priori estimates from [21]. We notice that

$$G_{\infty}^{hyp}(\varphi_d - \varphi_\infty) = G_{d}^{hyp} \varphi_d - G_{\infty}^{hyp} \varphi_\infty + O\left(\frac{1}{d^2}\right)$$

$$= \beta_d^0 \varphi_d - \beta_\infty^0 \varphi_\infty + O\left(\frac{1}{d^2}\right) = O\left(\frac{1}{d^2}\right),$$

in the $H^{\frac{1}{2}}(\Gamma)$ norm. But here again, the a priori estimates of Theorem 1.8 of [21] show that we must have $\varphi_d - \varphi_\infty = O\left(\frac{1}{d^2}\right)$ in the sup norm.

Just as in the previous subsection, we carried out numerical computations of eigenvectors pertaining to the same two line segments faults of length 2. The first eigenfunction $\varphi_d$ was computed for different values of the depth $d$. Formula (34) is verified in Figure 5, where we have plotted the remainder $\max_{v \in [-1, 1]} |\varphi_d(v) - \varphi_\infty(v)|$ versus the the depth $d$ in a log10 scale. Here too, a decay of order 2 can be observed, just as expected.

It is interesting to see how different $\varphi_d$ appears for small values of $d$. We plotted in Figure 6, profiles for $\varphi_d$ for $d = 0.8$ for two rotation angles $\theta$. In one case $\theta = 0$, and the other case $\theta = 0.5$. Note that for $\theta = .5$, the distance from the fault to the surface is about 0.32, which is small compared to the length of the fault (2). The profiles for $\varphi_d$ and $\varphi_\infty$ still appear very similar.

### 5.3 Asymptotic behavior of the surface observation

In the remainder of the paper, we choose to normalize the eigenvectors $\varphi_d$ and $\varphi_\infty$ by setting

$$\max_{[-1, 1]} \varphi_d = \max_{[-1, 1]} \varphi_\infty = 1.$$ 

Define the surface dislocation function associated to the first eigenvector $\varphi_d$ as

$$\psi(\Gamma_d)(y) := \Phi_d^0(y, 0) = \int_{\Gamma_d} -\partial_{n_x} G(x_1(v), x_2(v), y, 0) \varphi_d(v) d\sigma(v).$$

(37)
**Fig. 5** – The remainder $\max_{v \in [-1,1]} |\varphi_d(v) - \varphi_\infty(v)|$ versus the depth $d$ in a log$_{10}$ scale for two line segment faults ($\theta = 0$ and $\theta = \pi/3$).

**Fig. 6** – Plot of the distribution of the normalized eigenfunctions $\varphi_d$ on the unit fault $\Gamma_d$ at small depth ($d = .8$), for two different values of rotation angle $\theta$ ($\theta = 0, \theta = .5$) compared to $\varphi_\infty$. 
Calculations show that
\[ \psi(\Gamma_d)(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{-n_1 y - n_2 d + x_1 n_1 + x_2 n_2}{(x_1 - y)^2 + (x_2 \cos \theta - d)^2} \varphi_d(v) d\sigma(v), \]
where \( x_1 \) and \( x_2 \) are short for \( x_1(v) \), \( x_2(v) \), the chosen parametric equation for \( \Gamma \), and \( n = (n_1, n_2) \) is the oriented unit normal vector at \( v \).

We are now able to prove the main asymptotic formula for this paper.

**Proposition 5.3.** The "observable surface" eigenfunction \( \psi(\Gamma_d) = \Phi^d(\cdot, 0) \) can be estimated as follows
\[ \psi(\Gamma_d)(y) = \Psi^d_N(y) + O\left( \frac{1}{|y| + d^2} \right), \quad (38) \]
where
\[ \Psi^d_N(y) := \frac{1}{\pi} \frac{(y, d) \cdot (n_1, n_2) + x_1 n_1 + x_2 n_2}{y^2 + d^2} \varphi_{\infty}(v) d\sigma(v). \quad (39) \]
and \( N = N(\Gamma) \) is the "normalized seismic moment" associated to the free space problem defined by
\[ N := \int_{-1}^{1} n(v) \varphi_{\infty}(v) d\sigma(v). \quad (40) \]

**Proof.** Recalling (34), we write
\[ \psi(\Gamma_d)(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{-(y, d) \cdot (n_1, n_2) + x_1 n_1 + x_2 n_2}{(x_1 - y)^2 + (x_1 - d)^2} \varphi_{\infty}(v) d\sigma(v) \]
\[ + O\left( \frac{1}{d^2(|y| + |d|)} \right). \]
We notice that
\[ \frac{1}{(x_1 - y)^2 + (x_2 - d)^2} = \frac{1}{y^2 + d^2} + \]
\[ + O\left( \frac{1}{(y^2 + d^2)^2} \right). \]
Finally, as
\[ -(y, d) \cdot (n_1, n_2) + x_1 n_1 + x_2 n_2 = O((y^2 + d^2)^{\frac{1}{2}}), \]
the asymptotic formula (38) follows.

In many instances, the curve \( \Gamma \) is symmetric about its midpoint \((x_1(0), x_2(0))\). This is true, for example, if \( \Gamma \) is a line segment, using a suitable parametrization. In those symmetric cases, the remainder in asymptotic formula (38) has a higher order.

**Proposition 5.4.** If \( \Gamma \) is symmetric about its midpoint \((x_1(0), x_2(0))\), then the expansion for the surface eigenfunction \( \psi(\Gamma_d) = \Phi^d(\cdot, 0) \) has a remainder of higher order, that is,
\[ \psi(\Gamma_d)(y) = \Psi^d_N(y) + O\left( \frac{1}{|y| + d^2} \right). \quad (41) \]

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Proof. We recall that $\Phi_0^\infty$ was denoted to be a eigenvector for the linear problem (1),(6) in the eigenspace attached to the first eigenvalue $\beta_0^\infty$. That eigenspace was assumed to be one dimensional in that section. We assume that the parametrization for $\Gamma$ satisfies

$$(x_1(-v), x_2(-v)) = (-x_1(v), -x_2(v)), \quad v \in [-1, 1].$$

By symmetry $\Phi_0^\infty(-x_1, -x_2)$ is also an eigenvector for the linear problem (1),(6) corresponding to the first eigenvalue $\beta_0^\infty$. As we made $\phi_\infty$ unique by setting $\max \phi_\infty = 1$ and as $\phi_\infty \geq 0$, we conclude

$$\phi_\infty(-v) = \phi_\infty(v), \quad v \in [-1, 1].$$

The normal vector satisfies

$$(n_1(-v), n_2(-v)) = (n_1(v), n_2(v)), \quad v \in [-1, 1],$$

so does the length element,

$$\sigma(-v) = \sigma(v), \quad v \in [-1, 1].$$

We then go back to expanding

$$\frac{1}{(x_1 - y)^2 + (x_2 - d)^2} = \frac{1}{y^2 + d^2} + \frac{2(y \cdot d \cdot (x_1, x_2)}{(y^2 + d^2)^2} + O\left(\frac{1}{(y^2 + d^2)^2}\right).$$

Finally, as by symmetry

$$\int_{-1}^{1} \frac{2(y \cdot d \cdot (x_1, x_2)}{(y^2 + d^2)^2} (y \cdot d \cdot (n_1, n_2) \phi_\infty(v) d\sigma(v) = 0,$$

one order of magnitude is gained in expanding $\psi(\Gamma_d)$.  

We verify on numerical runs the convergence of the surface dislocation function. As in the previous subsection, we carried out computations for the same two line segments faults of length 2 of rotation angle 0 and $\pi/3$. The observable part of the first eigenfunction $\psi(\Gamma_d) = \Phi_0^\infty(\cdot, 0)$ was computed for different values of the depth $d$. In Figure 7 we have plotted the remainder $\max_{y \in \mathbb{R}} |\psi(\Gamma_d)(y) - \Psi_N^d(y)|$ versus the depth $d$ in a log$_{10}$ scale. As announced in formula (41), the convergence in $d$ is in this case of order 3.
6 Fault recovery from surface measurements

In this section we add a horizontal translation parameter $a$ to the fault. In other words, the fault, thereafter denoted $\Gamma_{d,a}$ is defined by the parametric equations

$$\begin{align*}
(x_1(v) + a, x_2(v) - d), \quad v \in [-1, 1].
\end{align*}$$

(42)

As in the previous section we assume that the first eigenspace for the linear problem (1),(6), in the half plane ruptured by the fault $\Gamma_{d,a}$ is one dimensional. We remark that the first dislocation eigenvector $\varphi_d$ is independent of $a$, but the first eigenfunction, denoted by $\Phi_{0,a}^d$, is obtained from $\Phi_{0}^d$ by translation. In the following formula, homogeneity is achieved by imposing the condition $\max \varphi_d = 1$. The associated surface displacement $\Phi_{0,a}^d(\cdot, 0)$ will be denoted $\psi(\Gamma_{d,a}^a)$, whose expression is

$$\psi(\Gamma_{d,a}^a)(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{-n_1(y - a) - n_2d + x_1n_1 + x_2n_2}{(x_1 + a - y)^2 + (x_2 - d)^2} \varphi_d(v) \alpha d\sigma(v),$$

(43)

After recentering at $(a, 0)$ our main asymptotic formula (38) reads

$$\psi(\Gamma_{d,a}^a)(y) = \Psi_N^d(y - a) + O\left( \frac{1}{|y - a| + d^2} \right).$$

(44)

We assume in what follows that the normalized seismic moment $N$ defined in (40) is not zero. Notice that it may very well be that one of the two components of $N$ is 0.

The aim of this section is to recover the fault $\Gamma_{d,a}$ from a surface observation through the use of the above asymptotic formula. Denote $\omega : \mathbb{R} \rightarrow \mathbb{R}$ such an observation. We
assume that this observation belongs to $H_{obs}$, i.e. there corresponds a dislocation $m\varphi_d$ along the fault $\Gamma_d^a$ associated to the first eigenvalue $\beta_0^d$ with the amplitude $m$. We denote now by $M := mN(\Gamma)$, "the (dimensional) seismic moment", given by

$$M := m \int_{-1}^{1} n(v) \varphi_\infty(v) d\sigma(v), \quad (45)$$

which corresponds to a dislocation $m\varphi_\infty$ on $\Gamma$ in the free space having the amplitude $m$. According to the asymptotic formula (44) we have

$$\omega(y) \approx \Psi^d_M(y - a). \quad (46)$$

Following this approximation, recovering $\Gamma_d^a$ consists of finding the depth $d > 0$, the horizontal location parameter $a \in \mathbb{R}$ and the seismic moment $M \in \mathbb{R}^2$. Note that among those three parameters, the only one conveying some information about the shape of $\Gamma$ is $M$.

A least square minimization can be used to recover the above physical parameters from a set of discrete observation points. For this purpose, let $y_1, y_2, \ldots, y_p$ be the observation points and let $\omega_i = \omega(y_i), i = 1, \ldots, p$ be the observations. The unknown parameters $d, a$ and $M$ can be estimated by the minimum of

$$J(d, a, M) := \sum_{i=1}^{p} [\omega_i - \Psi^d_M(y_i - a)]^2. \quad (47)$$

If a continuous set of observation $\omega$ is available, it is possible to take advantage of exact inversion formulas based on moments of $\omega$. Set $i_1, i_2, i_3, i_4$ to be the following four integrals computed from the observation $\psi$,

$$
\begin{align*}
  i_1 &:= \int_{-\infty}^{\infty} \omega(y) + \omega(-y) \, dy, & \quad i_2 &:= \int_{-\infty}^{\infty} \omega^2(y) \, dy, \\
  i_3 &:= \int_{-\infty}^{\infty} \omega^3(y) \, dy, & \quad i_4 &:= \int_{-\infty}^{\infty} \omega^3(y) y \, dy
\end{align*}
$$

If we replace $\psi$ from the above integrals by its approximation (46), after some algebra, we get the following nonlinear system for $d, a$ and $M = (M_1, M_2)$:

$$
\begin{align*}
  \begin{cases}
    -2M_2 = i_1 \\
    M_1^2 + M_2^2 = i_2 \\
    -3M_2(M_1^2 + M_2^2) = i_3 \\
    -\frac{3}{8\pi^2 d^2}(M_1^4 + M_2^4)(aM_2 + dM_1) = i_4
  \end{cases}
\end{align*}

(48)

We find $M_2 = -\frac{i_1}{2}$. Two cases arise. In the first case $M_2 \neq 0$. Then $d = -\frac{3i_2 M_2}{4\pi i_3}$, and

$$M_1^2 = 2\pi i_2 d - M_2^2.$$ 

This leads to two possible values for $M_1$. But

$$\int_{-\infty}^{\infty} \omega(y) + \frac{M_1 y}{\pi(y^2 + 1)} \, dy = -M_2, \quad (49)$$
and
\[ \int_{-\infty}^{\infty} \omega(y) - \frac{M_1 y}{\pi (y^2 + 1)} \, dy \] (50)
diverges, this way \( M_1 \) is fully determined. From the last equation we find
\[ a = -\frac{1}{M_2} (\frac{8\pi^2 d^2}{3(M_1^2 + M_2^2)} i_4 + dM_1). \]

In the second case \( M_2 = 0 \). We set in that case
\[ i_5 := \int_{-\infty}^{\infty} \omega^4(y) \, dy, \quad i_6 := \int_{-\infty}^{\infty} y^2 \omega^5(y) \, dy, \]
and replacing \( \psi \) through (46) after some algebra we get
\[ i_5 = \frac{M_1^4}{16\pi^3 d^3}, \quad i_6 = \frac{-5M_1^5 a}{64\pi^4 d^3}. \]

We obtain
\[ d = \frac{i_2^2}{4\pi i_5}, \quad M_1^2 = \frac{i_2^3}{2i_5}, \]
as in the previous case, the sign of \( M_1 \) is given by examining the two integrals (49) and (50). Finally \( a \) is given by
\[ a = -\frac{64\pi^4 d^3}{5M_1^5}. \]

### 7 Numerical inversion for the line segment fault

In this section we assume that the fixed geometry \( \Gamma \) is the line segment \([-1, 1] \times \{0\}\). In addition to a translation of vector \((0, -d)\), we apply a rotation of angle \( \theta \) to \( \Gamma \), to obtain the line segment \( \Gamma_{d, \theta} \) (see figure 8)
\[ (\cos \theta v, \sin \theta v - d), \quad v \in [-1, 1]. \] (51)

We are also careful to choose \( d \) large enough for a given \( \theta \) in order to have \( \Gamma_{d, \theta} \) included in the half plane \( x_2 < 0 \). We denote in the rest of the paper by \( \beta_{0}^{d, \theta} \) the first eigenvalue for the linear problem (1),(6), and by \( E_{d, \theta} \) the first eigenspace. For a line segment fault the assumptions of the previous section concerning the the dimension of \( E_{d, \theta} \) turns out to be always true. Indeed, as it follows from section 9.1 for \( d \) large enough (\( d > 1.274474... \)) \( E_{d, \theta} \) is one dimensional for all \( \theta \). Moreover since the dislocation \( \varphi_{d, \theta} = [\Phi_{0}^{d, \theta}] \) associated to the eigenfunction \( \Phi_{0}^{d, \theta} \) has a constant signum on \( \Gamma_{d, \theta} \), one can deduce that \((\beta_{0}^{d, \theta}, \Phi_{0}^{d, \theta})\) is a solution to the nonlinear Rayleigh problem (5).

Most of the asymptotic formulas and moment formulas previously derived can be significantly simplified in case of a line segment fault. In particular, the normalized seismic moment vector \( N \) simplifies as,
\[ N = (\sin \theta, -\cos \theta) \int_{-1}^{1} \varphi_\infty(v) \, dv, \] (52)
thus the vector $N$ gives the angle $\theta$ between the fault line and the horizontal line $x_2 = 0$. Notice that $\varphi_\infty$ simply refers here to the scaled first eigenvector for the hypersingular integral operator $G^{\text{hyp}}_\infty$, operating on the line segment $[-1,1] \times \{0\}$.

Another advantage to studying line segment faults, is that we were able to efficiently compute the eigenvalues and eigenvectors $\beta_{0,d,\theta}^{d,\theta}, \varphi_{d,\theta}$ (see section 9.2 for a description of the numerical scheme). That facilitated the numerical verification of our asymptotic theory and the numerical implementation of the recovery formulas presented in the previous section.

Let $\Gamma_{a,d,\theta}$ be the fault $\Gamma_{d,\theta}$ with a horizontal translation of parameter $a$

$$(\cos \theta v + a, \sin \theta v - d), \quad v \in [-1,1].$$

Since the first dislocation eigenvector $\varphi_{d,\theta}$ is independent of $a$, we denoted by $\Phi_{0,a}^{d,\theta}$ the translation of $\Phi_{0,0}^{d,\theta}$.

Recovering the fault $\Gamma_{a,d,\theta}$ from a surface observation $\omega$ corresponding to a dislocation $m \varphi_{d,\theta}$ along the fault $\Gamma_{d,\theta}$ associated to the first eigenvalue $\beta_{0,d}^d$ with the amplitude $m$, consists of finding the depth $d > 0$, the horizontal location $a \in \mathbb{R}$, the rotation angle $\theta \in [0, \pi)$ and the amplitude $m$. At this stage we ignore issues of sensitivity of measuring devices and number of measurement points. These will be considered later in this paper. In this section we only give numerical examples of reconstruction of $(d, \theta, a, m)$ from $\omega$ through the approximation (57). To do this, let $y_1, y_2, \ldots, y_p$ be the observation points and let $\omega_i, i = 1, \ldots, p$ be the observations.

The first method for recovering the fault $\Gamma_{a,d,\theta}$ that comes to mind relies on least square minimization. The unknown parameters $d, a, \theta$ and $m$ are found by minimizing

$$U(d, a, \theta, m) := \sum_{i=1}^p [\omega_i - m \Phi_{0,a}^{d,\theta}(y_i, 0)]^2.$$  

This method is sharp but computationally very expensive. We cannot fathom applying this method in three dimensional real life problems. Indeed, computational time for
earthquake nucleation could be much longer than nucleation time. That would kill the purpose of issuing short time warnings for seismic events.

We thus propose another inversion method, this one based on the asymptotic estimate (38), in an effort to reduce computational time. As the line segment $\Gamma$ is symmetric about its midpoint, the remainder in the asymptotic formula (38) decays as $d^{-3}$, as shown earlier. After recentering at $(a,0)$ our main asymptotic formula (41) for the surface observation reads

$$
\psi(\Gamma_{a,d,\theta}(y) = \Phi_{0,a}^{d,\theta}(y,0) = \Psi_{\theta}^{d}(y-a) + O\left(\frac{1}{|y-a|+d^{3}}\right),
$$

(55)

where the leading term $\Psi_{\theta}^{d}$ is obtained from (39) and (52)

$$
\Psi_{\theta}^{d}(y) := -\frac{y \sin \theta - d \cos \theta }{\pi(y^{2}+d^{2})} \int_{-1}^{1} \varphi_{\infty}(v)dv.
$$

(56)

We will make use of the asymptotic approximation

$$
\omega(y) \approx m \Psi_{\theta}^{d}(y-a)
$$

(57)

for recovering a fault $\Gamma_{a,d,\theta}$ from a surface observation $\omega : \mathbb{R} \to \mathbb{R}$ corresponding to a dislocation $m \varphi_{d,\theta}$ associated to the first eigenvalue $\beta_{0}^{d}$ and the amplitude $m$.

We now show numerical runs illustrating the use of this asymptotic approximation. In each run, we first created data by solving the direct problem. To do so, we picked values for $d, a$ and $\theta$, and we computed the first eigenfunction $\Phi_{0,a}^{d,\theta}$ following the numerical method outlined in section 9.2. We then picked a sampling interval $[-y_{\text{max}}, y_{\text{max}}]$ and a stepsize $h$. This way we defined $p = \lceil 2y_{\text{max}}/h \rceil + 1$ uniformly distributed observation points $y_{i}, i = 1, ..., p$. We picked an amplitude $m$ and we computed the observation data as $\omega_{i} = m \Phi_{0,a}^{d,\theta}(y_{i},0)$. We possibly blurred our data by adding random noise: this is for assessing robustness. We simply modeled noise by multiplying $m \Phi_{0,a}^{d,\theta}(y_{i},0)$ by $(1 + r/10)$ where $r$ is picked from a uniform distribution of numbers between -1 and 1. Each line in table 1 contains corresponds to chosen values for $h, y_{\text{max}}$, and chosen noise level. Columns 2 through 5 of Table 1 indicate set values for $d, \theta, a, m$ that were used to produce data.

The first inversion method illustrated in Table 1 relies on formulas for moments of $\omega$. We used the trapezoidal rule to evaluate numerically the integrals in the expressions for $i_{1}, ..., i_{6}$ from the observation data $\psi_{i}$. After that, we have computed directly $d, a$ and $M$ as solutions to the system (48). The rotation angle $\theta$ for the line segment that models the fault was then determined by $\tan \theta = -\frac{M_{1}}{M_{2}}$. If $\tan \theta$ is very large, it is then preferable to assume that $M_{2}$ is very close to 0, and then to follow formulas given in the relevant case. Columns 6 through 9 of Table 1 carry computed values for those parameters. We notice on lines 3, 6, and 9 from Table 1 that the recovery of $(d, \theta, a, m)$ based on the observation momentum only, can be very poor for $d = 50$ with the choice $y_{\text{max}} = 100$. This is easily explained by the fact that the truncation of $[-\infty, \infty]$ by $[-y_{\text{max}}, y_{\text{max}}]$ is
unsatisfactory in that case. Line 10 shows improved results for $y_{\max} = 200$.

The first inversion method illustrated in Table 1 relies on least square minimization of the quantity (47). At each step of the search, only the function $\Psi_d^\theta$ as defined in formula (56) was evaluated at the observations points, which is computationally inexpensive. We used the values of $d, a, m$ and $\theta$ obtained above as starting points. Columns 10 through 13 of Table 1 contain the output values after least square minimization.

We show a graphic for Line 8 of Table 1 in Figure 9. The graph of the noisy surface observation $\omega_i$ is plotted as '+' points, the graph of $m\Psi_d^\theta$ with $(d, \theta, a, m)$ computed following the momentum method, is plotted in a solid line, and the graph of $m\Psi_d^\theta$ with $(d, \theta, a, m)$ obtained after least square minimization, is plotted as 'o' points.

Finally, we report that even for very small values of the distance from the unit length line segment $\Gamma$ to the surface $\Gamma_{\text{obs}}$, the asymptotic formula (41) is still effective for solving the inverse problem. For example, we tested our method for the values $d = .58, \theta = \frac{\pi}{6}, a = 5, m = 2$. For $y_{\max} = 30, h = 2$, the reconstruction formulas based on the momentum of the observation lead to $d = .981, \theta = 4.90, a = 4.27, m = 1.75$; using the least square minimization with the asymptotic approximation (47) lead to $d = 0.604, \theta = 5.62, a = 5.05, m = 1.94$; finally, a least square minimization process (54) involving at each step the solution to the eigenvalue problem yielded $d = .5800, \theta =$. 

<table>
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<th>$d$</th>
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<th>$m$</th>
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<td>1.24</td>
<td>-45.9</td>
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Tab. 1 — Recovery of $(d, \theta, a, m)$ based on our integral formulas and asymptotic formula.
5.24, $a = 5.00, m = 2.00$, all of which have 3 correct decimals. The plots of the corresponding surface function and reconstructed surface function appear in Figure 10, where we have used the same legend as that for Figure 9 and we have added the graph of $m\Phi_{0,a}(y,0)$ with $(d, \theta, a, m)$ obtained after the least square minimization (54), plotted with of '□' points. Note the very good agreement between the last two methods. Thus, for faults near the surface, although asymptotic formula (41) is not expected to hold, the inversion method based on the least square minimization (47) still gives very good results while requiring a very small computational effort.

To close this section, we report that measurement on the surface points need not form a regular grid. The trapezoidal rule is still efficient on non uniform grids. We obtained numerical values, using non uniform grids, that were just as good, provided the spacing between grid points be small enough.

8 Detecting active faults from GPS observations

This section attempts to model real life situations. Under some threshold, surface displacements cannot be measured. Moreover, only a limited number of measurement points, on a fixed, irregular grid, can be used, regardless of the geometry and the size of the fault, and regardless of seismic intensity. The aim of this section is to account for the sensitivity of measuring apparatus and for the observation grid stepsize, and to understand how those may affect the detection of active faults.
8.1 Admissible stepsize of the observation grid

The goal of this subsection is to give a precise meaning to the notion of sensitivity of measuring devices and to discuss how, due to rescaling properties, observability should not depend on depth at fixed sensitivity, at least at the first order.

At fixed sensitivity $S$, a point $y$ on the grid of measurement points is indicative if

$$|m\psi(\Gamma^a_{d,\theta})(y)| \geq S,$$

where $\psi(\Gamma^a_{d,\theta})$ is the surface displacement generated by the first normalized eigenvector on the fault $\Gamma^a_{d,\theta}$, i.e. $\psi(\Gamma^a_{d,\theta})(y) = \Phi_{0,0}(y,0)$. Accordingly, at fixed sensitivity $S$, a fixed fault $\Gamma^a_{d,\theta}$ and fixed seismic amplitude $m$, we define the indicative set $I = I(S, d, a, \theta, m)$ to be

$$I := \{y \in \mathbb{R} : |m\psi(\Gamma^a_{d,\theta})(y)| \geq S\}.$$

The maximum grid stepsize $\text{Step}_{\text{max}} = \text{Step}_{\text{max}}(S, d, \theta, m)$ is defined to be the maximum length of all intervals included in the indicative set $I$. Accordingly, if an observation grid has a stepsize $\Delta$ greater than $\text{Step}_{\text{max}}$ then the indicative set $I$ may contain no observable points: in that case the active fault $\Gamma^a_{d,\theta}$ can not be detected. We illustrate the definition of $\text{Step}_{\text{max}}$ on a numerical example in Figure 11. We have plotted the distribution of the surface displacement $y \rightarrow m\psi(\Gamma^a_{d,\theta})(y)$ and the indicative set which in this case is an interval of length $\text{Step}_{\text{max}}$.

Since the displacement $\psi(\Gamma^a_{d,\theta})(y)$ is vanishing for $|y| \to \infty$ all the intervals included in $I$ are bounded: $\text{Step}_{\text{max}}$ is well defined. Moreover, we remark that $\text{Step}_{\text{max}}$ does not depend on the translation parameter $a$ and

$$\text{Step}_{\text{max}}(S, d, \theta, m) = \text{Step}_{\text{max}}(S/m, d, \theta, 1).$$
We now make the following observation: as \( d \) grows large, the surface displacement \( m\psi(\Gamma_{a,d,\theta}) \) decays as \( O(\frac{1}{d}) \). However, it is also rescaled in the \( y \) direction by a factor of \( O(\frac{1}{d}) \). Therefore, at fixed sensitivity \( S \), the maximum grid stepsize \( \text{Step}_{\text{max}} \) should be close to being constant for a whole range of depth \( d \). We put this idea to a test, and in Figure 12, we plotted \( \text{Step}_{\text{max}} \) against depth \( d \) for different values of \( \theta \) and for \( m = 1 \), for the fixed sensitivity \( S = 0.005 \). As expected we remark that \( \text{Step}_{\text{max}} \) belongs to the interval (20., 130.) for almost all depth \( d \) and all rotations angle \( \theta \).

### 8.2 Fault recovery at fixed observation sensitivity

The aim of this subsection is to evaluate how the sensitivity of observations affect our inversion techniques. We know from the previous subsection that sensitivity is related to the stepsize of the grid. Understanding this relation will prove useful in the numerical simulations that follow. The geometric parameters \( a, d, \theta \) may vary within a wide range. We picked a set of three possible faults \( \Gamma_{a,d,\theta}^a \) for our numerical simulations, representing different depths, orientations and horizontal translations. These three faults were respectively characterized by \( d = 2, \theta = \pi/2, a = 5 \), \( d = 10, \theta = \pi/6, a = 5 \), and \( d = 50, \theta = \pi/10, a = 15 \). In either case, the sensitivity \( S \) was fixed and equal to .003 and the amplitude was fixed at \( m = 1 \). We then chose the width of the sampling interval \([-y_{\text{max}}, y_{\text{max}}]\) to be \( y_{\text{max}} = 100 \). We denoted \( y_i \in [-y_{\text{max}}, y_{\text{max}}] \), \( i = 1, \ldots, p \) be the observation points. We computed the observation data \( \omega_i \) by numerically solving for displacements along active faults (i.e. \( \omega_i = m\Phi_{\theta,a}(y_i,0) \)), and possibly blurring the computed values by noise.
Let $K$ be the set of observations in the measurable range

$$K := \{i ; |\omega_i| \geq S\}.$$

In the numerical runs for this section, we retained only those observations $(\omega_i)_{i \in K}$ in the indicative set $I$.

In a first run, we used a grid of uniformly distributed observations points $y_i, i = 1, \ldots, p$, with stepsize $h = 8$. Data was free of noise in this first run. Recovered values after inversion appear in Table 2 and corresponding graphs are plotted in figures 13, 14 and 15. Only those points plotted as squares in Figures 13, 14, 15 were used in the inversion procedure: they are on the grid of observation points and the value of the surface displacement at those points is in the measurable range.

The first tested inversion method involves computing the momentum integrals $i_1, \ldots, i_6$ from observation data. Columns 6 through 9 of Table 2 show computed values of the parameters and the dashed line in figures 13, 14, 15 stands for the graph of $m \Psi^d_\theta$ with $(d, \theta, a, m)$ computed following the momentum method.

The second method that we tested is the least square minimization (47). Columns 10 through 13 of Table 1 contain the output values for that method and the solid line in figures 13, 14, 15 represents the graph of $m \Psi^d_\theta$ with $(d, \theta, a, m)$.

We noticed that increasing the stepsize for the grid of observation points might lead us to “skip” shallow faults, and having a higher sensitivity threshold might make us miss deeper faults.

Finally, we tried to assess the effects of noise and of having an irregular grid of observation points. We produced a random grid of observation points by fixing a length $l$. The distance between two consecutive points was taken to be $l(1 + \frac{1}{4}r)$, where $r$ is a random number between -1 and 1 from a uniform distribution. We define the stepsize
parameters | \( d \) | \( \theta \) | \( a \) | \( m \) | \( d \) | \( \theta \) | \( a \) | \( m \) | \( d \) | \( \theta \) | \( a \) | \( m \)  
--- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | ---  
Figure 13 | 2 | \( \frac{\pi}{2} \) | 5 | -1 | 2.24 | -1.38 | 1.27 | 1.11 | 2.01 | -1.56 | 4.97 | .999  
Figure 14 | 10 | \( \frac{\pi}{6} \) | 5 | 1 | 11.5 | 0 | -.51 | 1.03 | 10.0 | .525 | 5.03 | 1.00  
Figure 15 | 50 | \( \frac{\pi}{10} \) | 15 | 1 | 50.1 | 0 | 3.03 | .886 | 50.0 | .314 | 15.0 | 1.00

**Tab. 2** – Recovery of \((d, \theta, a, m)\) for \( S = 0.003 \), a stepsize equal to 8, \( y_{\text{max}} = 100 \).

**Fig. 13** – Recovery of a fault for a fixed grid of observation points at a fixed sensitivity threshold.

**Fig. 14** – Recovery of a fault for a fixed grid of observation points at a fixed sensitivity threshold.
\[ \begin{array}{cccccc|cccccc}
\Delta=11.5 & 2 & \frac{\pi}{5} & 5 & -1 & 4.12 & .99 & 7.82 & -1.42 & 1.75 & 1.78 & 4.31 & -1.02 \\
\Delta=10.3 & 10 & \frac{\pi}{5} & 5 & 1 & 10.8 & 0 & .691 & .98 & 9.82 & .50 & 4.67 & 9.74 \\
\Delta=11.3 & 50 & \frac{\pi}{10} & 15 & 1 & 48.0 & 0 & 3.55 & .807 & 47.4 & 209 & 11.1 & .896 
\end{array} \]

Tab. 3 – Recovery of \((d, \theta, a, m)\) for \(S = 0.003\), for noisy data, \(y_{\max} = 100\), and randomly perturbed observation grid.

\[ \Delta := \max_{1 \leq i \leq p-1} (y_{i+1} - y_i). \]

Next, each measurement of surface displacement \(\omega_i\) was perturbed by adding at random \(\pm 10\%\) of the sensitivity \(S\). We present in Table 3 numerical values for the recovery of \(d, \theta, a, m\), for the three geometries already considered in Table 2. The first column contains the actual value for \(\Delta\). The value of the sensitivity \(S\), was the same, that is .003. These values show robustness of the reconstruction method to random noise perturbations.

In conclusion, these numerical simulations showed that if the seismic amplitude \(m\) is fixed and equal to 1 in absolute value, if the sensitivity \(S\) is fixed at .003, if the stepsize is equal to 10, and if \([-100, 100]\) is the interval where the observation points lie, then any fault segment line can be recovered from the displacement values at the observation points, provided that \(2 \leq d \leq 60\), and \(-30 \leq a \leq 30\).

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References


9 Appendix

9.1 The line segment fault analysis

We will assume in this section that $\Gamma$ is the line segment $[-1, 1] \times \{0\}$. We denote $V$ the closure of smooth compactly supported in $\mathbb{R}^2 \setminus \Gamma$ functions for the norm

$$\|v\| = \sqrt{\int_{\mathbb{R}^2} |\nabla v|^2 dx}.$$  

We will denote $E_\infty$, the eigenspace of functions for the first eigenvalue $\beta_0^\infty$ for the linear problem (1),(6), and $\tilde{E}_\infty$ the related eigenspace for the first eigenvalue for the operator $G_{hyp}^\infty$ on $H^{1/2}(\Gamma)$.

**Lemma 9.1. All functions in $E_\infty$ are odd in the second variable.**
Proof. If \( u \) is in \( E_\infty \), define \( v \) by setting
\[
v(x_1, x_2) = u(x_1, -x_2).
\]
Set \( w = v + u \). \( w \) satisfies \( \Delta w = 0 \), in \( \mathbb{R}^2 \setminus \Gamma \), \( \partial_n w = 0 \), on \( \Gamma \), \([\partial_n w] = 0 \) across \( \Gamma \), and \([w] = 0 \) across \( \Gamma \). As \( w \) is in the functional space \( V \), we conclude that \( w = 0 \).

By symmetry, we may now examine a simpler problem in the upper half space only. Denote
\[
\Pi^+ = \{(x_1, x_2) : x_2 > 0\},
\]
\( W^+ = \{u \in H(\Pi^+) : u(x) = 0 \text{ if } |x| \geq R, \text{ and } u(x_1, 0) = 0 \text{ if } |x_1| > 1\}, \) and \( W^+ \) the completion of \( W^+ \) under the norm
\[
\sqrt{\int_{\Pi^+} |\nabla u|^2}.
\]
It is clear that by symmetry \( \beta_0^\infty \) is also the minimum of the Rayleigh quotient
\[
\beta_0^\infty = \min_{u \in W^+, u \neq 0} \frac{\int_{\Pi^+} |\nabla u|^2}{4 \int_\Gamma |u|^2}. \tag{58}
\]
The actual minimum is achieved by the restriction to \( \Pi^+ \) of a non zero function in \( E_\infty \). This new Rayleigh quotient for defining \( \beta_0^\infty \) proves useful for showing that \( E_\infty \) is one dimensional.

**Lemma 9.2.** Let \( v \) in \( W^+ \) be a minimizer for the Rayleigh quotient
\[
\frac{\int_{\Pi^+} |\nabla u|^2}{4 \int_\Gamma |u|^2}. \tag{59}
\]
The sign of \( v \) is constant in \( \Pi^+ \).

**Proof.** This proof follows the classical theory found in many PDE’s textbooks. We decompose a minimizer \( v \), in its positive and negative parts \( v = v^+ - v^- \). Assume that neither \( v^+ \) nor \( v^- \) is uniformly zero. As
\[
\beta_0^\infty = \frac{\int_{\Pi^+} |\nabla v^+|^2 + |\nabla v^-|^2}{4 \int_\Gamma |v^+|^2 + |v^-|^2}, \quad \frac{\int_{\Pi^+} |\nabla v^+|^2}{4 \int_\Gamma |v^+|^2} \geq \beta_0^\infty, \quad \frac{\int_{\Pi^+} |\nabla v^-|^2}{4 \int_\Gamma |v^-|^2} \geq \beta_0^\infty \tag{60}
\]
\( v^+ \) and \( v^- \) are also minimizers for (59). Recall the \( C^{1,2} \) regularity on \( \Gamma \) for jumps of eigenvectors for the linear problem (1),(6). By symmetry, it follows that \( v^+ \) and \( v^- \) are of class \( C^{1,2} \) on \( \Gamma \). On the portion of \( \Gamma \) where \( v^+ \) is zero, as \( v^+ \) satisfies \( 2\beta_0^\infty v^+ = \partial_n v^+ \), we have \( v^+ = \partial_n v^+ = 0 \). As \( \Delta v^+ = 0 \), \( v^+ \) must be uniformly null, which is a contradiction.

It now follows that the space of minimizers for (59) is one dimensional, and due to lemma 9.1 we get the following result.
Proposition 9.1. The solution $\Phi_0^\infty \in V$ to the linear Rayleigh problem (8) in free space ($D = \mathbb{R}^2$) is unique, i.e. $E_\infty = \text{Sp}\{\Phi_0^\infty\}$ is a one dimensional vector space. Consequently, $E_\infty$ is also one dimensional. Moreover $\Phi_0^\infty$ belongs to the space $V_+$ and $\Phi_0^\infty$ is the solution of the nonlinear Rayleigh problem (5).

If the fixed curve $\Gamma$ is the line segment $[-1, 1] \times \{0\}$, in addition to a translation of vector $(0, -d)$, we apply a rotation of angle $\theta$ to $\Gamma$, to obtain the line segment $\Gamma_{d,\theta}$ (see figure 8). We are also careful to choose $d$ large enough for a given $\theta$ in order to have $\Gamma_{d,\theta}$ included in the half plane $x_2 < 0$. We denote in the remainder of this section $\beta_{d,\theta}^0$ the first eigenvalue for the linear problem (1),(6), and by $\widetilde{E}_{d,\theta}$ the first eigenspace for the operator $G_{d,\theta}^{\text{hyp}}$ on $H^2(\Gamma)$. Due to the strong convergence $G_{d,\theta}^{\text{hyp}} - G_\infty^{\text{hyp}} \to 0$, it is clear that $\widetilde{E}_{d,\theta}$ is also a one dimensional space for $d$ large enough. In fact, it is possible to estimate a depth $d_0$ such that for all $d > d_0$, $\widetilde{E}_{d,\theta}$ is one dimensional. We propose to briefly outline how that can be done.

Just like in the proof of Proposition 5.2, we denote $P$ the orthogonal projection onto the nullspace of $G_\infty^{\text{hyp}} - \beta_0^\infty I$. ($G_\infty^{\text{hyp}} - \beta_0^\infty I)^{-1}$ is continuous from the range of $(I - P)$ into $H^2(\Gamma)$. Let $A$ be the norm of that operator. If we denote $\beta_1^\infty$ the second eigenvalue of $G_\infty^{\text{hyp}}$, $A$ is equal to $(\beta_1^\infty - \beta_0^\infty)^{-1}$. Dascalu et al. estimated $\beta_1^\infty$ and $\beta_0^\infty$ in [4]. As $\beta_1^\infty = 2.75475474\ldots$ and $\beta_0^\infty = 1.15777388\ldots$, we find

$$A = .626181581\ldots \quad (61)$$

Assume now that $\|G_{d,\theta}^{\text{hyp}} - G_\infty^{\text{hyp}}\| \leq B$, in the $L^\infty([-1, 1]^2)$ norm. Following the proof exposed in the proof of Proposition 5.1,

$$\beta_0^d = < G_{d,\theta}^{\text{hyp}} \varphi_d, \varphi_d > = < G_{d,\infty}^{\text{hyp}} \varphi_d, \varphi_d > + < (G_{d,\theta}^{\text{hyp}} - G_{d,\infty}^{\text{hyp}}) \varphi_d, \varphi_d > \leq \beta_0^\infty + B,$$

and as similarly,

$$\beta_\infty^d \leq \beta_0^d + B,$$

we conclude

$$|\beta_0^d - \beta_0^\infty| \leq B. \quad (62)$$

We now estimate the distance between any vector $\varphi$ in $\widetilde{E}_{d,\theta}$, of $L^2$ norm 1, and its analog $\varphi_\infty$ in $\widetilde{E}_\infty$. As

$$G_{d,\infty}^{\text{hyp}} \varphi = G_{d,\theta}^{\text{hyp}} \varphi + (G_{d,\infty}^{\text{hyp}} - G_{d,\theta}^{\text{hyp}}) \varphi,$$

we derive

$$\|G_{d,\infty}^{\text{hyp}} \varphi - \beta_0^d \varphi\| \leq B$$

and due to (62),

$$\|G_{d,\infty}^{\text{hyp}} \varphi - \beta_0^\infty \varphi\| \leq 2B.$$
or
\[ \| (G_{d,\infty}^{hyp} \varphi - \beta_0^\infty)(I - P)\varphi \| \leq 2B. \]

This in turn implies
\[ \| (I - P)\varphi \| \leq 2AB, \]
or
\[ 1 - 4A^2B^2 \leq < \varphi, \varphi_{\infty} >^2 \leq 1. \] (63)

By possibly changing \( \varphi \) into \(-\varphi\), we infer,
\[ \sqrt{1 - 4A^2B^2} \leq < \varphi, \varphi_{\infty} > \leq 1. \] (64)

Assume that \( \tilde{E}_{d,\theta} \) is at least two dimensional. Pick \( \varphi_1 \) and \( \varphi_2 \) in \( \tilde{E}_{d,\theta} \) of \( L^2 \) norm 1 satisfying (64), such that \( < \varphi_1, \varphi_2 > = 0 \). Then
\[ \sqrt{2} = \| \varphi_1 - \varphi_2 \| \leq \| \varphi_1 - \varphi_{\infty} \| + \| \varphi_1 - \varphi_{\infty} \|, \] (65)

but (63) implies
\[ \| \varphi - \varphi_{\infty} \| \leq 2\sqrt{2(1 - \sqrt{1 - 4A^2B^2})} \] (66)

and after combining (65) and (66),
\[ \frac{\sqrt{7}}{8} \leq AB. \] (67)

We now need to estimate the constant \( B \). A calculation shows that
\[ (G_{d,\theta}^{hyp} - G_{\infty}^{hyp})(t, v) = \frac{-1}{2\pi} \frac{(v - t)^2 \cos^2 \theta - 4(d - v \sin \theta)^2}{(v - t)^2 \cos^2 \theta + 4(d - v \sin \theta)^2}; \]
thus the supremum \( B \) of \( (G_{d,\theta}^{hyp} - G_{\infty}^{hyp}) \) is estimated as follows,
\[ B \leq \frac{1}{8\pi (d - 1)^2}. \] (68)

Combining (61, 67, 68), we find that for \( \tilde{E}_{d,\theta} \) to be more than one dimensional, the inequality
\[ \frac{\sqrt{7}}{8A} \leq \frac{1}{8(d - 1)^2\pi}, \] (69)
have to be satisfied. We have shown,

**Proposition 9.2.** If the distance \( d \) from the center of the line segment \( \Gamma_{d,\theta} \) is greater than \( \sqrt{\frac{A}{\sqrt{7}\pi}} + 1 \), which is about 1.274474..., then the first eigenspace \( \tilde{E}_{d,\theta} \) is one dimensional, for any rotation angle \( \theta \).

Since \( \tilde{E}_{d,\theta} \) is one dimensional we can denote now by \( \Phi_{0}^{d,\theta} \) and by \( \varphi_{d,\theta} = [\Phi_{0}^{d,\theta}] \) the first eigenfunction on \( \Omega \) and on the fault \( \Gamma \) respectively, i.e. \( \tilde{E}_{d,\theta} = Sp\{ \varphi_{d,\theta} \} \) and \( E_{d,\theta} = Sp\{ \Phi_{0}^{d,\theta} \} \).
9.2 Numerical solution to the direct problem

The objective of this section is to describe the numerical discretization that we used for solving the eigenvalue problem for the hypersingular operator $G_{d,\theta}^{hyp}$. We will focus on line segments only, although this method can be generalized to curves. The fault $\Gamma_{d,\theta}$ is given by the parametric equations

$$\begin{pmatrix} \cos \theta v, \\ \sin \theta v - d \end{pmatrix}, \quad v \in [-1, 1]. \quad (70)$$

The subscripts $d, \theta$ refer to the depth $d$ and the incline angle $\theta$ (see figure 8).

We propose to solve numerically the eigenvalue problem

$$\int_{-1}^{1} G_{d,\theta}^{hyp}(t, v) \varphi(v) dv = \beta(t), \quad \varphi \in \tilde{H}^{\frac{1}{2}}([-1, 1]), \quad (71)$$

where we are interested in computing the first eigenvalue $\beta_{0}^{d,\theta}$ and the associated eigenvector $\varphi_{d,\theta}$, scaled by the condition $\max_{[-1,1]} \varphi_{d,\theta} = 1$.

Dascalu and Ionescu proposed in [4] a numerical method for an analogous eigenvalue problem in free space, for the Helmholtz operator. After a trigonometric substitution and the use of the so called Glauert formula, a discrete linear eigenvalue problem was derived. This numerical scheme had excellent convergence properties. However it involved the computation of highly oscillatory double integrals.

Hsiao, Stephan and Wendland considered in [21] a related Dirichlet problem for the two-dimensional linear elasticity equations in the domain exterior to an open arc in the plane. They added special singular elements to the regular splines as test and trial functions, to use an augmented Galerkin procedure for the corresponding boundary integral equations thus obtaining a quasi-optimal rate of convergence for the approximate solutions.

We propose to discretize (71) by quadrature. The a priori estimates in [21] assert that the singularity of an eigenvector $\varphi(v)$ at the endpoints -1 and 1 is a sum of positive integer powers of $\sqrt{1 - v^2}$. Accordingly, we make the substitutions $v = f(w)$, $t = f(u)$, where $f(x) = \sin \frac{\pi}{2} x$. The eigenvalue problem (71) rewrites as,

$$\int_{-1}^{1} G_{d,\theta}^{hyp}(f(u), f(w)) \varphi(f(w)) f'(w) dw = \lambda \varphi(f(u)). \quad (72)$$

We use the following the decomposition

$$G_{d,\theta}^{hyp}(t, v) = \frac{1}{\pi(t - v)^2} + G_{d,\theta}(t, v),$$

where $G_{d,\theta}$ is a smooth function. We want to find quadrature coefficients for the hypersingular part of (72). In practice, we fix the grid of points $\frac{j}{n}$ for $j = -n + 1, \ldots, n - 1$ and we compute coefficients $\gamma_{j,l}$

$$\int_{-1}^{1} g(f(w)) f'(w) dw \approx \sum_{l=-n+1}^{n-1} \gamma_{j,l} g(f(\frac{l}{n})) + O\left(\frac{1}{n^4}\right), \quad (73)$$

$$\int_{-1}^{1} \frac{g(f(w)) f'(w) dw}{\pi(f(\frac{j}{n}) - f(w))^2} \approx \sum_{l=-n+1}^{n-1} \gamma_{j,l} g(f(\frac{l}{n})) + O\left(\frac{1}{n^4}\right), \quad (74)$$

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for a smooth function \( g \) in \([-1, 1]\). To do so, we first isolate the singularities by writing

\[
\int_{-1}^{1} g(w)f'(w)dw = \int_{-1}^{1} \frac{g(f(w)) - g(f(\frac{1}{n})) - g'(f(\frac{1}{n}))(f(\frac{1}{n}) - f(w))}{(f(\frac{1}{n}) - f(w))^2} f'(w)dw
\]

\[
+ \frac{2g(f(\frac{1}{n}))}{f(\frac{1}{n})^2 - 1} + g'(f(\frac{1}{n}))f'(\frac{1}{n}) \log \frac{|1 + f(\frac{1}{n})|}{|1 - f(\frac{1}{n})|}
\]

Order 4 schemes are used to estimate \( g'(f(\frac{1}{n})) \) and \( g''(f(\frac{1}{n})) \), which is needed for smoothly continuing the fraction

\[
\frac{g(f(w)) - g(f(\frac{1}{n})) - g'(f(\frac{1}{n}))(f(\frac{1}{n}) - f(w))}{(f(\frac{1}{n}) - f(w))^2},
\]  

(74)

at \( w = \frac{1}{n} \).

Finally, an order 4 method was used for the quadrature of the integral between -1 and 1 of the smooth function in (74). The same order 4 method is used for approximating

\[
\int_{-1}^{1} g(f(w))G_{d,\theta}(f(\frac{j}{n}), v)f'(w)dw.
\]

We then derive a discrete linear operator for discretizing (72), in the form of a \((2n - 1) \times (2n - 1)\) matrix. Finally, a standard routine was employed for finding eigenvalues and eigenvectors for that matrix.

As a test, we proceed to recover the first eigenvalue corresponding to the free space case. The first eigenvalue was computed in [4]. Its numerical value is, within 9 digits of accuracy, \( \beta_0^\infty = 1.15777388... \). We demonstrate in Table 4 the numerical convergence of the first eigenvalue as \( n \), the number of gridpoints, increases.

<table>
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<th>( n )</th>
<th>Computed ( \beta )</th>
<th>Relative Error</th>
</tr>
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<td>1.182608201</td>
<td>2.09996e-02</td>
</tr>
<tr>
<td>5</td>
<td>1.157517450</td>
<td>-2.2153400e-04</td>
</tr>
<tr>
<td>10</td>
<td>1.157761174</td>
<td>1.097436e-05</td>
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<tr>
<td>20</td>
<td>1.157774028</td>
<td>1.27836e-07</td>
</tr>
</tbody>
</table>

Tab. 4 – Numerical convergence of the first eigenvalue \( \beta_0^\infty \) as the number of gridpoints increases.