Earth surface effects on active faults: An eigenvalue asymptotic analysis

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Abstract

We study in this paper an eigenvalue problem (of Steklov type), modeling slow slip events (such as silent earthquakes, or earthquake nucleation phases) occurring on geological faults. We focus here on a half space formulation with traction free boundary condition: this simulates the earth surface where displacements take place and can be picked up by GPS measurements. We construct an appropriate functional framework attached to a formulation suitable for the half space setting. We perform an asymptotic analysis of the solution with respect to the depth of the fault. Starting from an integral representation for the displacement field, we prove that the differences between the eigenvalues and eigenfunctions attached to the half space problem and those attached to the free space problem, is of the order of $d^{-2}$, where $d$ is a depth parameter: intuitively, this was expected as this is also the order of decay of the derivative of the Green’s function for our problem. We actually prove faster decay in case of symmetric faults. For all faults, we rigorously obtain a very useful asymptotic formula for the surface displacement, whose dominant part involves a so called seismic moment. We also provide results pertaining to the analysis of the multiplicity of the first eigenvalue in the line segment fault case. Finally we explain how we derived our numerical method for solving for dislocations on faults in the half plane. It involves integral equations combining regular and Hadamard’s hypersingular integration kernels.

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1. Introduction

Mathematical and numerical modeling are important tools in modeling and investigating earthquake phenomena. Typically, seismic activity is related to the presence of faults buried beneath the earth surface, and yet all measurable physical parameters are available only at the surface. We are interested in this paper in slow seismic events: they are characterized by important slip taking place on an intermediate time scale (i.e. minutes to months). Two types of phenomena can be related to slow slip events: silent earthquakes and nucleation (or initiation) phases for (ordinary) earthquakes. Either phenomenon can be modeled using the same physics (slip-weakening of friction force) in association to the same mathematics which involve eigenvalue analysis.

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Accounts of silent earthquakes in subduction zones near Japan [13,18], New Zealand, Alaska [7], and Mexico [15,14] were recently reported in the literature. Silent earthquakes are rather large (6 ≤ Mw ≤ 8) and produce surface displacements (range about 2–6 cm) that can be picked up by GPS techniques.

The earthquake nucleation (or initiation) phase, which precedes dynamic rupture, was uncovered by detailed seismological observations [8,6] and recognized in laboratory experiments [5,17]. Important physical properties of the nucleation phase (characteristic time, critical fault length, etc.) were obtained in [2,9,4,1,19] through simple mathematical properties of unstable evolution. In all these papers the spectral analysis played a key role in the description of the nucleation phase. However, since the eigenvalue analysis was performed in the free space case, the effect of the earth’s free surface was not accounted for.

Our goal in this paper is to analyze the case of a fault buried in a half plane by considering an eigenvalue problem which is derived from the stability analysis of displacement fields near equilibrium. The analog in free space was studied in [4]. We focus here on how traction free conditions on the surface of the earth affect eigenfunctions. The trace of the first eigenfunction on the top surface can then be used for recovery of faults from surface displacements: this was done in [11]. We believe that this recovery technique will be useful in detecting active faults and localizing them using GPS measurements. In the special case of an elastic half plane ruptured by a straight line fault, we want to find semi-analytical techniques for computing eigenvalues and profiles of eigenfunctions, related to the quasi-static slow slip displacement equation.

We now give an outline of this paper. The eigenvalue problem describing the slow evolution of the slip is stated in Section 2. In Section 3 we provide physical background for modeling anti-plane configurations. We recall some mathematical properties of the related elastic energy and we derive the eigenvalue problem for the spectral stability analysis of the slip. In the following section we give an integral representation for the displacement field in the half plane and we indicate analogs of those results in the free space case.

In Section 5, we assume that the first eigenspace is one-dimensional and we perform an asymptotic analysis for the corresponding first eigenvector with respect to fault depth. We prove that eigenvalues and eigenfunctions for the half space differ from those in free space by a quantity of order d^−2, where d is a depth parameter. We derive an asymptotic formula for the observed surface displacement, valid within the same order. That formula then serves as the starting point for devising an efficient recovery method for faults (see [11]). We illustrate the previous asymptotic analysis by direct computations of eigenfunctions.

In the following section we discuss the case specific to line segment faults. We prove that if such faults are far enough from the surface, then the first eigenspace for the displacement eigenvalue problem is one-dimensional.

Finally we explain in the appendix how we derived our numerical method for solving for the dislocation on faults in the half plane. It involves solving integral equations combining regular and Hadamard’s hypersingular integration kernels.

2. Problem statement

We denote $\mathcal{D}$ the lower half plane $\mathcal{D} = \{(x_1, x_2) : x_2 < 0\}$ in the non-dimensional coordinate system $Ox_1x_2$. Its boundary, denoted by $\Gamma_{\text{obs}} := \{(x_1, x_2) : x_2 = 0\}$, is called the “surface observation” boundary. Let $\Gamma$ be a bounded connected arc, called cut, crack or fault, included in $\mathcal{D}$, which will be assumed to be a smooth oriented curve with no double points. Our problem is formulated in a non-dimensional coordinates system, which means that we chose a characteristic length $L$. A natural choice for $L$ is provided by relating it to the physical length of the fault. In our coordinate system we decide to fix the length of the fault, by imposing $|\Gamma| = 2$. Let

$$(x_1(v), x_2(v)), \quad v \in [-1, 1],$$

be the arc length parametric equations for $\Gamma$. We take the unit normal $n$ to be indirectly perpendicular to the tangent vector. We denote $\Omega = \Omega(\Gamma)$ the open set, $\Omega := \mathcal{D} \setminus \Gamma$: it has the fault $\Gamma$ as an inner boundary. We consider the following (Steklov type) eigenproblem involving the Laplace operator: find $\Phi : \Omega \rightarrow \mathbb{R}$ and $\beta \in \mathbb{R}$ such that

$$\text{div}(\nabla \Phi) = 0 \quad \text{in} \quad \Omega, \quad \partial_n \Phi = 0 \quad \text{on} \quad \Gamma_{\text{obs}},$$

$$[\partial_n \Phi] = 0, \quad \partial_n \Phi - \beta [\Phi] = 0 \quad \text{on} \quad \Gamma,$$

where $\Phi$ satisfies some decay at infinity discussed in the next paragraph, and where we have denoted [ ] the jump across $\Gamma$ (i.e. $[u] = u^+- u^-$), and $\partial_{\nu} = \nabla \cdot n$ the corresponding normal derivative, with the unit normal $n$ pointing toward the positive side. Let us remark that the above eigenvalue problem, associated to the wave equation with a special boundary condition (i.e. Robin-type with opposite sign), depends only on the position and shape of $\Gamma$. All the physical properties (elasticity, friction, loads, etc.) of the system are concentrated in the non-dimensional parameter $\beta$ and its associated eigenvector.

Let us now give the variational formulation for the above eigenvalue problem. We introduce, as in [16], the space $V$ of functions of finite elastic energy. Let $\mathcal{V}$ be the following subspace of $H^1(\Omega)$:
\[
\mathcal{V} = \{ v \in H^1(\Omega) ; \text{ there exists } R > 0 \text{ such that } v(x) = 0 \text{ if } |x| > R \}
\]
endowed with the norm $\| \cdot \|_V$ defined by the following dot product:
\[
(u, v)_V = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad \|u\|^2_V = (u, u)_V, \quad \forall u, v \in \mathcal{V}.
\] (4)

We define $V$ as the closure of $\mathcal{V}$ in the norm $\|u\|_V$. The dot product $(u, v)_V$ in $V$ is still defined by $\int_{\Omega} \nabla u \cdot \nabla v \, dx$. The space $V$ is continuously embedded in $H^1(\Omega_R)$ for all $R > 0$, with $\Omega_R := \{ x \in \Omega / |x| < R \}$. $V$ is not a subspace of $H^1(\Omega)$. Indeed, if $v \in V$ then $v(x)$ is not necessarily vanishing for $|x| \to +\infty$.

Eigenproblem (2)–(3) can be equivalently stated in its variational form: find $\Phi \in V$, $\Phi \neq 0$ and $\beta \in \mathbb{R}_+$ such that
\[
\int_{\Omega} \nabla \Phi \cdot \nabla v \, dx = \beta \int_{\Gamma} [\Phi][v] \, d\sigma, \quad \forall v \in V.
\] (5)

Eigenproblem (2)–(3) was analyzed in [10] in the case of bounded domains. We will provide in this paper the analog for unbounded domains. In either case the spectrum consists of a non-decreasing and unbounded positive sequence of eigenvalues $\beta$. More precisely,

**Theorem 2.1.** The eigenvalues and eigenfunctions of (5) consists of a sequence $(\beta_n, \Phi_n)_{n \in \mathbb{N}}$ such that $0 < \beta_0 \leq \beta_1 \leq \cdots$ and $\beta_n \to +\infty$. Moreover the first eigenvalue is given by the Rayleigh quotient,
\[
\beta_0 = \frac{\int_{\Omega} |\nabla \Phi|^2 \, dx}{\int_{\Gamma} [\Phi]^2 \, d\sigma} = \min_{v \in V} \frac{\int_{\Omega} |\nabla v|^2 \, dx}{\int_{\Gamma} [v]^2 \, d\sigma}.
\] (6)

**Proof.** Let us denote $T : L^2(\Gamma) \to V$ the linear and bounded operator which maps $f \in L^2(\Gamma)$ to the unique solution $T(f) \in V$ of the following linear equation
\[
(T(f), v)_V = \int_{\Gamma} f[v] \, dx, \quad \forall v \in V.
\] (7)

We can define now the linear bounded operator $K : L^2(\Gamma) \to L^2(\Gamma)$ by setting $K(f) = [T(f)]$, the jump of $T(f)$ across $\Gamma$. Note that the range of $K$ is in $H^{1/2}(\Gamma)$. From (7) we get
\[
\int_{\Gamma} K(f) g \, dx = (T(f), T(g))_V = \int_{\Gamma} f K(g) \, dx,
\] (8)

for all $f, g \in L^2(\Gamma)$, which implies that $K$ is symmetric and non-negative. Due to the compact embedding of $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ we deduce that $K : L^2(\Gamma) \to L^2(\Gamma)$ is compact. Let us remark that if $K(f) = 0$ then $T(f) = 0$. The nullspace of $K$ is thus 0. This yields the existence of a sequence of paired eigenvalues eigenfunctions $(\delta_n, h_n)_{n \in \mathbb{N}} \subset \mathbb{R} \times L^2(\Gamma)$ such that $\lim_{n \to \infty} \delta_n = 0$ and all the $\delta_n$ are real and positive. We may also suppose that $\delta_n$ is a non-increasing sequence. If we define $(\beta_n, \Phi_n)$ as $\beta_n := 1/\delta_n$ and $\Phi_n := T(h_n)$ then (5) holds and (6) is proved. □

$\Phi_0$ can be normalized in different ways. We will use two possible normalizations: one relative to the $L^2(\Gamma)$ norm,
\[
\int_{\Gamma} [\Phi_0]^2 = 1,
\]
and the other relative to the maximum slip,
\[
\max_{x \in I}[\Phi_0](x) = 1.
\]

3. Physical motivation

Consider, as in [3,4,19], the anti-plane shearing on a fault (or a system of finite faults) under a slip-dependent friction in a linear elastic domain \( \Omega \times \mathbb{R} \), in non-dimensional coordinates \( O_{x_1,x_2,x_3} \), for which a characteristic length \( L \) was chosen. It is assumed in this model that the displacement field \( u = (u_1, u_2, u_3) \) is zero in the \( O_{x_1} \) and \( O_{x_2} \) directions and that \( u_3 \) does not depend on \( x_3 \). The displacement is therefore simply denoted by \( w = w(t, x_1, x_2) \). Assume that the elastic medium has shear rigidity \( G \), density \( \rho \) and shear velocity \( \varepsilon = \sqrt{G/\rho} \). The non-vanishing shear stress components are \( \sigma_{31} = \tau_1^\infty + (G/L)\hat{\nu}_1 w, \sigma_{32} = \tau_2^\infty + (G/L)\hat{\nu}_2 w, \) and \( \sigma_{11} = \sigma_{22} = -S \), where \( \tau_\infty \) is the pre-stress and \( S > 0 \) is the normal stress on the faults. We assume that \( S, \tau_1^\infty, \tau_2^\infty \) are continuous in \( \bar{\Omega} \).

Let us now describe the static (or quasi-static) problem associated to this friction law. These processes correspond to “slow” slip events which characterize crustal displacements developing on intermediate time scales (days, month). Compared to geological time scales, these phenomena are sufficiently rapid to have been referred to as “silent earthquakes”, because at their time scale the crust is essentially behaving elastically, as for earthquakes. Note that the time scale governing usual earthquakes is of the order of seconds: the process is then fully dynamic. Even if the formulation is quite different in that case, the same approach is valid during the first part of the initiation (or nucleation) phase. The dynamical process is then quite slow and the same eigenvalue analysis is applicable, see [1,2,4,19].

The equilibrium equation reads
\[
\text{div} \left( \frac{G}{L} \nabla w \right) = 0 \quad \text{in} \ \Omega, 
\]
while on the boundary \( \Gamma_{\text{obs}} \times \mathbb{R} \), which corresponds to the surface of the earth where a stress free condition is imposed, and where \( \tau_\infty^2 = 0 \), we have
\[
[\hat{\nu}_n w] = 0 \quad \text{on} \ \Gamma_{\text{obs}}.
\]

On the interface \( \Gamma \), the shear stress has no jumps \( [G \hat{\nu}_n w] = 0 \) and a frictional contact is supposed to act. We now introduce a friction type constitutive law described, in the static case, by
\[
\frac{G}{L} \hat{\nu}_n w + q = -\mu([w(t)]) S \text{sign}([w]) \quad \text{if} \ [w] \neq 0,
\]
\[
\left| \frac{G}{L} \hat{\nu}_n w + q \right| \leq \mu([w]]) S \quad \text{if} \ [w] = 0,
\]
where \( q := \tau_\infty \cdot n \) is the tangential pre-stress acting on the fault. The above equations assert that the tangential (frictional) stress is bounded by the normal stress \( S \) multiplied by the value of the friction coefficient \( \mu \). If such a limit is not attained, sliding does not occur. Otherwise the frictional stress is opposed to the slip \([w]\) and its absolute value depends on the slip through \( \mu \).

We assume that the friction coefficient is a Lipschitz continuous function, with respect to the slip. Let \( H \) be the anti-derivative
\[
H(x, u) := S(x) \int_0^{|u|} \mu(x, s) \, ds.
\]
We suppose that there exist some constants \( l, a, x \geq 0 \), such that
\[
|\mu(x, s_1) - \mu(x, s_2)| \leq l |s_1 - s_2|, \quad H(x, s) - S(x) \mu(x, 0)s + \frac{1}{2} ax^2 + as^3 \geq 0, 
\]
for almost all \( x \in \Gamma \), and for all \( s, s_1, s_2 \in \mathbb{R}_+ \). If the friction coefficient \( \mu \) has a smooth dependence on the slip then the parameter \( x \), which plays a crucial role in the analysis of stability, is related to the slip rate at the beginning of
the slip process, i.e.
\[ \alpha = \sup_{x \in \Gamma} [\partial_u \mu(x, 0)]. \]

We suppose that we can choose the orientation of the unit normal of the fault (or cut) \( \Gamma \) such that \( q(x) = r^\infty(x) \cdot n(x) \lesssim q_0 < 0 \) almost everywhere in \( \Gamma \). This choice is possible in many concrete applications, where the pre-stress \( r^\infty \) gives a dominant direction of slip.

It is possible to state the following variational problem for the displacement: find \( w \in V \) such that
\[
\int_{\Omega} G \frac{\partial w}{\partial x} \cdot \nabla (v - w) \, dx + \int_{\Gamma} S \mu(||w||)((||v|| - ||w||)) \, d\sigma + \int_{\Gamma} q(||v| - |w||) \, d\sigma \geq 0,
\]
for all \( v \in V_+ \). If we consider \( \mathcal{W} : V \to \mathbb{R} \) the energy functional:
\[
\mathcal{W}(v) := \frac{1}{2} \int_{\Omega} G \frac{\partial w}{\partial x}^2 \, dx + \int_{\Gamma} H(||v||) + q||v|| \, d\sigma,
\]
and if \( w \in V \) is a local extremum for \( \mathcal{W} \), then \( w \) is a solution of (14) (see \([10,12]\)). Moreover, there exists at least a global minimum for \( \mathcal{W} \) on \( V \). Let us now analyze the stability of the equilibrium \( w \equiv 0 \). To this end, we will suppose that \( q(x) + S(x)\mu(x, 0) \lesssim 0 \), for almost all \( x \) in \( \Gamma \). This is true if and only if \( w \equiv 0 \) is a solution of (14).

The first eigenvalue \( \beta_0 \) for problem (2)–(3) can be related to the stability analysis near equilibrium: that was done in \([12]\). More precisely if
\[
\frac{\alpha L}{G} < \beta_0
\]
then \( w \equiv 0 \) is an isolated local minimum for \( \mathcal{W} \), i.e. there exists \( \delta > 0 \) such that \( \mathcal{W}(0) < \mathcal{W}(v) \) for all \( v \) in \( V_+ \) such that \( v \neq 0 \) and \( ||v||_V < \delta \). This means that \( \beta_0 \), may be regarded as the stability threshold. Indeed, if for some reason the stability condition \( \frac{\alpha L}{G} < \beta_0 \) is no longer valid, then the part of the solution associated with the first (positive) eigenvalue of the associated dynamical problem will have an exponential growth in time. Thus, after some time, this part will become dominant, while the other modes will undergo a wave-type evolution. The propagative terms are rapidly negligible and the shape of the slip distribution is fairly well approximated by the first eigenfunction \( \Phi_0 \) during all of the nucleation phase of an earthquake. The accuracy of the approximation of the dominant part (i.e. the first unstable eigenfunction) was illustrated by many numerical comparisons. The dominant part was compared in \([2,4]\) with the full solution computed by a finite difference method. In each case the difference was found to be of the order of the initial perturbation, which is negligible with respect to the final amplitude of the solution at the end of the initiation phase. In conclusion, the distribution of the displacement on the earth surface (i.e. \( x \rightarrow w(t, x) \) on \( \Gamma_{obs} \)) is fairly well captured by \( x \rightarrow \exp(\lambda t)\Phi_0^q(x) \), where \( \Phi_0^q \) can be approximated by the static eigenfunction \( \Phi_0 \) given by (6), during a “long” period of time \( t \in [0, T_L] \), called the nucleation phase. During that nucleation phase the slip \( [w(t, x)] \) is less than the critical slip \( D_c \) everywhere on the fault \( \Gamma \). At the beginning of the initiation phase the exponential growth exponent \( \lambda \) is small enough, so \( \exp(\lambda t)\Phi_0^q(x) \) is roughly the same as \( \Phi_0(x) \).

4. Integral formulation

Our starting point is the readily available Green’s function in half space that satisfies the Neumann condition at the line \( x_2 = 0 \). Denoting \( G_0 \) the free space Green’s function for the Laplacian,
\[
G_0(x_1, x_2, y_1, y_2) = \frac{1}{4\pi} \log \frac{1}{(x_1 - y_1)^2 + (x_2 - y_2)^2},
\]
the half space Green’s function \( G \) with zero normal derivative at the line \( x_2 = 0 \) is
\[
G(x_1, x_2, y_1, y_2) = G_0(x_1, x_2, y_1, y_2) + G_0(x_1, x_2, y_1, -y_2).
\]
Let
\[
(x_1(v), x_2(v)), v \in [-1, 1],
\]

be the arc length parametric equations for $\Gamma$. We take the unit normal $n$ to be indirectly perpendicular to the tangent vector.

We can now reformulate the characterization of the first eigenvalue $\beta_0$ defined by the Rayleigh quotient (6), in association with the linear eigenvalue problem (2)–(3), using integral operators on the curve $\Gamma$. We first review basic properties of double layer potentials, and normal derivatives of double layer potentials. The latter have to be understood in Hadamard’s finite part sense for hypersingular integrals. We will throughout this paper use the work by Wendland et al. [20] to refer to regularity properties for hypersingular integrals.

We define $\tilde{H}^{1/2}(\Gamma)$ as in [20]: let $\tilde{\Gamma}$ be a simple smooth closed curve in $\mathbb{R}^2$ such that $\Gamma \subset \tilde{\Gamma}$. Then,

$$\tilde{H}^{1/2}(\Gamma) = \{ u \in H^{1/2}(\tilde{\Gamma}) | \text{supp}(u) \subset \Gamma \}.$$  

The norm on $\tilde{H}^{1/2}$ is defined by $\| u \|_{\tilde{H}^{1/2}(\Gamma)} = \| u \|_{H^{1/2}(\tilde{\Gamma})}$.

**Lemma 4.1.** Let $\varphi$ in $\tilde{H}^{1/2}(\Gamma)$ be non-zero. Set

$$u(y_1, y_2) = - \int_\Gamma \partial_n G \varphi(u) \, dv.$$  

Then $u$ satisfies

$$\Delta u = 0 \quad \text{in } \Omega \setminus \tilde{\Gamma},$$  

$$\partial_n u = 0 \quad \text{along } \partial \varphi,$$

$$[u] = \varphi \quad \text{across } \Gamma,$$  

$$[\partial_n u] = 0 \quad \text{across } \Gamma.$$  

\(\nabla u\) is in $L^2(\Omega)$ and finally, if $\varphi \neq 0$,

$$\int_\Gamma \frac{\partial u}{\partial n} \varphi \, dv > 0.$$  

**Proof.** Identities (16)–(17) are obvious. To derive the other identities, we first extend $\Gamma$ to $\tilde{\Gamma}$, the boundary of a smooth domain $U$ whose closure is included in $\Omega$. We can do it in a such a way that $n$ be the interior normal on the $\Gamma'$ part of $\tilde{\Gamma}$, and that the orientation defined by parameterizing $\Gamma$ as $v$ increases be positive. $\varphi$ is extended to $\tilde{\Gamma}$ by $0$. Classical potential theory indicates that properties (18)–(19) hold. We now apply Green’s theorem:

$$\int_\Gamma \partial_n u \varphi \, dv = \int_\Gamma \partial_n u (u^+ - u^-) \, dv = \int_U |\nabla u|^2 + \int_{\Omega \setminus U} |\nabla u|^2,$$

where we have used that $\partial_n u = 0$ on $\partial \varphi$, and the fact that $u(y)$ decays as $y$ approaches infinity.

Now, if $\int_U |\nabla u|^2 + \int_{\Omega \setminus U} |\nabla u|^2 = 0$, then $u$ is a constant in $U$ and is zero in $\Omega \setminus U$. By making a second choice for $U$, we can argue that $u$ is zero everywhere in $\Omega \setminus \Gamma$. Recalling $[u] = \varphi$, this implies that $\varphi$ is equal to zero. \(\square\)

We are now ready to reformulate the characterization of the first eigenvalue $\beta_0$ defined by the Rayleigh quotient (6), in association with the linear eigenvalue problem (2)–(3), using integral operators on the curve $\Gamma$. This is done in the following proposition.

**Proposition 4.1.** The first eigenvalue $\beta_0$ defined by the Rayleigh quotient (6), associated to the linear eigenvalue problem (2)–(3), can also be defined by this other quotient,

$$\beta_0 = \inf_{\varphi \in \tilde{H}^{1/2}(\Gamma), \varphi \neq 0} \frac{-\int_\Gamma \partial_n G \varphi(v) \, d\sigma(v) \varphi(u) \, d\sigma(u)}{\int_\Gamma \varphi^2 \, d\sigma},$$

where $G$ stands for $G(x_1(v), x_2(v), y_1(u), y_2(u))$.  

Proof. Set

\[ \beta'_0 = \inf_{\varphi \in H^{1/2}(\Gamma), \varphi \neq 0} \frac{-\int_{\Gamma} \partial_{n_y} \int_{\Gamma} \partial_{n_x} G \varphi(v) \sigma(v) \varphi(u) \, d\sigma(u)}{\int_{\Gamma} \varphi^2 \, d\sigma}, \]  

(22)

and

\[ \beta_0 = \min_{v \in V, v \neq 0} \frac{\int_{\Omega} |\nabla v|^2 \, dx}{\int_{\Gamma} |v|^2 \, d\sigma}, \]  

(23)

where \( V \) was introduced in Section 2. We want to prove that \( \beta'_0 = \beta_0 \). Let us denote \( v_0 \) a function achieving the minimum in (23). \( v_0 \) is guaranteed to exist: this is shown in Ref. [10]. It is also known from [10] that \( v_0 \) satisfies

\[
\Delta v_0 = 0 \quad \text{in} \quad \Omega,
\]

\[
\partial_{n} v_0 = 0 \quad \text{along} \quad \partial \Omega,
\]

\[
\beta_0[v_0] = \partial_{n} v_0 \quad \text{across} \quad \Gamma,
\]

\[
[\partial_{n} v_0] = 0 \quad \text{across} \quad \Gamma.
\]

It follows that

\[
v_0(y_1, y_2) = -\int_{\Gamma} (\partial_{n_x} G)[v_0(x_1(v), y_1(v))] \, d\sigma(v).
\]

Therefore \( \beta_0 \geq \beta'_0 \).

Arguing by contradiction, assume that \( \beta_0 > \beta'_0 \). Then, for some positive \( \varepsilon \) there exists \( \varphi \) in \( H^{1/2}(\Gamma) \) such that \( \| \varphi \|_{L^2(\Gamma)} = 1 \) and

\[
-\int_{\Gamma} \partial_{n_y} \int_{\Gamma} \partial_{n_x} G \varphi(v) \sigma(v) \varphi(u) \, d\sigma(u) \leq \beta_0 - \varepsilon.
\]

Set

\[
u(y_1, y_2) = -\int_{\Gamma} \partial_{n_x} G \varphi(v) \, dv.
\]

Then

\[
\int_{\Gamma} \partial_{n} u(u^+ - u^-) \leq \beta_0 - \varepsilon,
\]

\[
\Delta u = 0 \quad \text{in} \quad \Omega,
\]

\[
\partial_{n} u = 0 \quad \text{on} \quad \partial \Omega,
\]

\[
[\partial_{n} u] = 0 \quad \text{across} \quad \Gamma.
\]

But then,

\[
\int_{\Omega} |\nabla u|^2 \leq \beta_0 - \varepsilon \quad \text{and} \quad \int_{\Gamma} [u]^2 = 1,
\]

which contradicts the definition of \( \beta_0 \). \( \Box \)

Remark. The proof of Proposition 4.1 also showed that the infimum of the Rayleigh quotient (21) is achieved.
Proposition 4.1 and the remark that follows have analogs in the free space case, which is the case where $\Omega = \mathbb{R}^2$. In that case the functional space $V$ is simply the closure of smooth and compactly supported in $\mathbb{R}^2 \setminus \Gamma$ functions for the norm

$$
\|v\| = \sqrt{\int_{\mathbb{R}^2} |\nabla v|^2 \, dx}.
$$

Accordingly, we introduce the following notations for the first eigenvalue for the linear problem (2)–(3) in free space given by (6),

$$
\beta_0^\infty = \frac{\int_{\mathbb{R}^2} |\nabla \phi_0^\infty|^2 \, dx}{\int_J |\phi_0^\infty|^2 \, d\sigma} = \min_{v \in V} \frac{\int_{\mathbb{R}^2} |\nabla v|^2 \, dx}{\int_J |v|^2 \, d\sigma}.
$$

(24)

The analog of Proposition 4.1 uses the free space Green’s function $G_0$ and states

$$
\beta_0^\infty = \min_{\varphi \in \tilde{H}^{1/2}(\Gamma)} -\int_J \partial_{n_1} \int_J \partial_{n_2} G_0 \varphi(v) \, d\sigma(v) \varphi(u) \, d\sigma(u) \int_J \varphi^2 \, d\sigma.
$$

(25)

5. Fault depth asymptotic analysis

We assume in this section that the parametric equations for $\Gamma$ are such that $(x_1(0), x_2(0)) = (0, 0)$ and we define $\Gamma_d$ to be the curve obtained from $\Gamma$ by translation of vector $(0, -d)$ (see Fig. 1). We will assume that $d$ is large enough to ensure that $\Gamma_d$ is included in the half plane $x_2 < 0$.

We denote $\beta_d^0$ the corresponding first eigenvalue given by Eq. (6), where $\Gamma$ is replaced by $\Gamma_d$. Let $\Phi_d^0$ be a function satisfying (2)–(3) with $\beta = \beta_d^0$ and such that $\int_J |\Phi_d^0|^2 = 1$. In this section we assume that the first eigenspace for the linear problem (2)–(3), in the half plane ruptured by the fault $\Gamma_d$ is one-dimensional. Similarly, we assume that the first eigenspace for the linear problem (2)–(3), in the whole plane ruptured by the fault $\Gamma$ is one-dimensional. We already denoted $\beta_0^\infty$ the corresponding eigenvalue. Let $\Phi_0^\infty$ be a function satisfying (2)–(3) where we set $\beta = \beta_0^\infty$ and such that $\int_J |\Phi_0^\infty|^2 = 1$.

We first analyze regularity properties for the jumps of $\Phi_d^0$ and $\Phi_0^\infty$ across the fault lines. We denote these jumps by

$$
\varphi_d := [\Phi_d^0], \quad \varphi_\infty := [\Phi_0^\infty].
$$

After a linear change of variables, $\varphi_d$ can also be regarded as a function in the space $\tilde{H}^{1/2}(\Gamma)$.

The goal of this section is first to prove the convergence $\beta_d^0 \to \beta_0^\infty$, and to prove convergence of scaled eigenvectors $\varphi_d = [\Phi_d^0]$ associated to $\beta_d^0$, to a scaled eigenvector $\varphi_\infty = [\Phi_0^\infty]$ associated to $\beta_0^\infty$.

![Fig. 1. The fault $\Gamma_d$ obtained from $\Gamma$ by translation of vector $(0, -d)$.](image-url)
From the previous section
\[
\beta_0^d = \min_{\varphi \in H^{1/2}(\Gamma)} \frac{-\int_{\Gamma_d} \hat{c}_{n_y} \int_{\Gamma_d} \hat{c}_{n_z} G \varphi(v) \, d\sigma(v) \varphi(u) \, d\sigma(u)}{\int_{\Gamma_d} \varphi^2 \, d\sigma}. \tag{26}
\]

A simple change of variables allows us to deal only with integral operators on the fixed curve \(\Gamma\). The change of variables induces an integration kernel \(G_d\). With that change of variables identity (26) becomes
\[
\beta_0^d = \min_{\varphi \in H^{1/2}(\Gamma)} \frac{-\int_{\Gamma} \hat{c}_{n_y} \int_{\Gamma} \hat{c}_{n_z} G_d \varphi(v) \, d\sigma(v) \varphi(u) \, d\sigma(u)}{\int_{\Gamma} \varphi^2 \, d\sigma}. \tag{27}
\]

It will also prove convenient to adopt a simpler notation for the hypersingular operators of interest acting on \(H^{1/2}(\Gamma)\). We denote,
\[
G_d^{\text{hyp}} \varphi := -\hat{c}_{n_y} \int_{\Gamma} \hat{c}_{n_z} G_d \varphi(v) \, d\sigma(v),
\]
\[
G^{\infty}_{\text{hyp}} \varphi := -\hat{c}_{n_y} \int_{\Gamma} \hat{c}_{n_z} G \varphi(v) \, d\sigma(v).
\]

5.1. Asymptotic behavior of the first eigenvalue

**Proposition 5.1.** Let \(\beta_0^d\) be the first eigenvalue for the linear problem (2)–(3) in the half plane ruptured by the fault \(\Gamma_d\), and \(\beta_0^{\infty}\) the first eigenvalue for the linear problem (2)–(3) in the whole plane ruptured by the fault \(\Gamma\). Then there exists a constant \(C\) depending only on the fixed curve \(\Gamma\) such that
\[
|\beta_0^d - \beta_0^{\infty}| \leq \frac{C}{d^2}. \tag{28}
\]

**Proof.** Calculations show that \(G_d^{\text{hyp}} - G^{\infty}_{\text{hyp}}\) is smooth and that, for all \(t, v\) in \([-1, 1]\),
\[
|G_d^{\text{hyp}}(x_1(v), x_2(v), y_1(t), y_2(t)) - G^{\infty}_{\text{hyp}}(x_1(v), x_2(v), y_1(t), y_2(t))| \leq \frac{C_2}{d^2}. \tag{29}
\]

Let \(\varphi_d\) achieve the minimum for defining \(\beta_0^d\), that is \(\|\varphi_d\|_{L^2(\Gamma)} = 1\) and
\[
\beta_0^d = \int_{\Gamma} (G_d^{\text{hyp}} \varphi_d(v)) \varphi_d(v) \, d\sigma(v) = \min_{\|\varphi\|_{L^2(\Gamma)} = 1} \int_{\Gamma} (G_d^{\text{hyp}} \varphi(v)) \varphi(v) \, d\sigma(v). \tag{30}
\]

Similarly, define \(\varphi_{\infty}\) in \(H^{1/2}(\Gamma)\) such that \(\|\varphi_{\infty}\|_{L^2(\Gamma)} = 1\) and
\[
\beta_0^{\infty} = \int_{\Gamma} (G^{\infty}_{\text{hyp}} \varphi_{\infty}(v)) \varphi_{\infty}(v) \, d\sigma(v) = \min_{\|\varphi\|_{L^2(\Gamma)} = 1} \int_{\Gamma} (G^{\infty}_{\text{hyp}} \varphi(v)) \varphi(v) \, d\sigma(v). \tag{31}
\]

According to estimate (29),
\[
\beta_0^{\infty} + \frac{C_2}{d^2} |\Gamma| \geq \int_{\Gamma} (G_d^{\text{hyp}} \varphi_{\infty}(v)) \varphi_{\infty}(v) \, d\sigma(v) \geq \beta_0^d,
\]
where \(|\Gamma|\) is the arc length of \(\Gamma\), and similarly
\[
\beta_0^d + \frac{C_2}{d^2} |\Gamma| \geq \int_{\Gamma} (G_{\infty}^{\text{hyp}} \varphi_d(v)) \varphi_d(v) \, d\sigma(v) \geq \beta_0^{\infty}.
\]

The last two estimates lead to the estimate for the first eigenvalues, (28). \(\Box\)
We now present two numerical runs which illustrate the derived asymptotic behavior. Each of these two runs involve faults that are line segments of length 2. In the first run, the line segment is parallel to the observation surface, or in other words, the inclination angle \( \theta \) is 0. In the second run, the inclination angle \( \theta \) is \( \pi/3 \). The first eigenvalue \( \beta_0^d \) was computed following the numerical scheme presented in appendix for different values of the depth \( d \). In Fig. 2 we have plotted the remainder \( |\phi_d^d - \phi_0^\infty| \) versus the depth \( d \) in a log\(_{10} \) scale. The announced decay of order 2 given in (28) is clearly observed in each case. It is noteworthy that for \( d = 1 \), the numerical values within three decimals are \( \phi_0^d = 1.627 \) for the first fault \( (\theta = 0) \) and \( \phi_0^d = 1.567 \) for the second fault \( (\theta = \pi/3) \) which are already somewhat close to \( \phi_0^\infty = 1.158 \ldots \).

5.2. Asymptotic behavior of the first eigenfunction

We first analyze regularity properties for the functions \( \phi_d \) and \( \phi_\infty \). As

\[
G_d^{\text{hyp}} \varphi_d = \beta_0^d \varphi_d
\]

and

\[
G_\infty^{\text{hyp}} \varphi_\infty = \beta_0^\infty \varphi_\infty,
\]

the a priori estimates stated in Theorem 1.8 by Wendland et al. [20], ensure that \( \varphi_d \) and \( \varphi_\infty \) are in \( C^{1/2}(\Gamma) \). Furthermore, the singularities of \( \varphi_d \) and \( \varphi_\infty \) at the tips of \( \Gamma \) are exactly of square root type.

**Proposition 5.2.** There exists a constant \( C \) depending only on the curve \( \Gamma \) such that

\[
\max_{v \in [-1,1]} |\phi_d(v) - \varphi_\infty(v)| \leq \frac{C}{d^2},
\]  
(32)

**Proof.** Using (29) and (28), we may write

\[
G_\infty^{\text{hyp}} \varphi_d = \beta_0^d \varphi_d + O\left(\frac{1}{d^2}\right) = \beta_0^\infty \varphi_d + O\left(\frac{1}{d^2}\right).
\]

Let \( P \) the orthogonal projection onto the nullspace of \( G_\infty^{\text{hyp}} - \beta_0^\infty I \). We have the following estimate

\[
(G_\infty^{\text{hyp}} - \beta_0^\infty I)(I - P)\varphi_d = O\left(\frac{1}{d^2}\right).
\]
Noticing that \((G_{\infty}^{\text{hyp}} - \rho_{0}^{\infty} I)^{-1}\) is continuous from the range of \((I - P)\) into \(H^{1/2}(\Gamma)\), we derive
\[
(I - P)\varphi_{d} = \mathcal{O}\left(\frac{1}{d^2}\right),
\]
in the \(H^{1/2}(\Gamma)\) norm. Equivalently,
\[
\varphi_{d} - \langle \varphi_{d}, \varphi_{\infty} \rangle \varphi_{\infty} = \mathcal{O}\left(\frac{1}{d^2}\right),
\tag{33}
\]
thus, taking the dot product by \(\varphi_{d}\),
\[
\langle \varphi_{d}, \varphi_{\infty} \rangle^2 = 1 + \mathcal{O}\left(\frac{1}{d^2}\right).
\]
As we chose \(\varphi_{d}\) and \(\varphi_{\infty}\) to be non-negative, we infer,
\[
\langle \varphi_{d}, \varphi_{\infty} \rangle = 1 + \mathcal{O}\left(\frac{1}{d^2}\right)
\]
and plugging back into (33),
\[
\varphi_{d} - \varphi_{\infty} = \mathcal{O}\left(\frac{1}{d^2}\right),
\tag{34}
\]
in the \(H^{1/2}(\Gamma)\) norm. As \(H^{1/2}(\Gamma)\) is not included in \(L^{\infty}(\Gamma)\), we need to do more work. We will use once again the a priori estimates from [20]. We notice that
\[
G_{\infty}^{\text{hyp}} (\varphi_{d} - \varphi_{\infty}) = G_{d}^{\text{hyp}} \varphi_{d} - G_{\infty} \varphi_{\infty} + \mathcal{O}\left(\frac{1}{d^2}\right)
\]
\[
= \rho_{0}^{d} \varphi_{d} - \rho_{\infty} \varphi_{\infty} + \mathcal{O}\left(\frac{1}{d^2}\right) = \mathcal{O}\left(\frac{1}{d^2}\right),
\]
in the \(H^{1/2}(\Gamma)\) norm. But here again, the a priori estimates of Theorem 1.8 of [20] show that we must have \(\varphi_{d} - \varphi_{\infty} = \mathcal{O}(1/d^2)\) in the sup norm. □

Just as in the previous subsection, we carried out numerical computations of eigenvectors pertaining to the same two line segments faults of length 2. The first eigenfunction \(\varphi_{d}\) was computed for different values of the depth \(d\). Formula (32) is verified in Fig. 3, where we have plotted the remainder \(\max_{v \in [1, 1]} |\varphi_{d}(v) - \varphi_{\infty}(v)|\) versus the depth \(d\) in a log10 scale. Here too, a decay of order 2 can be observed, just as expected.

It is interesting to see how different \(\varphi_{d}\) appears for small values of \(d\). We plotted in Fig. 4, profiles for \(\varphi_{d}\) for \(d = 0.8\) for two rotation angles \(\theta\). In one case \(\theta = 0\), and the other case \(\theta = 0.5\). Note that for \(\theta = 0.5\), the distance from the fault to the surface is about 0.32, which is small compared to the length of the fault (2). The profiles for \(\varphi_{d}\) and \(\varphi_{\infty}\) still appear very similar: see the final remark in the last section for an explanation.

5.3. Asymptotic behavior of the surface observation

In the remainder of the paper, we choose to normalize the eigenvectors \(\varphi_{d}\) and \(\varphi_{\infty}\) by setting
\[
\max_{[-1, 1]} \varphi_{d} = \max_{[-1, 1]} \varphi_{\infty} = 1.
\]
Remark that this normalization is possible because, as proved in the next section, \(\varphi_{\infty}\) is of constant sign, and the previous section shows convergence of \(\varphi_{d}\) to \(\varphi_{\infty}\).

Define the surface dislocation function associated to the first eigenvector \(\varphi_{d}\) as
\[
\psi(\Gamma_{d})(y) := \Phi_{0}^{d}(y, 0) = \int_{\Gamma_{d}} -\delta_{n_{x}} G(x_{1}(v), x_{2}(v), y, 0) \varphi_{d}(v) \, d\sigma(v).
\tag{35}
\]
Calculations show that

\[
\psi(\Gamma_d)(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{-n_1 y - n_2 d + x_1 n_1 + x_2 n_2}{(x_1 - y)^2 + (x_2 - d)^2} \varphi_d(v) \, d\sigma(v),
\]

where \(x_1\) and \(x_2\) are short for \(x_1(v), x_2(v)\), the chosen parametric equation for \(\Gamma\), and \(n = (n_1, n_2)\) is the oriented unit normal vector at \(v\).

We are now able to prove the main asymptotic formula for this paper.

**Proposition 5.3.** The “observable surface” eigenfunction \(\psi(\Gamma_d) = \Phi^d_0(\cdot, 0)\) can be estimated as follows:

\[
\psi(\Gamma_d)(y) = \Psi^d_N(y) + O\left(\max \left\{ \frac{1}{y^2 + d^2}, \frac{1}{d^2(|y| + |d|)} \right\}\right),
\]

where
\[ \psi_N^d(y) := \frac{1}{\pi} \frac{(y, d) \cdot N}{y^2 + d^2}, \] (38)
and \( N = N(\Gamma) \) is the “normalized seismic moment” associated to the free space problem defined by
\[ N := \int_{-1}^{1} n(v) \phi_\infty(v) \, d\sigma(v). \] (39)

**Proof.** Recalling (32) and (36), we write
\[ \psi(\Gamma_d)(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{-(y, d) \cdot (n_1, n_2) + x_1 n_1 + x_2 n_2}{(x_1 - y)^2 + (x_2 - d)^2} \phi_\infty(v) \, d\sigma(v) + O \left( \frac{1}{d^2(|y| + |d|)} \right). \]
As
\[ \frac{-(y, d) \cdot (n_1, n_2) + x_1 n_1 + x_2 n_2}{(x_1 - y)^2 + (x_2 - d)^2} = \frac{-(y, d) \cdot (n_1, n_2)}{y^2 + d^2} + O \left( \frac{1}{y^2 + d^2} \right) \]
asymptotic formula (37) follows. \( \square \)

In many instances, the curve \( \Gamma \) is symmetric about its midpoint \((x_1(0), x_2(0))\). This is true, for example, if \( \Gamma \) is a line segment, using a suitable parametrization. In those symmetric cases, the remainder in asymptotic formula (37) has a higher order.

**Proposition 5.4.** If \( \Gamma \) is symmetric about its midpoint \((x_1(0), x_2(0))\), then the expansion for the surface eigenfunction \( \psi(\Gamma_d) = \Phi_0^\infty(\cdot, 0) \) has a remainder of higher order, that is,
\[ \psi(\Gamma_d)(y) = \psi_N^d(y) + O \left( \frac{1}{d^2(|y| + |d|)} \right). \] (40)

**Proof.** We recall that \( \Phi_0^\infty \) was denoted to be an eigenvector for the linear problem (2)–(3) in the eigenspace attached to the first eigenvalue \( \beta_0^\infty \). That eigenspace was assumed to be one-dimensional in that section. We assume that the parametrization for \( \Gamma \) satisfies
\[ (x_1(-v), x_2(-v)) = (-x_1(v), -x_2(v)), \quad v \in [-1, 1]. \]
By symmetry \( \Phi_0^\infty(-x_1, -x_2) \) is also an eigenvector for the linear problem (2)–(3) corresponding to the first eigenvalue \( \beta_0^\infty \). As we made \( \phi_\infty \) unique by setting max \( \phi_\infty = 1 \) and as \( \phi_\infty \geq 0 \), we conclude
\[ \phi_\infty(-v) = \phi_\infty(v), \quad v \in [-1, 1]. \]
The normal vector satisfies
\[ (n_1(-v), n_2(-v)) = (n_1(v), n_2(v)), \quad v \in [-1, 1], \]
so does the arc length,
\[ \sigma(-v) = \sigma(v), \quad v \in [-1, 1]. \]
We then go back to expanding
\[ \frac{1}{(x_1 - y)^2 + (x_2 - d)^2} = \frac{1}{y^2 + d^2} + \frac{2(y, d) \cdot (x_1, x_2)}{(y^2 + d^2)^2} + O \left( \frac{1}{(y^2 + d^2)^2} \right). \]
Finally, as by symmetry
\[ \int_{-1}^{1} \frac{2(y, d) \cdot (x_1, x_2)}{(y^2 + d^2)^2} (y, d) \cdot (n_1, n_2) \phi_\infty(v) \, d\sigma(v) = 0 \]
Fig. 5. The remainder \( \max_{y \in \mathbb{R}} |\psi(\Gamma_d)(y) - \Psi_N^d(y)| \) versus the depth \( d \) in a log\(_{10}\) scale for two line segment faults (\( \theta = 0 \) and \( \theta = \pi/3 \)).

and

\[
\int_{-1}^{1} \frac{x_1 n_1 + x_2 n_2}{y^2 + d^2} \varphi_\infty(v) \, d\sigma(v) = 0,
\]

one order of magnitude is gained in expanding \( \psi(\Gamma_d) \).

We verify on numerical runs the convergence of the surface dislocation function. As in the previous subsection, we carried out computations for the same two line segments faults of length 2 of rotation angle 0 and \( \pi/3 \). The observable part of the first eigenfunction \( \psi(\Gamma_d) = \Phi_0^d(\cdot, 0) \) was computed for different values of the depth \( d \). In Fig. 5 we have plotted the remainder \( \max_{y \in \mathbb{R}} |\psi(\Gamma_d)(y) - \Psi_N^d(y)| \) versus the depth \( d \) in a log\(_{10}\) scale. As announced in formula (40), the convergence in \( d \) is in this case of order 3.

6. The line segment fault analysis

We will assume in this section that \( \Gamma \) is the line segment \([-1, 1] \times \{0\}\). We denote \( V \) the closure of smooth compactly supported in \( \mathbb{R}^2 \setminus \Gamma \) functions for the norm

\[
\|v\| = \sqrt{\int_{\mathbb{R}^2} |\nabla v|^2 \, dx}.
\]

We will denote \( E_\infty \) the eigenspace of functions for the first eigenvalue \( \beta_\infty^0 \) for the linear problem (2)–(3), and \( \tilde{E}_\infty \) the related eigenspace for the first eigenvalue for the operator \( G_{\infty}^{\text{hyp}} \) on \( H^{1/2}(\Gamma) \).

**Lemma 6.1.** All functions in \( E_\infty \) are odd in the second variable.

**Proof.** If \( u \) is in \( E_\infty \), define \( v \) by setting

\[
v(x_1, x_2) = u(x_1, -x_2).
\]

Set \( w = v + u \). \( w \) satisfies \( \Delta w = 0 \), in \( \mathbb{R}^2 \setminus \Gamma \), \( \partial_n w = 0 \), on \( \Gamma \), \( [\partial_n w] = 0 \) across \( \Gamma \), and \( [w] = 0 \) across \( \Gamma \). As \( w \) is in the functional space \( V \), we conclude that \( w = 0 \).
By symmetry, we may now examine a simpler problem in the upper half space only. Denote
\[ \Pi^+ = \{(x_1, x_2) : x_2 > 0\}, \]
\[ \mathcal{H}^+ = \{u \in H(\Pi^+) : u(x) = 0 \text{ if } |x| \geq R, \text{ and } u(x_1, 0) = 0 \text{ if } |x_1| > 1\}, \]
and \( W^+ \), the completion of \( \mathcal{H}^+ \) under the norm
\[ \sqrt{\int_{\Pi^+} |\nabla u|^2}. \]

It is clear that by symmetry \( \beta_0^\infty \) is also the minimum of the Rayleigh quotient
\[ \beta_0^\infty = \min_{u \in W^+, u \neq 0} \frac{\int_{\Pi^+} |\nabla u|^2}{4 \int_{\Gamma} |u|^2}. \] (41)
The actual minimum is achieved by the restriction to \( \Pi^+ \) of a non-zero function in \( E_\infty \). This new Rayleigh quotient for defining \( \beta_0^\infty \) proves useful for showing that \( E_\infty \) is one-dimensional.

**Lemma 6.2.** Let \( v \) in \( W^+ \) be a minimizer for the Rayleigh quotient
\[ \frac{\int_{\Pi^+} |\nabla u|^2}{4 \int_{\Gamma} |u|^2}. \] (42)
The sign of \( v \) is constant in \( \Pi^+ \).

**Proof.** This proof follows the classical theory found in many PDEs textbooks. We decompose a minimizer \( v \), in its positive and negative parts \( v = v^+ - v^- \). Assume that neither \( v^+ \) nor \( v^- \) is uniformly zero. As
\[ \beta_0^\infty = \frac{\int_{\Pi^+} |\nabla v^+|^2 + |\nabla v^-|^2}{4 \int_{\Gamma} |v^+|^2 + |v^-|^2}, \]
\[ \beta_0^\infty \geq \frac{\int_{\Pi^+} |\nabla v^+|^2}{4 \int_{\Gamma} |v^+|^2}. \] (43)
\( v^+ \) and \( v^- \) are also minimizers for (42). Recall the \( C^{1/2} \) regularity on \( \Gamma \) for jumps of eigenvectors for the linear problem (2)–(3). By symmetry, it follows that \( v^+ \) and \( v^- \) are of class \( C^{1/2} \) on \( \Gamma \). On the portion of \( \Gamma \) where \( v^+ \) is zero, as \( v^+ \) satisfies \( 2\beta_0^\infty v^+ = \partial_n v^+ \), we have \( v^+ = \partial_n v^+ = 0 \). As \( \Delta v^+ = 0 \), \( v^+ \) must be uniformly null, which is a contradiction. □

It now follows that the space of minimizers for (42) is one-dimensional, and due to Lemma 6.1 we get the following result.

**Proposition 6.1.** The solution \( \Phi_0^\infty \in V \) to the linear Rayleigh problem (6) in free space \( (\mathcal{D} = \mathbb{R}^2) \) is unique, i.e. \( E_\infty = Sp(\Phi_0^\infty) \) is a one-dimensional vector space. Consequently, \( E_\infty \) is also one-dimensional. Moreover \( [\Phi_0^\infty] \geq 0 \) a.e. on \( \Gamma \).

If the fixed curve \( \Gamma \) is the line segment \([-1, 1] \times \{0\} \), in addition to a translation of vector \((0, -d)\), we apply a rotation of angle \( \theta \) to \( \Gamma \), to obtain the line segment \( \Gamma_{d,\theta} \) (see Fig. 6). We are also careful to choose \( d \) large enough for a given \( \theta \) in order to have \( \Gamma_{d,\theta} \) included in the half plane \( x_2 < 0 \). We denote in the remainder of this section \( \beta_{0,d,\theta} \) the first eigenvalue for the linear problem (2)–(3), and by \( \hat{E}_{d,\theta} \) the first eigenspace for the operator \( G_{d,\theta}^{hyp} \) on \( \hat{H}^{1/2}(\Gamma) \). Due to the strong convergence \( G_{d,\theta}^{hyp} - G_{\infty}^{hyp} \to 0 \), it is clear that \( \hat{E}_{d,\theta} \) is also a one-dimensional space for \( d \) large enough. In fact, it is possible to estimate a depth \( d_0 \) such that for all \( d > d_0 \), \( \hat{E}_{d,\theta} \) is one-dimensional. We propose to briefly outline how that can be done.

Just like in the proof of Proposition 5.2, we denote \( P \) the orthogonal projection onto the nullspace of \( G_{\infty}^{hyp} - \beta_0^\infty \). \( (G_{\infty}^{hyp} - \beta_0^\infty I)^{-1} \) is continuous from the range of \((I - P)\) into \( H^{1/2}(\Gamma) \). Let \( A \) be the norm of that operator. If we
denote $\beta_1^\infty$ the second eigenvalue of $G_\infty^{\text{hyp}}$, $A$ is equal to $(\beta_1^\infty - \beta_0^\infty)^{-1}$. Dascalu et al. estimated $\beta_1^\infty$ and $\beta_0^\infty$ in [4]. As $\beta_1^\infty = 2.75475474 \ldots$ and $\beta_0^\infty = 1.15777388 \ldots$, we find

$$A = 0.626181581 \ldots$$  \hspace{1cm} (44)

Assume now that $\|G_{d,0}^{\text{hyp}} - G_\infty^{\text{hyp}}\| \leq B$, in the $L^\infty([-1,1]^2)$ norm. Following the proof exposed in the proof of Proposition 5.1,

$$\beta_0^d = (G_{d,0}^{\text{hyp}} \phi_d, \phi_d) = (G_\infty^{\text{hyp}} \phi_d, \phi_d) + ((G_\infty^{\text{hyp}} - G_{d,0}^{\text{hyp}}) \phi_d, \phi_d) \leq \beta_0^\infty + B,$$

and as similarly,

$$\beta_\infty^d \leq \beta_0^d + B,$$

we conclude

$$|\beta_0^d - \beta_0^\infty| \leq B.$$  \hspace{1cm} (45)

We now estimate the distance between any vector $\phi$ in $\tilde{E}_{d,0}$, of $L^2$ norm 1, and its analog $\phi_\infty$ in $\tilde{E}_{\infty}$. As

$$G_\infty^{\text{hyp}} \phi = G_{d,0}^{\text{hyp}} \phi + (G_\infty^{\text{hyp}} - G_{d,0}^{\text{hyp}}) \phi,$$

we derive

$$\|G_\infty^{\text{hyp}} \phi - \beta_0^d \phi\| \leq B$$

and due to (45),

$$\|G_\infty^{\text{hyp}} \phi - \beta_0^\infty \phi\| \leq 2B,$$

or

$$\|(G_\infty^{\text{hyp}} \phi - \beta_0^\infty)(I - P)\| \leq 2B.$$

This in turn implies

$$\|(I - P)\phi\| \leq 2AB,$$  \hspace{1cm} (46)

thus

$$\|P \phi\| \geq 1 - 2AB,$$  \hspace{1cm} (47)
an by possibly changing $\varphi$ into $-\varphi$, we infer,

$$1 - 2AB \leq \langle \varphi, \varphi_\infty \rangle \leq 1,$$

(48)

thus

$$\|\varphi - \varphi_\infty\| \leq 2\sqrt{AB}.$$  

(49)

Assume that $\tilde{E}_{d,0}$ is at least two-dimensional. Pick $\varphi_1$ and $\varphi_2$ in $\tilde{E}_{d,0}$ of $L^2$ norm 1 satisfying (48), such that $\langle \varphi_1, \varphi_2 \rangle = 0$. Then

$$\sqrt{2} = \|\varphi_1 - \varphi_2\| \leq \|\varphi_1 - \varphi_\infty\| + \|\varphi_2 - \varphi_\infty\| \leq 4\sqrt{AB},$$

(50)

from which it follows that

$$\frac{1}{8} \leq AB.$$  

(51)

We now need to estimate the constant $B$. A calculation shows that

$$(G_{d,0}^{\text{hyp}} - G_\infty^{\text{hyp}})(t, v) = -\frac{1}{2\pi} \frac{(v - t)^2 \cos^2 \theta - ((t + v) \sin \theta - 2d)^2}{((v - t)^2 \cos^2 \theta + ((t + v) \sin \theta - 2d)^2)},$$

thus the supremum $B$ of $(G_{d,0}^{\text{hyp}} - G_\infty^{\text{hyp}})$ is estimated as follows:

$$B \leq \frac{1}{8\pi} \frac{1}{(d - 1)^2},$$

(52)

which is achieved for $\sin \theta = 1$, $v = t = 1$. Combining (44), (51), (52), we find that for $\tilde{E}_{d,0}$ to be more than one-dimensional, the inequality

$$\frac{1}{8A} \leq \frac{1}{8(d - 1)^2},$$

(53)

has to be satisfied. We have shown:

**Proposition 6.2.** If the distance $d$ from the center of the line segment $\Gamma_{d,0}$ is greater than $\sqrt{A/\pi} + 1$, which is about 1.4464524474..., then the first eigenspace $\tilde{E}_{d,0}$ is one-dimensional, for any rotation angle $\theta$.

If the rotation angle $\theta$ is zero, estimate (52) can be greatly improved. Indeed in that case,

$$|(G_{d,0}^{\text{hyp}} - G_\infty^{\text{hyp}})(t, v)| \leq \frac{1}{2\pi} \frac{(v - t)^2}{((v - t)^2 + 4d^2)^2} \leq \frac{1}{2\pi} \max \left\{ \frac{1}{16d^2}, \frac{1}{4(1 + d^2)} \right\}.$$  

We then find that if $d$ is greater than $\frac{1}{2} \sqrt{A/\pi}$ which is about 0.2232262237..., then the first eigenspace $\tilde{E}_{d,0}$ is one-dimensional. Note that this depth is small compared to the length of the fault, which is 2.

**Remark.** From the above estimates we can explain why the plots for $\theta = 0$, $d = 0.8$ and $d = \infty$ in Fig. 4 look so similar. Firstly, the corresponding eigenfunctions $\varphi_\infty$ and $\varphi_{0.8,0}$ must have square root singularities at the endpoints $-1$ and 1; this is known from [20]. Secondly, we can estimate $\int_{-1}^1 \varphi_\infty \varphi_{0.8,0}$: if that integral is close to being 1, then the two unit vectors $\varphi_\infty$ and $\varphi_{0.8,0}$ must be close to each other in the $L^2([-1, 1])$ norm. For the values $d = 0.8$, $\theta = 0$, constant $B$ from previous appendix can be chosen to be $\|G_{d,0}^{\text{hyp}} - G_\infty^{\text{hyp}}\|_{L^1([-1, 1])}$, which is about 0.01085845671... . Now due to inequality (48), we find

$$0.9864012688 \ldots \leq \int_{-1}^1 \varphi_\infty \varphi_{0.8,0} \leq 1.$$  

(54)

This last estimate confirms that the difference between $\varphi_\infty$ and $\varphi_{0.8,0}$ is small.
7. Conclusion

The eigenvalue problem (of Steklov type) associated to a quasi-static frictional sliding problem in elasticity turns out to model slow slip events (such as silent earthquakes, or earthquake nucleation phases) occurring on geological faults. This model has been extensively studied in free space. We focused on faults in half planes and we proceeded to analyze displacements produced on the surface, which can be picked up by GPS measurements. The trace of the first eigenfunction on the top surface can then be used for recovery of faults from surface displacements (see [11]). This recovery technique is helpful for detecting active faults and localizing them using GPS measurements.

As the first eigenfunction is the solution of an elliptic PDE eigenproblem that cannot in general be written in closed form, we found in this present paper a convenient approximation formula for surface displacements, valid if the fault is deep enough. For more shallow faults the same approximation is still valid at surface points that are far enough from the fault. This formula serves as the basis for a robust and computationally inexpensive method for solving the fault inverse problem, which can be found in [11]. We have also obtained an “uniqueness” result of prime importance for the inverse problem: the eigenspace for the first eigenvalue of the quasi-static slip on geological faults problem is one-dimensional, if the fault is linear and not too close to the surface. An important implication of this uniqueness result is that only “one family of surface displacement patterns” is possible in the nucleation phase of earthquakes or for slow slip events: this is useful in particular for solving the fault inverse problem.

Acknowledgments

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Appendix. Numerical solution

The objective of this section is to describe the numerical discretization that we used for solving the eigenvalue problem for the hypersingular operator \( G_{d,\theta}^{hyp} \). We will focus on line segments only, although this method can be generalized to curves. The fault \( \Gamma_{d,\theta} \) is given by the parametric equations

\[
(cos \theta v, \sin \theta v - d), \quad v \in [-1, 1].
\]  

The subscripts \( d, \theta \) refer to the depth \( d \) and the incline angle \( \theta \) (see Fig. 6).

We propose to solve numerically the eigenvalue problem

\[
\int_{-1}^{1} G_{d,\theta}^{hyp}(t, v) \varphi(v) \, dv = \beta \varphi(t), \quad \varphi \in \tilde{H}^{1/2}([-1, 1]),
\]  

where we are interested in computing the first eigenvalue \( \beta_{0}^{d,\theta} \) and the associated eigenvector \( \varphi_{d,\theta} \), scaled by the condition \( \max_{[-1,1]} \varphi_{d,\theta} = 1 \).

Dascalu and Ionescu proposed in [4] a numerical method for an analogous eigenvalue problem in free space, for the Helmholtz operator. After a trigonometric substitution and the use of the so-called Glauert formula, a discrete linear eigenvalue problem was derived. This numerical scheme had excellent convergence properties. However, it involved the computation of highly oscillatory double integrals.

Hsiao, Stephan and Wendland considered in [20] a related Dirichlet problem for the two-dimensional linear elasticity equations in the domain exterior to an open arc in the plane. They added special singular elements to the regular splines as test and trial functions, to use an augmented Galerkin procedure for the corresponding boundary integral equations thus obtaining a quasi-optimal rate of convergence for the approximate solutions.

We propose to discretize (56) by quadrature. The a priori estimates in [20] assert that the singularity of an eigenvector \( \varphi(v) \) at the endpoints \(-1 \) and \( 1 \) is a sum of positive integer powers of \( \sqrt{1 - v^2} \). Accordingly, we make the substitutions \( v = f(w), t = f(u) \), where \( f(x) = \sin(\pi/2) x \). The eigenvalue problem (56) rewrites as

\[
\int_{-1}^{1} G_{d,\theta}^{hyp}(f(u), f(w)) \varphi(f(w)) f'(w) \, dw = \lambda \varphi(f(u)).
\]
Table 1  
Numerical convergence of the first eigenvalue \( \beta_0^\infty \) as the number of gridpoints increases

<table>
<thead>
<tr>
<th>Value for ( n )</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computed ( \beta )</td>
<td>1.182608201</td>
<td>1.157517450</td>
<td>1.157761174</td>
<td>1.157774028</td>
</tr>
<tr>
<td>Relative error</td>
<td>2.09996e−02</td>
<td>−2.2153400e−04</td>
<td>1.097436e−05</td>
<td>1.27836e−07</td>
</tr>
</tbody>
</table>

We use the following the decomposition:

\[
G_{d,0}(t, v) = \frac{1}{2\pi(t - v)^2} + G_{d,0}(t, v),
\]

where \( G_{d,0} \) is a smooth function. We want to find quadrature coefficients for the hypersingular part of (57). In practice, we fix the grid of points \( j/n \) for \( j = -n + 1, \ldots, n - 1 \) and we compute coefficients \( c_{j,l} \)

\[
\int_{-1}^{1} \frac{g(f(w))f'(w)dw}{(f(j/n) - f(w))^2} \approx \sum_{l=-n+1}^{n-1} c_{j,l} g\left(\frac{j}{n}\right) + O\left(\frac{1}{n^4}\right),
\]

for a smooth function \( g \) in \([-1, 1]\). To do so, we first isolate the singularities by writing

\[
\int_{-1}^{1} \frac{g(f(w))f'(w)dw}{(f(j/n) - f(w))^2} = \int_{-1}^{1} \frac{g(f(w)) - g(f(j/n))}{(f(j/n) - f(w))^2} f'(w)dw
\]

\[
+ \frac{2g(f(j/n))}{f'(j/n)^2 - 1} + g'\left(\frac{j}{n}\right) f'\left(\frac{j}{n}\right) \log \left|\frac{1 + f(j/n)}{1 - f(j/n)}\right|.
\]

Order 4 schemes are used to estimate \( g'(f(j/n)) \) and \( g''(f(j/n)) \), which is needed for smoothly continuing the fraction

\[
\frac{g(f(w)) - g(f(j/n)) - g'(f(j/n))f'(j/n)(f(j/n) - f(w))}{(f(j/n) - f(w))^2},
\]

at \( w = j/n \).

Finally, an order 4 method was used for the quadrature of the integral between \(-1\) and 1 of the smooth function in (59). The same order 4 method is used for approximating

\[
\int_{-1}^{1} g(f(w))G_{d,0}\left(\frac{j}{n}, v\right) f'(w)dw.
\]

We then derive a discrete linear operator for discretizing (57), in the form of a \((2n - 1) \times (2n - 1)\) matrix. Finally, a standard routine was employed for finding eigenvalues and eigenvectors for that matrix.

As a test, we proceed to recover the first eigenvalue corresponding to the free space case. The first eigenvalue was computed in [4]. Its numerical value is, within nine digits of accuracy, \( \beta_0^\infty = 1.15777388 \ldots \). We demonstrate in Table 1 the numerical convergence of the first eigenvalue as \( n \), the number of gridpoints, increases.

References
