

# Quasi-static fracture evolution and local minimization

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## Quasi-static evolution

The problem that we consider is to predict crack paths in a quasi-static setting. More precisely, we have an elastic material occupying a domain  $\Omega$  and we suppose that the material is in equilibrium subject to a boundary condition  $f(t)$ . Then, if there is no crack, the displacement  $u(t)$  minimizes

$$E_{el}(v) = \int_{\Omega} |\nabla v|^2 \quad \text{subject to } v = f(t) \text{ on } \partial\Omega$$

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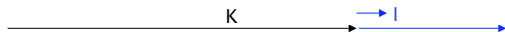
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If there is a fixed crack  $K$ , each displacement would solve the same Dirichlet problem, but in the space  $H^1(\Omega \setminus K)$  instead of  $H^1(\Omega)$ , which implies that the stored elastic energy can only be lower if there is a crack.

## Griffith's criterion

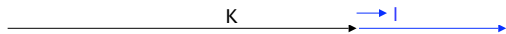
The starting point for predicting crack growth is Griffith's criterion (1920). Griffith considered a pre-existing crack  $K$  with a specified future path (here in blue).



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For a crack increment of length  $l$ ,  $E(l)$  is the elastic energy of the corresponding elastic equilibrium. The criterion states that the crack can only grow if the rate of decrease of elastic energy as  $l$  increases is large enough, i.e., if

$$\begin{aligned} -\frac{dE(l)}{dl} &< G_c && \text{the crack can not run} \\ &= G_c && \text{the crack can run} \\ &> G_c && \text{the crack is unstable.} \end{aligned}$$

# The static problem

Formulated by Ambrosio and Braides (1995): If  $u$  minimizes

$$v \mapsto \int_{\Omega} |\nabla v|^2 + \mathcal{H}^1(S_v)$$

over  $v \in SBV_f(\Omega)$ , then the crack  $K := S_u$  is stable (taking  $G_c = 1$ ). The reason is that each increment in length  $l$  cannot reduce the energy, i.e.,

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Note: This and all that follows is in any dimension, but we will assume 2-D throughout.

## Quasi-static formulation

This led to the following quasi-static formulation (Francfort-Marigo, 1998), where  $f = f(t)$ .

For discrete times  $\{t_i\}$ ,  $u(t_i)$  minimizes

$$v \mapsto \int_{\Omega} |\nabla v|^2 + \mathcal{H}^1(S_v \setminus \bigcup_{j < i} S_{u(t_j)})$$

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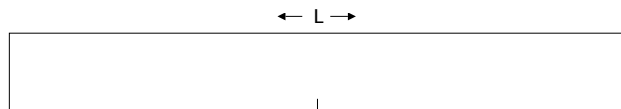
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The plan was then to take a sequence of discretizations  $\{t_i^n\}$  with, e.g.,  $t_i^n - t_{i-1}^n = \frac{1}{n}$ , resulting in a sequence  $\{u_n\}$  that hopefully converges to a  $u$  that is a solution to a corresponding continuous-time problem. This has been carried out (Dal Maso, Francfort, L., Toader).

## Connection to Griffith

The resulting solution  $u(t)$  with  $K(t) := \cup_{\tau \leq t} S_{u(\tau)}$  satisfies Griffith's criterion if  $t \mapsto \mathcal{H}^1(K(t))$  is continuous.

Problem:

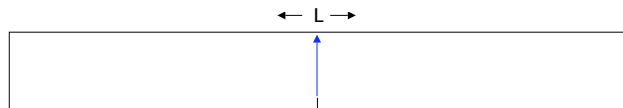


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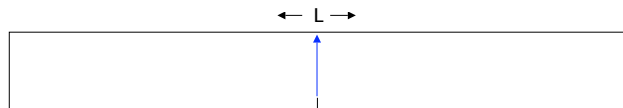


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Note the connection to local vs. global minimality – the initial crack was a local minimizer and was stable in the sense of Griffith.

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We want to have that if  $u$  is a strict local minimizer, then there is no accessible state with lower energy  $\iff$  strict local minimizers are stable.

This suggests a definition of accessibility:

$v$  is accessible from  $u \iff$  there exists a continuous path  $\phi$  from  $u$  to  $v$  along which the total energy is nonincreasing.

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An idea like this has been tried by Dal Maso and Toader (2002), based on following gradient flows from  $u(t_{i-1})$  to find  $u(t_i)$ .

Unfortunately, there are technical difficulties in proving that, when  $u_n(t) \rightarrow u(t)$ , we have the properties we want for  $u(t)$ . In particular, local minimality is a problem.

In fact, the same problem occurs with accessibility – we would need to show that since the  $u_n$  were stable, so is  $u$ , i.e., if there is a  $v$  that is accessible from  $u$  and has lower energy, then there are  $v_n$  that are accessible from  $u_n$  and have lower energy.

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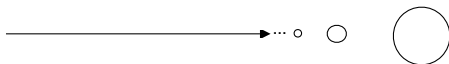
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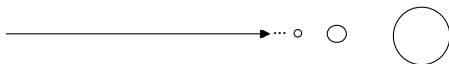
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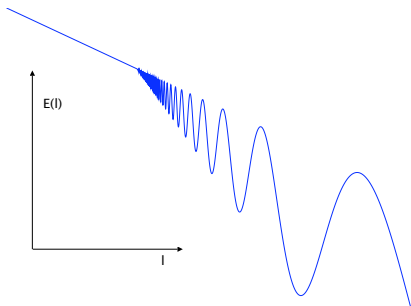


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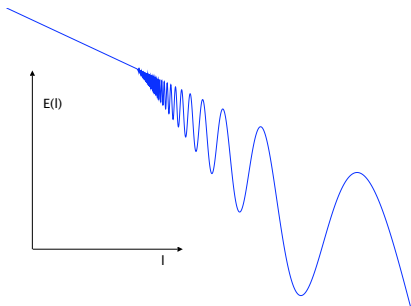


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so that we cannot move to the state that we want without the energy initially increasing by a small amount. Allowing small energy increases in our definitions of accessibility and stability overcomes all mathematical issues.

### Definition ( $\varepsilon$ -accessible)

$v$  is  $\varepsilon$ -accessible from  $u$  if there exists a continuous function  $\phi: [0, 1] \rightarrow SBV(\Omega)$  such that  $\phi(0) = u$ ,  $\phi(1) = v$ , and

$$\sup_{\tau_1 < \tau_2} [E(\phi(\tau_2)) - E(\phi(\tau_1))] < \varepsilon.$$

We then have the corresponding definition of stability:

### Definition ( $\varepsilon$ -stability)

$u$  is  $\varepsilon$ -stable if there does not exist an  $\varepsilon$ -accessible  $v$  with strictly lower energy. The path to such a  $v$  is called an  $\varepsilon$ -slide.

We also define  $\bar{\varepsilon}$ -accessibility, where the inequality is not strict.

# Existence theorem

## Theorem

Given  $f(t)$  with sufficient regularity, there exists a quasi-static evolution  $u(t)$  with the properties of a globally minimizing evolution, modified as follows:

- $u(t)$  is a local minimizer at every  $t$  (coming from being  $\bar{\varepsilon}$ -stable)
- Energy inequality:

$$E(u(t_2)) - E(u(t_1)) \leq \int_{t_1}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \dot{f} \, dx dt$$

for every  $t_1 \leq t_2$ .

- If  $u^-(t) \neq u^+(t)$ , then  $u^+(t)$  is  $\bar{\varepsilon}$ -accessible from  $u^-(t)$  and has lower energy than all states that are  $\varepsilon$ -accessible from  $u^-(t)$ .

## Proof.

- First issue: show that if  $u_n \rightarrow u$  and there exists an  $\varepsilon$ -slide for  $u$ , then for  $n$  sufficiently large, there exists an  $\varepsilon$ -slide for  $u_n$ .

Strategy: For an  $\varepsilon$ -slide  $(\phi, K)$ , “transfer”  $K(\tau) \cap S_u$  to  $S_{u_n}$ , leaving the rest alone. Precisely,

$$K_n(\tau) := \cup_i (K(\tau_i) \setminus K(\tau'_i)) \cup (K(\tau) \setminus S_u) \cup \mathcal{I}_n(K(\tau))$$

and define  $\phi_n(\tau)$  to be the elastic minimizer subject to  $S_{\phi_n(\tau)} \subset K_n(\tau)$ .

- Second: show that all drops in energy for  $u$  come from drops in energy for  $u_n$ .
- Smaller issue: show that  $\varepsilon$ -stability implies local minimality (quick for fracture).



## $\varepsilon$ -stable $\Gamma$ convergence

We define a corresponding general notion of Gamma convergence, so that (with some assumptions on  $E_n$  and  $E$ )

### Theorem (Braides & L.)

If  $E_n \xrightarrow{s-\Gamma} E$ , then if  $S$  is the set

$$\{u : \exists \{u_n\}, \varepsilon > 0 \text{ such that } u_n \rightarrow u \text{ and } u_n \varepsilon\text{-stable for } E_n\},$$

we have

$$\{\text{strict local minimizers of } E\} \subset S \subset \{\text{local minimizers of } E\}.$$