

Quasi-static fracture evolution and local minimization

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The problem that we consider is to predict crack paths in a quasi-static setting. More precisely, we have an elastic material occupying a domain Ω and we suppose that the material is in equilibrium subject to a boundary condition $f(t)$. Then, if there is no crack, the displacement $u(t)$ minimizes

$$E_{el}(v) = \int_{\Omega} |\nabla v|^2 \quad \text{subject to } v = f(t) \text{ on } \partial\Omega$$

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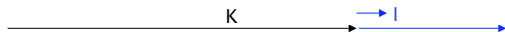
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If there is a fixed crack K , each displacement would solve the same Dirichlet problem, but in the space $H^1(\Omega \setminus K)$ instead of $H^1(\Omega)$, which implies that the stored elastic energy can only be lower if there is a crack.

Griffith's criterion

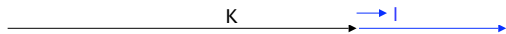
The starting point for predicting crack growth is Griffith's criterion (1920). Griffith considered a pre-existing crack K with a specified future path (here in blue).



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For a crack increment of length l , $E(l)$ is the elastic energy of the corresponding elastic equilibrium. The criterion states that the crack can only grow if the rate of decrease of elastic energy as l increases is large enough, i.e., if

$$\begin{aligned} -\frac{dE(l)}{dl} &< G_c && \text{the crack can not run} \\ &= G_c && \text{the crack can run} \\ &> G_c && \text{the crack is unstable.} \end{aligned}$$

The static problem

Formulated by Ambrosio and Braides (1995): If u minimizes

$$v \mapsto \int_{\Omega} |\nabla v|^2 + \mathcal{H}^1(S_v)$$

over $v \in SBV_f(\Omega)$, then the crack $K := S_u$ is stable (taking $G_c = 1$). The reason is that each increment in length l cannot reduce the energy, i.e.,

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Note: This and all that follows is in any dimension, but we will assume 2-D throughout.

Quasi-static formulation

This led to the following quasi-static formulation (Francfort-Marigo, 1998), where $f = f(t)$.

For discrete times $\{t_i\}$, $u(t_i)$ minimizes

$$v \mapsto \int_{\Omega} |\nabla v|^2 + \mathcal{H}^1(S_v \setminus \bigcup_{j < i} S_{u(t_j)})$$

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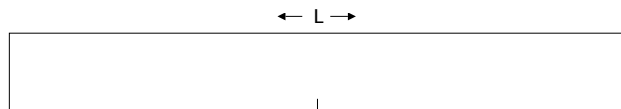
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The plan was then to take a sequence of discretizations $\{t_i^n\}$ with, e.g., $t_i^n - t_{i-1}^n = \frac{1}{n}$, resulting in a sequence $\{u_n\}$ that hopefully converges to a u that is a solution to a corresponding continuous-time problem. This has been carried out (Dal Maso, Francfort, L., Toader).

Connection to Griffith

The resulting solution $u(t)$ with $K(t) := \cup_{\tau \leq t} S_{u(\tau)}$ satisfies Griffith's criterion if $t \mapsto \mathcal{H}^1(K(t))$ is continuous.

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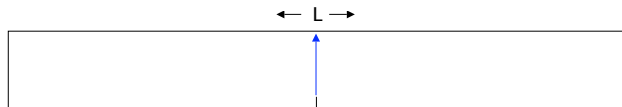


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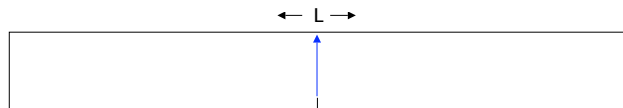


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Note the connection to local vs. global minimality – the initial crack was a local minimizer and was stable in the sense of Griffith.

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We want to have that if u is a strict local minimizer, then there is no accessible state with lower energy \iff strict local minimizers are stable.

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Unfortunately, there are technical difficulties in proving that, when $u_n(t) \rightarrow u(t)$, we have the properties we want for $u(t)$. In particular, local minimality is a problem.

In fact, the same problem occurs with accessibility – we would need to show that since the u_n were stable, so is u , i.e., if there is a v that is accessible from u and has lower energy, then there are v_n that are accessible from u_n and have lower energy.

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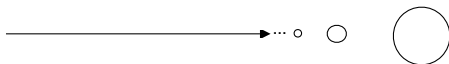
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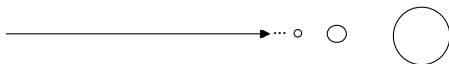
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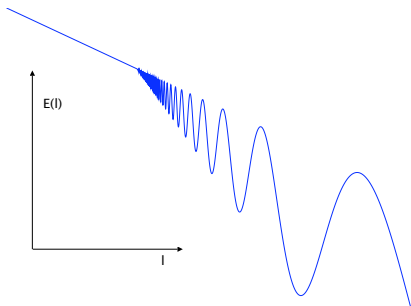


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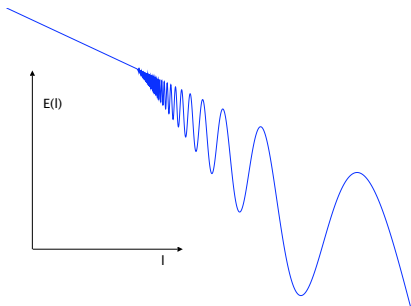


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so that we cannot move to the state that we want without the energy initially increasing by a small amount. Allowing small energy increases in our definitions of accessibility and stability overcomes all mathematical issues.

Definition (ε -accessible)

v is ε -accessible from u if there exists a continuous function $\phi: [0, 1] \rightarrow SBV(\Omega)$ such that $\phi(0) = u$, $\phi(1) = v$, and

$$\sup_{\tau_1 < \tau_2} [E(\phi(\tau_2)) - E(\phi(\tau_1))] < \varepsilon.$$

We then have the corresponding definition of stability:

Definition (ε -stability)

u is ε -stable if there does not exist an ε -accessible v with strictly lower energy. The path to such a v is called an ε -slide.

We also define $\bar{\varepsilon}$ -accessibility, where the inequality is not strict.

Existence theorem

Theorem

Given $f(t)$ with sufficient regularity, there exists a quasi-static evolution $u(t)$ with the properties of a globally minimizing evolution, modified as follows:

- $u(t)$ is a local minimizer at every t (coming from being $\bar{\varepsilon}$ -stable)
- Energy inequality:

$$E(u(t_2)) - E(u(t_1)) \leq \int_{t_1}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \dot{f} dx dt$$

for every $t_1 \leq t_2$.

- If $u^-(t) \neq u^+(t)$, then $u^+(t)$ is $\bar{\varepsilon}$ -accessible from $u^-(t)$ and has lower energy than all states that are ε -accessible from $u^-(t)$.

Proof.

- First issue: show that if $u_n \rightarrow u$ and there exists an ε -slide for u , then for n sufficiently large, there exists an ε -slide for u_n .

Strategy: For an ε -slide (ϕ, K) , “transfer” $K(\tau) \cap S_u$ to S_{u_n} , leaving the rest alone. Precisely,

$$K_n(\tau) := \cup_i (K(\tau_i) \setminus K(\tau'_i)) \cup (K(\tau) \setminus S_u) \cup \mathcal{I}_n(K(\tau))$$

and define $\phi_n(\tau)$ to be the elastic minimizer subject to $S_{\phi_n(\tau)} \subset K_n(\tau)$.

- Second: show that all drops in energy for u come from drops in energy for u_n .
- Smaller issue: show that ε -stability implies local minimality (quick for fracture).



ε -stable Γ convergence

We define a corresponding general notion of Gamma convergence, so that (with some assumptions on E_n and E)

Theorem (Braides & L.)

If $E_n \xrightarrow{s-\Gamma} E$, then if S is the set

$$\{u : \exists \{u_n\}, \varepsilon > 0 \text{ such that } u_n \rightarrow u \text{ and } u_n \varepsilon\text{-stable for } E_n\},$$

we have

$$\{\text{strict local minimizers of } E\} \subset S \subset \{\text{local minimizers of } E\}.$$