

Energy-dissipation functionals in fracture mechanics: crack branching and the mother-daughter mechanism

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Quasi-static evolution

Suppose that we have an elastic material initially occupying a domain Ω and for each time t the material is in equilibrium subject to a boundary condition $f(t)$. Without the possibility of fracture, we have that the displacement $u(t)$ minimizes

$$E_{el}(v) = \int_{\Omega} W(\nabla v) dx \quad \text{subject to } v = f(t) \text{ on } \partial\Omega$$

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If there is a fixed crack K , the displacement at each time would solve the same Dirichlet problem as above, but in the space $H^1(\Omega \setminus K)$ instead of $H^1(\Omega)$, which implies that the stored elastic energy can only be lower if there is a crack.

Griffith's criterion

The essence of Griffith's criterion for crack growth is that a crack grows only when the reduction in elastic energy exceeds the increase in surface area of the crack (which can be viewed as energy dissipation).

This led to modeling quasi-static evolution by time-incremental minimization problems, first in the engineering community and then more mathematically, using SBV , by Francfort and Marigo:

For discrete times $\{t_i\}$, $u(t_i)$ minimizes

$$v \mapsto \int_{\Omega} W(\nabla v) + \mathcal{H}^{N-1}(S_v \setminus \bigcup_{j < i} S_{u(t_j)})$$

over $v \in SBV_{f(t_i)}(\Omega)$. The technicality in the last term models irreversibility of fracture, so that only the new crack at t_i is penalized.

One can then take the limit as the time steps tend to zero, getting $u(t)$ having the expected properties.

Mielke-Ortiz dissipation functional

For a general stored energy $E(u)$ and a dissipation functional $D(\dot{u})$, Mielke and Ortiz proposed modeling quasi-static trajectories by minimizing

$$\int_0^T e^{-t/\varepsilon} \left\{ \frac{d}{dt} E(u(t)) + D(\dot{u}(t)) \right\} dt,$$

which is essentially the same as minimizing

$$\int_0^T e^{-t/\varepsilon} \left\{ \frac{1}{\varepsilon} E(u(t)) + D(\dot{u}(t)) \right\} dt. \quad (1)$$

Then let $\varepsilon \rightarrow 0$.

For rate-independent fracture, we would have $D(\dot{C}) = \frac{d}{dt} \mathcal{H}^{N-1}(C(t))$.

Comparison to discrete-time:

- Morally the same as discrete time, since it also balances decreases in the stored energy with dissipation
- The Mielke-Ortiz functional is defined on the entire trajectory, which seems more natural when penalizing dissipation (written as a time derivative)
- A bit of causality is sacrificed – at each time, the minimality is based on a small peek into the future
- For both, expect (hope) that the limits (either as $\Delta t \rightarrow 0$ or $\varepsilon \rightarrow 0$) $u(t)$ are such that every $\dot{u}(t)$ minimizes

$$v \mapsto \nabla E(u(t)) \cdot v + D(v).$$

Crack derivatives: fronts and front speeds

Back to fracture: we want to consider dissipation that is local to “where” the crack is growing and is nonlinear in the local speed (rate dependent). Hopefully, we will see some features that are experimentally observed, but do not occur in rate-independent fracture (branching, mother-daughter cracking).

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We need to define the crack front (or “tip” in 2-dimensions). This is a plus, since fracture mechanics relies heavily on the idea of a crack tip (e.g., stress intensity factors, J-integrals, etc.) without there being a mathematical definition.

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$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x) d\mathcal{H}^{N-1}(x) dt = - \int_0^T \varphi(t) \int_{F(t)} f(x) v(x, t) d\mathcal{H}^{N-2}(x) dt$$

$\forall \varphi \in C_0^1([0, T]), \forall f \in C_0(\Omega)$, then F is the front and v is the front speed.

First model

We consider first the problem of minimizing

$$\int_0^T e^{-t/\varepsilon} \left\{ \frac{1}{\varepsilon} \int_{\Omega} W(\nabla u) dx + \int_{F(t)} v^p d\mathcal{H}^{N-2}(x) \right\} dt$$

among an appropriate class of trajectories $(u, C) \in \mathcal{P}$.

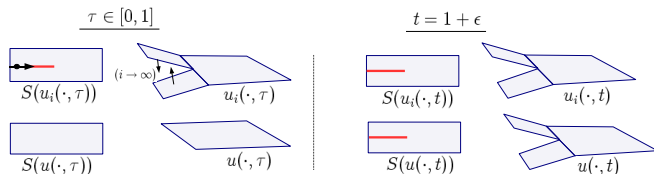
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We have serious existence issues, mostly due to the non-causality. For example, we cannot take $C(t) := \cup_{\tau \leq t} S_{u(\tau)}$ as is done in the discrete-time problem. That is, if we take a minimizing sequence u_i , and we have $u_i \rightarrow u$, then u might not have the front representation:



with $t \mapsto \mathcal{H}^{N-1}(C(t))$ not absolutely continuous in time.

Existence

We have existence in 2D of a minimizer for

$$I(p) := \int_{[0, T]} e^{-\frac{t}{\varepsilon}} \left\{ \frac{1}{\varepsilon} \int_{\Omega} W(\nabla u) dx + \int_{F(t)} v^p d\mathcal{H}^0 \right\} dt,$$

over the class $p = (u, C, F) \in \mathcal{P}$:

- $C : [0, T] \rightarrow \{K : K \subset \Omega, \mathcal{H}^1(K) < \infty\}$ such that:
 - ▶ C nondecreasing
 - ▶ $\forall \tau \leq t, S(u(\tau)) \subset C(t)$
 - ▶ (C, F) satisfy the front representation
- Plus constraints on how the crack can grow (e.g., a bound on the number of points in the front (1!)), $t \mapsto F(t)$ needs to be continuous)

Idea of Proof

- 1 $p_i = (u_i, C_i, F_i)$ minimizing
- 2 F_i bounded in $W^{1,p}$, $F_i \rightarrow F$ uniformly ($F \in W^{1,p}$)
- 3 $C(t) := \bigcup_{\tau < t} F(\tau)$
- 4 $u(t)$ defined as minimizer of

$$v \mapsto \int_{\Omega} W(\nabla v) : v \in SBV(\Omega), S_v \subset C(t)$$

- 5 The triple $p = (u, C, F)$ is a minimizer (real issue: show $p \in \mathcal{P}$).

Another Model: Nucleation Penalty

Instead of constraining the crack trajectories, we now try energy penalties. The underlying idea is that the issues we are trying to prevent are due to non-causality, so hopefully as $\varepsilon \rightarrow 0$ we can take the penalties to zero.

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We consider minimizing

$$I(u) := \int_{[0, T]} e^{-\frac{t}{\varepsilon}} \left\{ \frac{1}{\varepsilon} \int_{\Omega} W(\nabla u) dx + \int_{F(t)} (\alpha + v^p) d\mathcal{H}^{N-2} \right\} dt.$$

Features:

- Fixed penalty for each point in the front (2D)
- No constraints on number of fronts or jumps

This energy needs to be relaxed.

Relaxation

Convergence for $u_i, u \in \mathcal{P}$:

$$u_i \rightarrow u \iff u_i(\cdot, t) \xrightarrow{SBV} u(\cdot, t) \quad \forall t \in [0, T].$$

We then try to find the relaxation of I (with the above convergence):

$$I^*(u) := \inf_{\substack{u_i \in \mathcal{P} \\ u_i \rightarrow u}} \left\{ \liminf_{i \rightarrow \infty} I(u_i) \right\}. \quad (2)$$

Relaxation Result

The relaxed energy is:

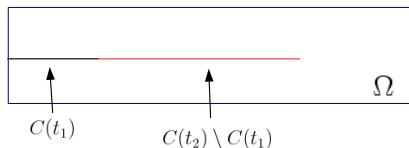
$$I^*(u) = \int_0^T e^{-t/\varepsilon} \left\{ \frac{1}{\varepsilon} \int_{\Omega} W(\nabla u) dx + \rho \left(\frac{\alpha}{\rho - 1} \right)^{\frac{\rho-1}{\rho}} \int_{F(t)} v d\mathcal{H}^{N-2} \right\} dt.$$

Notice:

- Rate independent!
- No front penalty
- $\rho \rightarrow 1$, α doesn't play any role.

Idea of Proof

Suppose C grows by an increment of length L in $[t_1, t_2]$:



By Jensen's inequality, the optimal front configuration minimizes, over n number of fronts, the quantity

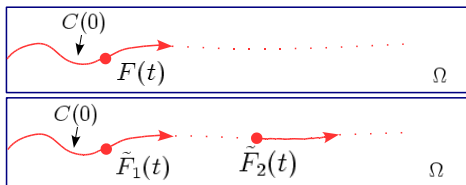
$$\begin{aligned} \int_{t_1}^{t_2} \int_{F(t)} (\alpha + v^p) d\mathcal{H}^0 dt &= n\Delta t\alpha + n\Delta t\bar{v}^p \\ &= n\Delta t\alpha + n\Delta t \left(\frac{L}{n\Delta t} \right)^p \end{aligned}$$

Idea of Proof

Minimizing the previous expression yields an effective dissipation of

$$I = p \left(\frac{\alpha}{p-1} \right)^{\frac{p-1}{p}} L.$$

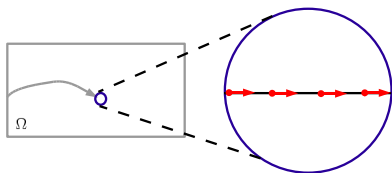
Reason: use multiple fronts ahead of the crack tip (daughter cracks), and use less time than Δt .



Idea of Proof

This generalizes by a covering argument:

- Take an arbitrary $u \in \mathcal{P}$.
- Break up $[0, T]$ into increments like $\frac{1}{i}$.
- Using Lebesgue points and Besicovitch covering, we get a countable disjoint collection of balls in which $C(t_i) \setminus C(t_{i-1})$ is close to a hyperplane
- In each ball use an optimal front configuration to build the crack increment



Issues

- We need a better penalty on nucleation – integrating a constant over the front is not enough. Likely idea: consider $t \mapsto F(t)$ in “SBV” and penalize jumps directly
- Real issue: as $\varepsilon \rightarrow 0$, can we take these penalties away?
- The big question: what properties does u have, if u_ε is a minimizer of the ε -dissipation functional, and $u_\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$?