EPSILON-STABLE QUASI-STATIC BRITTLE FRACTURE EVOLUTION

CHRISTOPHER J. LARSEN

Abstract

We introduce a new definition of stability, ε-stability, that implies local minimality and is robust enough for passing from discrete-time to continuous-time quasi-static evolutions, even with very irregular energies. We use this to give the first existence result for quasi-static crack evolutions that both predicts crack paths and produces states that are local minimizers at every time, but not necessarily global minimizers. The key ingredient in our model is the physically reasonable property, absent in global minimization models, that whenever there is a jump in time from one state to another, there must be a continuous path from the earlier state to the later along which the energy is almost decreasing. It follows that these evolutions are much closer to satisfying Griffith's criterion for crack growth than are solutions based on global minimization, and initiation is more physical than in global minimization models.

1. Introduction

Recent mathematical progress on fracture ([9], [5], [6], [8], [4]) is based on an attempt to turn Griffith's criterion for crack growth into a well-posed model for predicting crack paths, at least in the quasi-static case (by "well-posed" we mean that existence can be shown – "Only a mathematical existence proof can ensure that the mathematical description of a physical phenomenon is meaningful." R. Courant, see [11].) While the basic underlying idea, that crack increments should be optimal in reducing stored elastic energy, had existed in the engineering community, there had not been a continuous-time model amenable to mathematical analysis until [9] (together with refinements introduced in [5]).

There is, however, one main flaw commonly acknowledged in this model: it rests on global minimization. This results in a non-locality (in space) that is, in particular, at times inconsistent with Griffith's criterion for crack growth. Attempts at addressing this, and extending existence results based on global minimization to results based instead on local minimization, have met with difficulties, and have either sacrificed local minimality or the goal of predicting crack paths. Dal Maso and Toader, in [6], design a procedure that, instead of globally minimizing the total energy at discrete times (and then passing to the continuous-time limit), follows certain (approximate) gradient flows to get from one discrete time to the next. They succeed in the prediction of crack paths, together with a type of minimality that is close to, but does not imply, local minimality for the continuous-time limit.

On the other hand, in [14, 12], cracks based on local minimality are obtained, but only by sacrificing the main original goal of predicting crack paths – these papers need to specify, a priori, the crack path and are only able to predict the speed with which the crack moves along that given path. Until now, there have been no results predicting crack paths for which the displacements are local minimizers but not necessarily global minimizers.

In this paper we take a somewhat new point of view, focussing on what we call *accessibility* and its relationship to *stability*. The idea is that, when going from one discrete time to the next, the total energy should be minimized (subject to whatever changed, e.g., the Dirichlet data, from the earlier discrete time to the next), but with the restriction that only accessible states are admissible. Viewed in this way, in global minimization, all states are accessible from all other states, and for [6], a state v is accessible from a state u if and only if a certain gradient flow beginning at u approaches v in the long-time limit. Our corresponding notion of stability is simply that u is stable if and only if there is no lower-energy state that is accessible from u. So, with global minimality, only global minimizers are stable, and with the [6] model, only states u for which certain gradient flows that begin at u stay at u, are stable.

Further, when passing to the continuous-time limit from the discrete-time, we expect solutions to have a corresponding property of minimality and accessibility with respect to solutions at previous times. This was missing in [6], where the continuous-time stability is weaker than the discrete-time version.

The difficulty faced in [6] was that the corresponding definitions of accessible and stable were too strong to be preserved when passing to continuous-time limits. That is, if $u_n \to u$ (in the sense of SBV compactness, which we describe below) and the u_n are stable, this does not imply that u is stable. Furthermore, if $u_n \to u$, $v_n \to v$, with each v_n accessible from the corresponding u_n, this does not imply that v is accessible from u. Note that with global minimality, there is no issue of accessibility, and the stability of u when u_n are stable is exactly the point of the Jump Transfer method we introduced in [8].

If we weaken the idea of gradient flows, and instead consider v to be accessible from u if there is a continuous path from u to v along which the total energy is nonincreasing, we see that there is some mathematical difficulty in trying to use Jump Transfer to conclude that the corresponding notion of stability is maintained when taking limits. In fact, examples indicate that this stability will not, in general, be maintained. However, we notice that for an arbitrary $\varepsilon > 0$, if our notion of accessibility is modified to allow paths for which the total energy never increases by more than ε, Jump Transfer can be extended to show that the corresponding definition of stability *is* maintained when taking limits.

1.1. **Griffith's criterion and current models.** Griffith's criterion for crack growth states that a crack can only grow if its energy release rate equals the fracture toughness of the material [10]. More precisely, we suppose that there is a prescribed future crack path $C(s)$ (a curve) parameterized by arc-length, with crack at time zero given by $C(s_0)$. Each $C(s)$ determines an elastic equilibrium $u(s)$ (subject to some given boundary loads or Dirichlet data) defined on $\Omega \setminus C(s)$, with stored elastic energy $E(s)$. We can then define the elastic *energy release rate* as $-\frac{d}{ds}E(s)$, as $E(\cdot)$ is decreasing. Griffith's criterion then states that for a given fracture toughness G_c:

(1) if $-\frac{d}{ds}E(s) < G_c$, the crack cannot run
(2) if $-\frac{d}{ds}E(s) = G_c$, the crack can run

(3) if $-\frac{d}{ds}E(s) > G_c$, the crack is unstable.

The essence of the Griffith approach is that these conditions, together with the assumption that the crack is never unstable, determine C as a function of time.

The central idea behind turning this criterion into a method for predicting crack paths, and not just for determining whether a crack runs along a given path, is the fact that underlying the criterion is an energy comparison: $-\Delta E$ vs. $G_c \Delta s$. This led to the [9] approach: first, consider discrete times $0 = t_1 < t_2 < \ldots < t_n = T$, with $t_{i+1} - t_i = \Delta t$, and to find $u(t_i), C(t_i)$, minimize

$$(u, C) \mapsto \mathcal{E}(u, C \cup C(t_{i-1})) := \int_\Omega W(\nabla u)dx + G_c \mathcal{H}^{N-1}(C \cup C(t_{i-1}))$$

with $u \in H^1(\Omega \setminus C)$. Here the elastic energy is given by

$$E_{el}(u) := \int_\Omega W(\nabla u)dx.$$

Minimizing $\mathcal{E}$ then reflects Griffith's criterion: the crack path will grow by Δs only if the resulting elastic energy drop is at least $G_c \Delta s$. This formulation has the distinct advantage of using the energy to *choose* the crack path. Of course, the original criterion involves only a local comparison: Griffith's criterion concerns whether, for *infinitesimal* Δs, $-\Delta E \geq G_c \Delta s$. Not surprisingly, minimizing (globally) $\mathcal{E}$ does not necessarily result in crack growth that satisfies Griffith's criterion, as we illustrate below.

Example 1.1 (Long-bar paradox)**.** *Consider a rectangular domain Ω with height 1 and length L, with Dirichlet data at the ends such that the elastic equilibrium u (without a crack) satisfies $|\nabla u| \equiv \delta$. We see that for $\delta > 0$ arbitrarily small but fixed, by increasing L, the elastic energy in Ω can be made arbitrarily large. Therefore, minimizing $\mathcal{E}$ would prefer a vertical crack of length 1, resulting in a stored elastic energy of zero at a cost of only G_c.*

But, it is not hard to show that for δ small enough (independent of L), the energy release rate is small, and in particular, can be made to be less than G_c (see [3] *for a detailed study). Therefore, while globally minimizing $\mathcal{E}$ will result in a crack, this violates Griffith's criterion.*

More precisely we note that if cracks were forced to grow continuously in this example, then energy would increase initially, which is why such cracks are ruled out in [3]*. It follows that even if small energy increases were allowed, say by an amount ε, as the crack grew continuously, this crack growth would still be impossible for ε small enough.*

It is therefore natural to attempt a formulation based on local minimality, instead of global. First, we note that from now on, we will take $W(\cdot) := \frac{1}{2}|\cdot|^2$ (and $G_c = 1$), so that the ideas we describe are in the simplest reasonable setting, in the same spirit that in [8] we introduced Jump Transfer in a simple setting.

In more detail, the definition of globally minimizing quasi-static fracture evolution is:

(1) $(u(0), C(0))$ minimizes $\mathcal{E}$ (subject to boundary condition $g(0)$)
(2) $\mathcal{E}(u(t), C(t)) \leq \mathcal{E}(v, K)$ for all $v \in H^1(\Omega \setminus K)$, $K \supset C(t)$ (subject to boundary condition $g(t)$)
(3) $\mathcal{E}(u(t), C(t)) = \mathcal{E}(u(0), C(0)) + \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s) dx ds$,

where the last item represents energy conservation (see [5]). For a locally minimizing solution, we expect to have:

(1) $(u(0), C(0))$ locally minimizes $\mathcal{E}$ (subject to boundary condition $g(0)$)
(2) $\mathcal{E}(u(t), C(t)) \leq \mathcal{E}(v, K)$ for all $v \in H^1(\Omega \setminus K)$, $K \supset C(t)$ "close enough" to $(u(t), C(t))$ (subject to boundary condition $g(t)$)
(3) $\mathcal{E}(u(t_2), C(t_2)) \leq \mathcal{E}(u(t_1), C(t_1)) + \int_{t_1}^{t_2} \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s) dx ds$, for all $t_1 < t_2$.

Items (1) and (2) are natural, and item (3) comes from the fact, which we will examine later, that when a system evolves based on local minimality, it might at some point be in a local but not global minimum, yet at a later time the system might gain "access" to the global minimizer, resulting in an energy drop. Therefore, we only have the one inequality expressed in (3).

There is an immediate problem with adopting this definition of locally-minimizing quasi-static evolution: globally minimizing evolutions also satisfy it. What property is it that locally minimizing evolutions should have, and globally minimizing solutions do not have? It cannot be that the solution is a local but not a global minimizer, since, for example, at some times there might only exist a global minimizer. Looking at Example 1.1, it is when the solution jumps (in time) that the solution becomes non-physical and has a Griffith violation. These jumps are non-physical because global minimization has no regard for *how* the system got from one state to the next. If we follow a (continuous) path from the earlier state to the later, and insist that the energy must be decreasing along that path, then the jump from no crack to a complete crack in Example 1.1 would be ruled out, unless the applied strain were large enough and *Griffith's criterion were satisfied*. However, as the example below shows, there is also a problem in implementing this definition of accessibility.

Example 1.2. *Consider Ω to be the square $(-1, 1) \times (-1, 1)$ with pre-existing crack $K_0 := (-1, 0) \times \{0\}$. We suppose that Dirichlet conditions $-t$ along $(-1, 1) \times \{-1\}$ and t along $(-1, 1) \times \{1\}$ are applied, resulting in an energy release rate at the origin exceeding G_c for $t > t_c$. We further suppose that the resulting quasi-static crack runs along $[0, 1) \times \{0\}$.*

We then consider adding a disjoint sequence of circles to K_0, centered at $(x_i, 0)$, $x_i > 0$, with radii r_i going to zero as x_i goes to zero (in particular, we suppose $r_i = x_i^2$). A straightforward calculation shows that this only increases the energy release rate at the origin.

The question then is, what should a locally-minimizing quasi-static crack do as t passes t_c? If we insist that the crack can only follow paths of nonincreasing energy, then it cannot grow along $[0, 1) \times \{0\}$, since whenever the crack tip leaves a circle, the total energy increases because there is no singularity in ∇u (see [3] for details). Yet, the crack cannot be considered stable, since if we consider growing the crack along the curve $y = 2x^2$, we get the same energy release rate as for growing a straight crack, but bypass the circles. Of course, this path is not optimal either, as growing along $y = 3/2x^2$ would result in even lower energy.

Addressing this question is an issue of modeling, and here we take the point of view that the crack runs along the x-axis until it reaches a certain circle of small but finite size, and then is considered stable. As we will see below in the discussion of the mathematics, making this choice is exactly what is necessary to overcome the difficulties illustrated by this example, as well as those encountered in, e.g., [6].

1.2. **Mathematical Preliminaries.** The spaces $BV(\Omega)$ and $SBV(\Omega)$, $\Omega \subset \mathbb{R}^N$ bounded and Lipschitz, are defined in the usual way, see [2]. We also use the usual notation, found in [2], for the jump set S_u for a BV function u, ν the normal to the jump set, etc. We set $SBV_q(\Omega) := \{v \in SBV(\Omega) : \nabla v \in L^q(\Omega)\}$. We say that $u_n \overset{SBV}{\rightharpoonup} u$, or u_n converges to u in the sense of SBV convergence if

$$\begin{cases} \nabla u_n & \rightharpoonup & \nabla u \text{ in } L^1(\Omega); \\ [u_n]\nu_n \mathcal{H}^{N-1}\lfloor S_{u_n} & \overset{*}{\rightharpoonup} & [u]\nu\mathcal{H}^{N-1}\lfloor S_u \text{ as measures;} \\ u_n & \to & u \text{ in } L^1(\Omega); \text{ and} \\ u_n & \overset{*}{\rightharpoonup} & u \text{ in } L^\infty(\Omega). \end{cases}$$

The meaning of boundary conditions is a little unusual in fracture, since one must allow cracks to form along $\partial\Omega$. Therefore, one can either say that Dirichlet conditions can be ignored on part of $\partial\Omega$ at a cost of $\mathcal{H}^{N-1}$ of that part, or we can consider $\Omega \subset\subset \Omega'$ for some Ω', and consider $SBV(\Omega')$, with the constraint $S_u \subset \overline{\Omega}$. For simplicity, we adopt the latter, although we generally will just refer to $SBV(\Omega)$ and Dirichlet data.

Continuity is with respect to the L^1 strong topology, as is local minimality, although in this context these are essentially equivalent to, e.g., L^2, or the topology of SBV convergence.

The measure theoretic boundary of a set $A \subset \Omega$, $\partial_* A$ and its reduced boundary $\partial^* A$, are defined as in [7] and [15]. The t-super level set for a given function u, E_t, is defined by

$$E_t := \{x : u(x) > t\}.$$

For simplicity of notation, we usually write $S_v \subset C$ for $\mathcal{H}^{N-1}(S_v \setminus C) = 0$ and $S_v = C$ for $\mathcal{H}^{N-1}(S_v \triangle C) = 0$ (That is, we identify sets that are equal up to $\mathcal{H}^{N-1}$-measure zero in these relations). Also, from now on, we will write the usual $E(u, C)$ instead $\mathcal{E}(u, C)$.

2. Epsilon-Slides and Epsilon-Stability

Definition 2.1 (Slides)**.** *We say that a continuous map $\phi : [0,1] \to SBV$ is a* slide *for $u \in SBV$ if $\phi(0) = u$, $E_\phi(\tau_2) \le E_\phi(\tau_1)$ for all $\tau_1 < \tau_2$, and $E_\phi(0) > E_\phi(1)$.*

Here,

$$E_\phi(\tau) := E_{el}(\phi(\tau)) + \mathcal{H}^{N-1}\Big(C_\phi(\tau)\Big)$$

and

$$C_\phi(\tau) := \bigcup_{\substack{s\in\mathbb{Q} \\ s\le\tau}} S_{\phi(s)}.$$

We will also consider energies corresponding to

$$C^\Gamma_\phi(\tau) := \Gamma \cup \bigcup_{\substack{s\in\mathbb{Q} \\ s\le\tau}} S_{\phi(s)},$$

in which case we will refer to a *slide with respect to* Γ (though, when it seems clear, we will drop the ϕ subscript and Γ superscript).

Lemma 2.2. *For any slide ϕ, C_ϕ satisfies the following properties:*

(1) C_ϕ *is increasing* ($C_\phi(\tau_1) \subset C_\phi(\tau_2)$ *if* $\tau_1 < \tau_2$)
(2) $S_{\phi(\tau)} \subset C_\phi(\tau)$ *for all* τ
(3) C_ϕ *is the smallest (in the sense of inclusion) function satisfying the previous two conditions.*

Proof. In fact, it follows from the proof of Lemma 6.6 in [13] that, without any continuity assumption on ϕ, if there exists a bounded set function satisfying (1) and (2), then there exists a countable set $\mathcal{D}$ such that C_ϕ defined by

$$C_\phi(\tau) := \bigcup_{\substack{s\in\mathcal{D}\\ s\leq\tau}} S_{\phi(s)}$$

satisfies (1)-(3). Here, we can use the continuity of ϕ to prove the lemma holds using any countable dense set, for example $\mathbb{Q}$. (1) and (3) are immediate from the definition, so the only issue is (2), which follows from SBV compactness and the continuity of ϕ (see the proof of (3.20) in [8] for details). $\square$

As noted in the introduction, and in particular in Example 1.2, requiring accessibility to depend on the existence of slides will not work without regularity assumptions. We therefore introduce the following slides, which we will see allow us to overcome all mathematical difficulties:

Definition 2.3 (Epsilon-Slides)**.** *We say that a continuous map* $\phi\colon[0,1]\to SBV$ *is an* ε-slide *for* $u \in SBV$ *if* $\phi(0) = u$,

$$\sup_{\tau_1<\tau_2} \big[E_\phi(\tau_2) - E_\phi(\tau_1)\big] < \varepsilon, \tag{2.1}$$

and $E_\phi(0) > E_\phi(1)$*. A continuous map* $\phi\colon[0,1]\to SBV$ *is an* $\bar\varepsilon$-slide *for* $u \in SBV$ *if* $\phi(0) = u$,

$$\sup_{\tau_1<\tau_2} \big[E_\phi(\tau_2) - E_\phi(\tau_1)\big] \leq \varepsilon, \tag{2.2}$$

and $E_\phi(0) > E_\phi(1)$*.*

Of course, Lemma 2.2 holds for ε and $\bar\varepsilon$-slides as well as for slides. We also define ε slide with respect to Γ as we did for slides.

In what follows, we will further require ε-slides for a given u to respect the boundary conditions of u ($\phi(\tau) = u$ on $\Omega' \setminus \overline{\Omega}$). Also, for simplicity, we will without loss of generality assume that $\tau \mapsto \mathcal{H}^{N-1}(C_\phi(\tau))$ is affine.

Definition 2.4 (Epsilon Stability)**.** $u \in SBV$ *is* ε-stable *if it has no ε-slides, and* $\bar\varepsilon$-stable *if it has no $\bar\varepsilon$-slides.*

Definition 2.5 (Epsilon-Stable Fracture)**.** (u, C) *is an* ε-stable fracture *if*

(1) $t \mapsto C(t)$ *is monotonic*
(2) $u(t) = g(t)$ *on* $\partial\Omega \setminus C(t)$ *for all* $t \in [0, T]$
(3) $u(t)$ *is ε-stable (and a local minimizer) with respect to* $C(t)$ *for every* $t \in [0, T]$
(4) *if* $(u(t), C(t))^- \neq (u(t), C(t))^+$*, then there exists an $\bar\varepsilon$-slide with respect to* $C(t)^-$ *from* $u^-(t)$ *to* $u^+(t)$*. Furthermore,* $E(u^+(t), C^+(t)) \leq E(v, C_\phi(1))$ *for every v that is ε-accessible (with ε-slide ϕ) from $u^-(t)$ with respect to* $C(t)^-$*. Here,*

$$u(t)^- := \lim_{s\to t^-} u(s),\ C(t)^- := \bigcup_{s<t} C(s),\ \textit{etc.}$$

(5) *for every* $t_1 < t_2$,

$$E(u(t_2), C(t_2)) - E(u(t_1), C(t_1)) \le \int_{t_1}^{t_2} \int_\Omega \nabla u \cdot \nabla \dot{g} dx dt.$$

Remark 2.6. *It is immediate that if* ϕ *is an* ε*-slide from* u *to* v*, then there exists* $\delta > 0$ *such that* ϕ *is a* $(\varepsilon - \delta)$*-slide from* u *to* v*.*

3. The Algorithm

(1) We consider a countable and dense subset I_∞ in $[0, T]$, and, for each $n \in \mathbb{N}$, a subset $I_n = \{t_0^n = 0 < t_1^n < ... < t_n^n\}$, such that $\{I_n\}$ form an increasing sequence of nested sets whose union is I_∞. We set $\Delta_n := \sup_{k \in \{1,...,n\}} (t_k^n - t_{k-1}^n)$. Note that $\Delta_n \searrow 0$. As seen in the introduction we are given boundary data $g \in W^{1,\infty}((0,1); H^1(\Omega')) \cap L^\infty((0,1) \times \Omega')$, so that at time t, the admissible fields v should satisfy $v = g(t)$ in $\Omega' \setminus \overline{\Omega}$.

(2) We define the nth crack set at time t_k^n, $C_n(t_k^n)$, recursively by

$$C_n(t_k^n) := C_n(t_{k-1}^n) \cup C_{\phi_k^n}(1)$$

with $C_n(t_{-1}^n) = \emptyset$ and ϕ_k^n defined as follows. Set v_k^n to be the minimizer of $E_{el}(v)$ over v satisfying $v = g(t_k^n)$ on $\Omega' \setminus \overline{\Omega}$ and $S_v \subset C_n(t_{k-1}^n)$. We now choose $u_n(t_k^n)$ and ϕ_k^n to be a minimizer of

$$(u, \phi) \mapsto E_{el}(u) + \mathcal{H}^{N-1}(C_\phi(1) \setminus C_n(t_{k-1}^n))$$

over all functions u that are $\bar{\varepsilon}$-accessible from v_k^n (with the same Dirichlet conditions) with ϕ a corresponding $\bar{\varepsilon}$-slide ($\phi(\tau) = v_k^n$ on $\Omega' \setminus \overline{\Omega}$). The existence of such minimizers follows from the SBV compactness theorem of [1] and the proof of lemma 6.2. It follows quickly that each $u_n(t_k^n)$ is ε-stable.

(3) The basic idea is to take a diagonal subsequence such that $u_n(t)$ converges for each $t \in I_\infty$, and we call the limit $u(t)$. The limit $C(t)$ is a little bit more complicated, taking into account discontinuity sets along ε-slides, in addition to the discontinuity sets of u.

We then claim that the resulting (u, C) is a locally minimizing ε-stable fracture.

Theorem 3.1. *Given* $g \in W^{1,\infty}((0,1); H^1(\Omega')) \cap L^\infty((0,1) \times \Omega')$ *and* $\varepsilon > 0$*, there exists an* ε*-stable quasi-static evolution for* g*.*

The proof follows the above outline. We begin with some preliminary lemmas on epsilon-slides and stability, their "transferability", and their relation to local minimality.

4. Properties of epsilon-stability

We will find the following lemma useful, which gives a characterization of when an ε or $\bar{\varepsilon}$-slide exists, though for simplicity we state and prove it for $\bar{\varepsilon}$-slides.

Lemma 4.1. *There exists an* $\bar{\varepsilon}$*-slide with respect to* Γ *from* u *to* v*, where* $S_u \subset \Gamma$ *and* u *and* v *are in elastic equilibrium with* $E(v, \Gamma \cup S_v) < E(u, \Gamma)$*, if and only if there exists a pair* (ϕ, C) *with* $\phi : [0,1] \to SBV(\Omega)$*,* C *a crack set for* ϕ *(i.e.,* C *satisfies properties 1 and 2 in Lemma 2.2 with respect to* ϕ*),* $\phi(0) = u$*,*

(1) $\tau \mapsto \mathcal{H}^{N-1}(C(\tau))$ *is continuous,*

(2) $S_v \subset C(1)$*, and*

(3) *for all* τ,

$$E_{el}(\phi(\tau)) + \mathcal{H}^{N-1}(\Gamma \cup C(\tau)) \le E_{el}(u) + \mathcal{H}^{N-1}(\Gamma) + \varepsilon.$$

Proof. Again for simplicity we take $\Gamma = \emptyset$. Suppose first that we have an $\bar{\varepsilon}$-slide ψ from u to v. Then ψ automatically satisfies conditions 2 and 3, but not necessarily condition 1. We show that we can construct a ϕ that satisfies condition 1, while still satisfying the other two. This construction will be in stages.

In the first stage, we simply set $\phi_1(\tau)$ to be the elastic minimizer in $SBV(\Omega)$ subject to the constraint that $\mathcal{H}^{N-1}(S_{\phi_1(\tau)} \setminus C_\psi(\tau)) = 0$. It is then immediate that $E(\phi_1(\tau), C_\psi(\tau)) \le E_\psi(\psi(\tau), C_\psi(\tau))$ for all τ. In particular, both conditions (2) and (3) are satisfied. We suppose that $\tau \mapsto \mathcal{H}^{N-1}(C_\psi(\tau))$ has a jump discontinuity at τ' (without loss of generality, we assume that $\mathcal{H}^{N-1}(C_\psi(\tau') \setminus C_\psi(\tau)) \to 0$ as $\tau \to \tau'$ from below, i.e., we have continuity from the left). Define ϕ_2 by $\phi_2(\tau) := \phi_1(2\tau)$ for $\tau \le \frac{1}{2}\tau'$, and $\phi_2(\tau) = \phi_1(\tau)$ for $\tau > \tau'$. Then we choose a point $x \in \Omega$ such that each circle centered at x intersects $C_\psi(1)$ on a set of $\mathcal{H}^{N-1}$ measure zero. Choosing $R > 0$ such that $\Omega \subset B(x, R)$, we define $\phi_2(\tau)$ for $\tau \in (\frac{1}{2}\tau', \tau']$ to be the minimizer of E_{el} over functions with jump set inside

$$C(\tau) := C_\psi(\tau') \cup \left(\bigcap_{s > \tau'} C_\psi(s) \cap B\Big(x, \frac{2\tau - \tau'}{\tau'} R\Big) \right).$$

This removes the jump discontinuity at τ', but now we need to check that condition (3) is still satisfied (condition (2) is unaltered). The only possibility of a violation of condition (3) is for $\tau \in (\frac{1}{2}\tau', \tau']$. We note that for such τ, $E_{el}(\phi_2(\tau)) \le E_{el}(\psi(\tau'))$ by the minimality of ϕ_2 since $C(\tau) \supset C_\psi(\tau')$. We also have that $\psi(\tau') = \lim_{s \to \tau'} \psi(s)$ by continuity of ψ, and so

$$E_{el}(\psi(\tau')) \le \lim_{s \to \tau'} E_{el}(\psi(s)).$$

By monotonicity of C, we get the inequality

$$\mathcal{H}^{N-1}(C_{\phi_2}(\tau)) \le \lim_{s \searrow \tau'} \mathcal{H}^{N-1}(C_\psi(s))$$

so that

$$E(\phi_2(\tau), C_{\phi_2}(\tau)) \le \lim_{s \searrow \tau'} E(\psi(s), C_\psi(s)),$$

which implies condition (3). A similar alteration can be done for each jump discontinuity of $\tau \mapsto \mathcal{H}^{N-1}(C_\psi(\tau))$ (in order of decreasing jump size), resulting in a ϕ_2 such that $\tau \mapsto \mathcal{H}^{N-1}(C_{\phi_2}(\tau))$ is continuous, and ϕ_2 satisfies (2) and (3).

Now we suppose that we have a pair (ϕ, C) satisfying conditions (1)-(3), and we assume, without loss of generality, that at each τ, $\phi(\tau)$ minimizes the elastic energy over functions in $SBV(\Omega)$ with jump set in $C(\tau)$. If we can alter ϕ, creating ψ, so that $\tau \mapsto E(\psi(\tau), C(\tau))$ is increasing on an interval $[0, d]$ and decreasing on $[d, 1]$, and ψ still satisfies (1)-(3) and is continuous, then it will be an $\bar{\varepsilon}$-slide.

There are many ways to accomplish this monotonicity, and we outline here a simple one. Let $w \in H^1_0(\Omega)$, $w \neq 0$, and for each τ we will choose $\gamma(\tau) \in \mathbb{R}$ such that $\psi(\tau) := \phi(\tau) + \gamma(\tau) w$ has the properties we want. First, a calculation shows that

$$E_{el}(\psi(\tau)) = E_{el}(\phi(\tau)) + \gamma(\tau)^2 E_{el}(w). \tag{4.3}$$

Choose $d \in [0,1]$ such that $\limsup_{s\to d} E(\phi(s), C(s)) = \sup E(\phi(\cdot), C(\cdot))$ and for each τ choose $\gamma(\tau)$ so that

$$E(\psi(\tau), C(\tau)) = \begin{cases} \sup\limits_{s\le\tau} E(\phi(s), C(s)) & \text{if } \tau < d \\ \sup\limits_{s\ge\tau} E(\phi(s), C(s)) & \text{if } \tau \ge d. \end{cases}$$

This ψ is not necessarily continuous, but from its definition and (4.3) it is continuous wherever both ϕ and $E(\phi(\cdot), C(\cdot))$ (which we denote by $E \circ \phi$) are continuous. We claim also that if $E \circ \phi$ is continuous at some τ', then so is ϕ. To see this, suppose $\tau_n \to \tau'$ and $E_{el}(\phi(\tau_n)) \to E_{el}(\phi(\tau'))$. Take a convergent subsequence so that $\phi(\tau_n) \to \phi'$ in SBV. Then

$$E_{el}(\phi') \le \liminf_{n\to\infty} E_{el}(\phi(\tau_n)) = E_{el}(\phi(\tau'))$$

and $S(\phi') \subset C_\phi(\tau')$ by the continuity of $\mathcal{H}^{N-1}(C_\phi(\cdot))$ and SBV compactness. By the minimality of $\phi(\tau')$ and the convexity of E_{el}, we have $\phi(\tau') = \phi'$. Therefore, ϕ is continuous at τ'.

Next, we note that $E_{el}(\phi(\cdot))$ is nonincreasing, so it has only a countable number of (jump) discontinuities. $E \circ \phi$ must be continuous from the right, since if $\tau_n \searrow \tau'$ and $\phi(\tau_n) \to \phi'$, we have

$$E_{el}(\phi') \le \liminf_{n\to\infty} E_{el}(\phi(\tau_n))$$

and $E_{el}(\phi(\tau')) \le E_{el}(\phi')$ by the minimality of $\phi(\tau')$, as above. But, since $C_\phi(\tau') \subset C_\phi(\tau_n)$, we have $E_{el}(\phi(\tau')) \ge E_{el}(\phi(\tau_n))$. So

$$E_{el}(\phi(\tau')) = \liminf_{n\to\infty} E_{el}(\phi(\tau_n)).$$

We now go back and alter ϕ to remove these discontinuities, without altering its other properties. For each τ' at which there is a jump discontinuity, we insert a time interval into $[0,1]$ so that the series of all the lengths of these intervals is summable. The interval corresponding to τ' is of the form $[\tau', \tau'']$. We define $\phi_1(\tau') := \lim_{s\nearrow\tau'} \phi(s)$ and $\phi_1(\tau'') := \phi(\tau')$. Between τ' and τ'', ϕ_1 is given by the corresponding convex combination of $\phi_1(\tau')$ and $\phi_1(\tau'')$. This ϕ_1 is continuous, and so we create ψ as before, which is now an $\bar{\varepsilon}$-slide. $\square$

We then have the following theorem, from [8], to which we will have to make some relatively small alterations.

Theorem 4.2 (Jump Transfer)**.** *Let $\overline{\Omega} \subset \Omega'$, with $\partial\Omega$ Lipschitz, and let $\{u_n\} \subset SBV(\Omega')$ be such that*

- $S(u_n) \subset \overline{\Omega}$*;*
- $|\nabla u_n|$ *weakly converges in* $L^1(\Omega')$*; and*
- $u_n \to u$ *in* $L^1(\Omega')$*,*

where $u \in BV(\Omega')$ with $\mathcal{H}^{N-1}(S(u)) < \infty$. Then, for every $\varphi \in SBV_q(\Omega')$, $1 \le q < \infty$, with $\mathcal{H}^{N-1}(S(\varphi)) < \infty$, there exists a sequence $\{\varphi_n\} \subset SBV_q(\Omega')$ with $\varphi_n = \varphi$ on $\Omega' \setminus \overline{\Omega}$ such that

i) $\varphi_n \to \varphi$ *strongly in* $L^1(\Omega')$*;*
ii) $\nabla\varphi_n \to \nabla\varphi$ *strongly in* $L^q(\Omega')$*; and*
iii) $\mathcal{H}^{N-1}\Big([S(\varphi_n) \setminus S(u_n)] \setminus [S(\varphi) \setminus S(u)]\Big) \to 0.$

To prove lemma 4.4, which is the basis for much of this analysis, we need the following extension of Jump Transfer.

Remark 4.3. *We make a small refinement to the proof of Jump Transfer in* [8], *assuming the reader has familiarity with that somewhat lengthy proof and its notation. The starting point is a sequence $\{u_n\}$ of SBV functions that converges, in the sense of SBV compactness, to $u \in SBV$. For an arbitrary $\varphi \in SBV_q$, we construct $\varphi_n \in SBV_q$ in the following way:*

(1) *$\varphi_n = \varphi$ outside T_n, a set with $O(n)$ measure, which is a union of cubes $\cup_i Q_i^n$ that almost covers S_u and which comes from a covering argument. In fact, $\varphi_n = \varphi$ outside $\cup_i R_i^n$, where each R_i^n is a thin neighborhood of a hyperplane inside Q_i^n;*
(2) *$\varphi_n = r(\varphi)$ inside $\cup_i R_i^n$, where $r(\varphi)$ denotes a certain reflection of φ. The point is that the values φ_n takes in R_i^n come from values φ takes in Q_i^n (and similarly for $\nabla\varphi_n$).*

This construction potentially results in an increase in $\mathcal{H}^{N-1}(S_{\varphi_n} \setminus S_{u_n})$ over $\mathcal{H}^{N-1}(S_\varphi \setminus S_u)$ as follows. The reflection moves most of $S_\varphi \cap S_u \cap Q_i^n$ into $\partial_* E_{t_i}^n \cap Q_i^n$ for an appropriate level set $E_{t_i}^n$ of u_n. Precisely, we need to control:

(1') *$\partial_* E_{t_i}^n \cap Q_i^n \setminus S_{u_n}$ (which depends on u_n and not φ);*
(2') *potential new jumps created on the boundaries of Q_i^n due to redefining φ inside R_i^n (which again depends on the cover, and not φ);*
(3') *$S_{r(\varphi)}$, the jump set of φ_n coming from the reflection of φ inside $Q_i^n \setminus R_i^n$.*

The proof of Jump Transfer shows that (1') and (2') can be made to have arbitrarily small $\mathcal{H}^{N-1}$-measure independently of φ. *Only (3') depends on the particular φ. What we will want to do below is simultaneously apply the Jump Transfer construction to (ε-slides) $\varphi(\tau)$, $\tau \in [0,1]$ so that*

$$\mathcal{H}^{N-1}(C_{\varphi_n}(1) \cap T_n \setminus S_{u_n}) \leq O(n). \tag{4.4}$$

In fact, since $C_{\varphi_n}(\tau) = \cup_{\substack{\tau' \in D \\ \tau' \leq \tau}} S_{\varphi_n(\tau')}$ and (1') and (2') are independent of τ, the only issue is controlling (3') uniformly in τ. This is easily accomplished using the fact that $C_\varphi(1) \setminus S_u$ has zero $\mathcal{H}^{N-1}\lfloor S_u$ density $\mathcal{H}^{N-1}$-a.e. in S_u, so that we can originally choose the cover such that $\mathcal{H}^{N-1}(C_\varphi(1) \cap T_n \setminus S_u)$ is arbitrarily small (this is the same as [8] *equation (2.3) 6., with S_φ replaced by $C_\varphi(1)$).*

Lemma 4.4. *If $u_n \overset{SBV}{\rightharpoonup} u$, $v_n \overset{SBV}{\rightharpoonup} v$, and ϕ is an ε-slide for u with respect to S_v, then for n large enough, there exists an ε-slide for u_n with respect to S_{v_n}.*

Proof. Let ϕ be an ε-slide for u with respect to S_v, so that in particular it is also an $(\varepsilon - \delta)$-slide for some $\delta > 0$. Without loss of generality, assume also that each $\phi(\tau)$ minimizes the stored elastic energy E_{el} subject to its boundary conditions and

$$S_{\phi(\tau)} \subset C(\tau),$$

where $C(\tau) := S_v \cup C_\phi(\tau)$. Choose $\alpha \in \mathbb{R}$ such that, setting $w := u + \alpha v$, we have $S_w = S_u \cup S_v$ up to a set of $\mathcal{H}^{N-1}$-measure zero (see the proof of Lemma 3.1 in [8]). Note that $w_n := u_n + \alpha v_n \overset{SBV}{\rightharpoonup} w$ and $S_{w_n} \subset S_{u_n} \cup S_{v_n}$. From Remark 4.3, we can apply Jump Transfer simultaneously to $\phi(\tau)$ for all $\tau \in [0,1]$, creating $\phi_n(\tau)$ and a sequence of sets $\{T_n\}$ such that $\{\phi_n(\tau) \neq \phi(\tau)\} \subset T_n$, where $|T_n| \to 0$,

$$\int_{T_n} |\nabla\phi_n(\tau)|^2 \leq 2\int_{T_n} |\nabla\phi(\tau)|^2 \quad \forall\tau \in [0,1] \text{ by Remark 4.3 (2),} \tag{4.5}$$

and

$$\mathcal{H}^{N-1}([C_{\phi_n}(1) \setminus S_{w_n}] \cap T_n) \leq O(n) \text{ repeating (4.4).} \tag{4.6}$$

Then define $\phi'_n(\tau)$ to be the minimizer of E_{el} over functions with jump sets in $C_n(\tau) := S_{v_n} \cup C_{\phi_n}(\tau)$. We can suppose that $\tau \mapsto \mathcal{H}^{N-1}(C(\tau))$ is continuous, using lemma 4.1.

We now wish to show that ϕ'_n is an ε-slide for u_n with respect to S_{v_n} if n is large enough, but we will see that a small refinement is still necessary. Again from Lemma 4.1 it is enough to show that

$$\limsup_{n\to\infty} [E(\phi'_n(\tau), C_n(\tau)) - E(u_n, C_n(0))] \leq E(\phi(\tau), C(\tau)) - E(u, C(0))$$

uniformly for $\tau \in [0,1]$. Notice first that

(4.7)

$$\begin{aligned}
\mathcal{H}^{N-1}(C_n(\tau)) - \mathcal{H}^{N-1}(C_n(0)) &= \mathcal{H}^{N-1}(C_n(\tau) \backslash C_n(0)) \\
&\leq \mathcal{H}^{N-1}([C_n(1) \setminus S_{u_n}] \cap T_n) + \mathcal{H}^{N-1}(C(\tau) \setminus C(0)) \\
&\leq O(n) + \mathcal{H}^{N-1}(C(\tau) \setminus C(0)) \\
&= O(n) + \mathcal{H}^{N-1}(C(\tau)) - \mathcal{H}^{N-1}(C(0)),
\end{aligned}$$

where $O(n)$ is independent of τ.

Now, we also want to show that

$$\limsup_{n\to\infty} \int_\Omega |\nabla \phi'_n(\tau)|^2 < \int_\Omega |\nabla \phi(\tau)|^2 + \delta \tag{4.8}$$

uniformly in τ. We suppose first that

$$\limsup_{n\to\infty} \sup_\tau \int_{T_n} |\nabla \phi(\tau)|^2 < \delta/4, \tag{4.9}$$

which is a weakened version of equi-integrability since $|T_n| \to 0$. Then using (4.5) and (4.9) we have

$$\int_\Omega |\nabla \phi_n(\tau)|^2 \leq \int_\Omega |\nabla \phi(\tau)|^2 + \delta/2 + O(n),$$

so that, by the minimality of $\phi'_n(\tau)$, we get

$$\int_\Omega |\nabla \phi'_n(\tau)|^2 \leq \int_\Omega |\nabla \phi(\tau)|^2 + \delta/2 + O(n) \tag{4.10}$$

uniformly in τ, giving (4.8).

Now we need to consider what happens if (4.9) is not satisfied. In this case, there exists at least one sequence $\{\tau_n\}$ such that

$$\limsup_{n\to\infty} \int_{T_n} |\nabla \phi(\tau_n)|^2 \geq \delta/4$$

and without loss of generality we can assume that $\{\tau_n\}$ is monotonic and the above limit sup is a limit. We now show that such a sequence cannot be decreasing. If $\tau_n \searrow \tau$, then setting ψ to be the weak limit in SBV of (a subsequence of) $\phi(\tau_n)$, we have

$$\int_\Omega |\nabla \phi(\tau)|^2 \leq \int_\Omega |\nabla \psi|^2 \leq \liminf_{n\to\infty} \int_\Omega |\nabla \phi(\tau_n)|^2 - \delta/4,$$

where we use the minimality of $\phi(\tau)$ together with the fact that $S_\psi \subset C(\tau)$ by SBV compactness, and $|T_n| \to 0$. But

$$\int_\Omega |\nabla \phi(\tau_n)|^2 \leq \int_\Omega |\nabla \phi(\tau)|^2$$

for all $n \in \mathbb{N}$ by the minimality of $\phi(\tau_n)$, since $C(\tau) \subset C(\tau_n)$, a contradiction.

Hence we only need to consider sequences $\tau_n \nearrow \tau$ for some $\tau \in (0,1]$, and we note as above that for these sequences,

$$\int_\Omega |\nabla \phi(\tau)|^2 \leq \liminf_{n\to\infty} \int_\Omega |\nabla \phi(\tau_n)|^2 - \delta/4. \tag{4.11}$$

Since the stored elastic energy can only decrease as C grows, it follows that there can only be a finite set D of τ at which there is such an energy drop.

The idea for proving (4.8) under these conditions is as follows. These energy drops (which are due to energy concentrations within T_n) occur over arbitrarily small increases in $C(\tau)$. Hence, if $\tau \in D$ with $\tau_n \nearrow \tau$ satisfying (4.11), and we had added $C(\tau) \setminus C(\tau_n)$ earlier, at or before time τ_n, we would have $\phi(\tau_n) = \phi(\tau)$, and so an upper bound for the energy of $\phi_n'(\tau_n)$ would be the energy of $\phi_n'(\tau)$.

Specifically, for each $\tau_i \in D$, we choose $\tau_i' < \tau_i$ such that $\mathcal{H}^{N-1}(C(\tau_i) \setminus C(\tau_i')) < \frac{\delta}{4(\#D)}$, where $\#D$ is the number of elements in D. Then set

$$C_n^*(\tau) := \cup_i \Big(C(\tau_i) \setminus C(\tau_i') \Big) \cup C_n(\tau).$$

By (4.7) and the choice of the τ_i', we have that

$$\lim_{n\to\infty} \Big(\mathcal{H}^{N-1}(C_n^*(\tau)) - \mathcal{H}^{N-1}(C_n^*(0)) \Big) \leq \mathcal{H}^{N-1}(C(\tau)) - \mathcal{H}^{N-1}(C(0)) + \delta/4$$

uniformly in τ.

We now set $\phi_n^*(\tau)$ to be the elastic minimizer subject to its jump set being a subset of $C_n^*(\tau)$. Then

$$\limsup_{n\to\infty} \sup_{\tau \in [0,1] \setminus \cup_i [\tau_i', \tau_i]} \int_{T_n} |\nabla \phi(\tau)|^2 < \delta/4, \tag{4.12}$$

and so for $\tau \in [0,1] \setminus \cup_i [\tau_i', \tau_i)$, we have (4.10) just as before (indeed, the elastic energy can now only be lower). Given i and $\tau \in [\tau_i', \tau_i)$, we have

$$\int_\Omega |\nabla \phi_n^*(\tau)|^2 \leq \int_\Omega |\nabla \phi_n(\tau_i)|^2 \leq \int_\Omega |\nabla \phi(\tau_i)|^2 + \delta/2$$

as in 4.9. Hence, ϕ_n^* is an ε-slide for u_n, for n sufficiently large.

□

Lemma 4.5. *If $u_n \to u$ with u_n ε-stable for some $\varepsilon > 0$, then*

$$E_{el}(u) = \lim_{n\to\infty} E_{el}(u_n).$$

Proof. We suppose that

$$E_{el}(u) + \delta \leq \lim_{n\to\infty} E_{el}(u_n)$$

for some $\delta > 0$. Then applying Theorem 4.2 with $\phi = u$, we get ϕ_n such that

$$\mathcal{H}^{N-1}(S_{\phi_n} \setminus S_{u_n}) < \min\{\varepsilon/2, \delta\}$$

and

$$\lim_{n\to\infty} E_{el}(\phi_n) = E_{el}(u).$$

But then, just as in the proof of Lemma 4.6, for n large enough we can find an ε-slide from u_n to ϕ_n, contradicting the ε-stability of u_n. □

We now consider the relation between epsilon stability and local minimality.

Lemma 4.6. *If u is ε-stable with respect to some Γ, then it is a (unilateral with respect to Γ) local minimizer.*

Proof. For simplicity, we take $\Gamma = \emptyset$, with only very minor changes in the proof. We note first that if u is ε-stable, then it is minimal for its discontinuity set. We suppose that u is not a local minimizer, and show that we can build an ε-slide for u, contradicting the stability of u. If u is not a local minimizer, then there exists a sequence $\{u_n\}$ in SBV such that $\|u - u_n\| < \frac{1}{n}$ and

$$E_{el}(u_n)dx + \mathcal{H}^{N-1}(S(u_n) \setminus S(u)) < E_{el}(u)dx.$$

By the lower semicontinuity of the bulk energy, it follows that

$$\lim_{n\to\infty} \left[E_{el}(u_n)dx + \mathcal{H}^{N-1}(S(u_n) \setminus S(u))\right] = E_{el}(u)dx$$

and $\mathcal{H}^{N-1}(S(u_n)\setminus S(u)) \to 0$. We choose $n \in \mathbb{N}$ such that $\mathcal{H}^{N-1}(S(u_n)\setminus S(u)) < \varepsilon$ and now construct an ε-slide from u to u_n. Choose a point $x \in \Omega$ such that each circle centered at x intersects $S(u_n)$ on a set of $\mathcal{H}^{N-1}$ measure zero. We define $\phi(\tau)$ to be the minimizer of the elastic energy over SBV functions with jump set inside $S(u) \cup \Big(S(u_n) \cap B(x, \tau)\Big)$. This function ϕ satisfies the energy inequality necessary for ε-slides since the elastic energy cannot increase and the crack energy increases by less than ε. Then, by Lemma 4.1, since

$$\tau \mapsto \mathcal{H}^{N-1}\Big(S(u) \cup \Big[S(u_n) \cap B(x,\tau)\Big]\Big)$$

is continuous by the choice of x, there is an ε-slide from u.

□

In fact, it is quite quick to prove that strict local minimizers are ε-stable for ε small enough, since for such a u, minimizing the total energy on the boundary of a small enough ball in L^1 (for which we have strong compactness due to SBV compactness) results in an energy strictly larger than that of u. So, u is ε-stable for ε smaller than the difference in energy between that minimum, and that of u.

5. Definition and properties of (u, C)

In this section, we normalize the ε-slides from t_i^n to t_{i+1}^n by $\mathcal{H}^{N-1}(C_n(t_{i+1}^n) \setminus C_n(t_i^n))$, with uniform Lipschitz constant. The resulting functions we refer to as $U_n(\tau)$, and each $\tau \mapsto \mathcal{H}^{N-1}(C_n(\tau))$ is uniformly Lipschitz. More precisely, recalling that $\mathcal{H}^{N-1}(C_{\phi_i^n}(\cdot))$ are affine, we first extend each C_n, which so far is defined only on I_n, by

$$C_n(t) := C_n(t_i^n) \cup C_{\phi_i^n}\left(\frac{t - t_i^n}{t_{i+1}^n - t_i^n}\right)$$

for $t \in (t_i^n, t_{i+1}^n)$, and we similarly extend u_n: for $t \in (t_i^n, t_{i+1}^n)$

$$u_n(t) := \phi_i^n\left(\frac{t - t_i^n}{t_{i+1}^n - t_i^n}\right).$$

Then set

$$f_n(t) := t + \mathcal{H}^{N-1}(C_n(t)),$$

which has an inverse on $[0, T + \mathcal{H}^{N-1}(C_n(T))]$ since each f_n is continuous and strictly increasing. We define

$$U_n(\tau) := u_n(f_n^{-1}(\tau))$$

and

$$\mathcal{C}_n(\tau) := C_n(f_n^{-1}(\tau)) = C_{U_n}(\tau).$$

First, define u on I_∞ by $u_n(t) \overset{SBV}{\rightharpoonup} u(t)$ for every $t \in I_\infty$ (choosing a diagonal subsequence as necessary). Since the f_n are each monotonic, we then take a further subsequence and choose f such that $f_n \to f$ a.e., including on I_∞, with f monotonic. Set $\mathcal{T} := f(I_\infty)$ and choose a countable dense set $D \subset [0, f(T)] \setminus \overline{\mathcal{T}}$, a function U on $\mathcal{T} \cup D$, and a further subsequence such that $U_n(\tau) := u_n(f_n^{-1}(\tau)) \to U(\tau)$ on D, $U_n(f_n(t)) \to U(f(t))$ for $t \in I_\infty$. It follows that $u(t) = U(f(t))$ for all $t \in I_\infty$. We then extend U to $[0, f(T)]$ by continuity from below (the fact that this uniquely defines U is explained below). This U then defines $\mathcal{C} := C_U$. In turn, we define $C(t) := C_U(f(t^-))$.

We note that f is increasing and can have jump discontinuities. We can consider f to be set-valued, with $f(t)$ the closed interval $[f(t^-), f(t^+)]$. We then define $F(t) := t + \mathcal{H}^{N-1}(C(t))$.

In formulations based on global minimization, one can normally show that

$$\mathcal{H}^{N-1}(C(t)) = \lim_{n\to\infty} \mathcal{H}^{N-1}(C_n(t)) \tag{5.13}$$

basically because any increase, at the discrete level, of $\mathcal{H}^{N-1}(C_n(t))$ must be exactly offset by a reduction in the elastic energy, and we can easily keep track of changes in the elastic energy. It follows that

$$\mathcal{H}^{N-1}\big(C(t_2) \setminus C(t_1)\big) = \lim_{n\to\infty} \mathcal{H}^{N-1}\big(C_n(t_2) \setminus C_n(t_1)\big),$$

which is key to the energy equality. Here, we need a different argument than the usual one, when there are jumps in time. The reason is that in this setting, energy drops might occur when there are jumps, because the jumped-to state may not have been accessible earlier, a fact that rules out the usual arguments. So, we will need the following.

Lemma 5.1. *Suppose $f(t)$ is a nontrivial interval, and set $\tau_1 := \min f(t)$, $\tau_2 := \max f(t)$. Then*

$$\mathcal{H}^{N-1}(\mathcal{C}(\tau_2) \setminus \mathcal{C}(\tau_1)) = \lim_{n\to\infty} \mathcal{H}^{N-1}(C_n(t_j^n) \setminus C_n(t_i^n))$$

where $t_i^n, t_j^n \in I_n$, $f_n(t_i^n) \to \tau_1$, $f_n(t_j^n) \to \tau_2$.

Proof. Note first that since τ_1, τ_2 are endpoints of $f(t)$, they are elements of $\overline{\mathcal{T}}$, so that such t_i^n, t_j^n exist. To simplify, we assume that there are $\{t_i^n\}$ and $\{t_{i+1}^n\}$ such that $f_n(t_i^n) \to \tau_1$ and $f_n(t_{i+1}^n) \to \tau_2$, noting that for the general case, we require a straightforward concatenation. Let ϕ_n be $\bar{\varepsilon}$-slides from $v_n(t_{i+1}^n)$ to $u_n(t_{i+1}^n)$ such that

$$\mathcal{H}^{N-1}(C_n(t_{i+1}^n) \setminus C_n(t_i^n)) = \mathcal{H}^{N-1}(C_{\phi_n}(1) \setminus C_n(t_i^n)).$$

Without loss of generality, we assume that the maps $\tau \mapsto \mathcal{H}^{N-1}(C_{\phi_n}(\tau))$ are uniformly Lipschitz. For a subsequence, $\phi_n(\tau) \overset{SBV}{\rightharpoonup} \phi(\tau)$ just as with U_n above, and in particular, $\lim_{n\to\infty} \mathcal{H}^{N-1}(C_{\phi_n}(\tau))$ exists for every τ. We suppose for the sake of contradiction that

$$0 < \lim_{n\to\infty} \mathcal{H}^{N-1}\big(C_{\phi_n}(1) \setminus C_{\phi_n}(0)\big) - \mathcal{H}^{N-1}\big(C_\phi(1) \setminus C_\phi(0)\big) =: \delta.$$

We have that

$$\lim_{n\to\infty} [f_n(t^n_{i+1}) - f_n(t^n_i)] =: \gamma > 0.$$

We then claim that, for n large enough, ϕ_n is not an optimal slide for $v_n(t^n_{i+1})$. The plan is to transfer part of the path ϕ to ϕ_n, creating a new $\bar{\varepsilon}$-slide that has lower energy than ϕ_n.

First, define $d\colon[0,1]\to[0,\infty)$ (the drop in energy) by

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(C_{\phi_n}(\tau) \setminus C_{\phi_n}(0)) - \mathcal{H}^{N-1}(C_\phi(\tau) \setminus \mathcal{C}(\tau_1)).$$

Note that d satisfies

$$d(\tau + \Delta\tau) \le d(\tau) + c\Delta\tau \tag{5.14}$$

for $\Delta\tau > 0$ since

$$\mathcal{H}^{N-1}(C_n(\tau + \Delta\tau)) \le \mathcal{H}^{N-1}(C_n(\tau)) + c\Delta\tau$$

(from the Lipschitz property of $\mathcal{H}^{N-1}(C_n(\cdot))$) and $\mathcal{H}^{N-1}(C(\tau+\Delta\tau)) \ge \mathcal{H}^{N-1}(C(\tau))$. Now set

$$\tau' := \sup\{\tau : d(\tau) \le \delta/2\}.$$

We claim that $d(\tau') = \delta/2$. If $d(\tau') > \delta/2$, then we violate (5.14). If $d(\tau') < \delta/2$, then again using (5.14) we have $d(\tau' + \Delta\tau) < \delta/2$ for $\Delta\tau > 0$ small enough, contradicting the definition of τ'.

We then have that for $\tau \in [\tau', 1]$, $E(\phi(\tau), C_\phi(\tau)) < E(\phi(0), C_\phi(0)) + \gamma - \delta/2$, since $E(\phi_n(\tau), C_{\phi_n}(\tau)) \le E(\phi(\tau), C_\phi(\tau)) + \gamma$. Proceeding as in Lemma 4.4, we transfer the slide $\phi\colon[\tau', 1]\to SBV$ from $\phi(\tau')$ to $\phi_n(\tau')$. This produces an $\bar{\varepsilon}$-slide $\bar{\phi}_n$ with lower energy at $\tau = 1$ than the original ϕ_n, contradicting the minimality of ϕ_n.

□

Lemma 5.2. *Each $u(t)$ is ε-stable with respect to $C(t)$, for all $t \in I_\infty$.*

Proof. Suppose that $u(t)$ does not have this stability. Then there exists $v \in SBV$ and an ε-slide ϕ with respect to $C(t)$ from $u(t)$ to v. In particular, ϕ is also an $(\varepsilon-\delta)$-slide, for some $\delta > 0$. We have that $u(t) := \lim_{n\to\infty} u_n(t)$, and we know that each $u_n(t)$ is ε-stable with respect to $C_n(t)$. Choose a finite collection $\{\tau_i\} \subset \mathcal{T} \cup D$ with each $\tau_i \le f(t)$, such that

$$\mathcal{H}^{N-1}(C(t) \setminus \cup_i S_{U(\tau_i)}) < \delta, \tag{5.15}$$

where τ_I is the largest τ_i in the finite collection. As in the proof of Lemma 3.1 in [8], we can choose $\alpha_i \in \mathbb{R}$ such that for $w := \sum \alpha_i U(\tau_i)$, we have $S_w = \cup_i S_{U(\tau_i)}$. We then have that ϕ is an ε-slide for $u(t)$ with respect to S_w. Setting $w_n := \sum \alpha_i U_n(\tau_i)$, we have $w_n \overset{SBV}{\rightharpoonup} w$. But then, by Lemma 4.4, there exists an ε-slide ϕ_n with respect to S_{w_n}, for $u_n(t)$, and so these ϕ_n are also ε-slides for $u_n(t)$ with respect to $C_n(t)$, since $S_{w_n} \subset C_n(t)$. This contradicts the ε-stability of $u_n(t)$ with respect to $C_n(t)$. □

Lemma 5.3. *If $t_n \in I_n$ with $t_n \to t$, and f, F are continuous at t, then for $\tau_n = f_n(t_n) \to f(t) = \tau$, we have that, if $U_n(\tau_n) \overset{SBV}{\rightharpoonup} z$, then z is ε-stable with respect to $\mathcal{C}(t)$.*

Proof. Suppose that z does not have this stability. Then there exists $v \in SBV$ and an ε-slide ϕ with respect to $\mathcal{C}(\tau)$ from z to v. In particular, ϕ is also an $(\varepsilon-\delta)$-slide, for some $\delta > 0$. We have that $z := \lim_{n\to\infty} U_n(\tau_n)$, and we know that each $U_n(\tau_n)$ is ε-stable with respect to $\mathcal{C}_n(\tau_n)$. Choose $t' < t$ such that $t' \in I_\infty$ and $F(t) - F(t') < \delta/2$, and choose a finite collection $\{\tau^i\} \subset [0, f(t')] \cap D$ such that

$$\mathcal{H}^{N-1}(\mathcal{C}(f(t')) \setminus \cup_i S_{U(\tau^i)}) < \delta/2. \tag{5.16}$$

As in the proof of Lemma 3.1 in [8], we can choose $\alpha_i \in \mathbb{R}$ such that for $w := \sum \alpha_i U(\tau^i)$, we have $S_w = \cup_i S_{U(\tau_i)}$. Setting $w_n := \sum \alpha_i U_n(\tau_i)$, we have $w_n \overset{SBV}{\rightharpoonup} w$.

We then have that ϕ is an ε-slide for z with respect to S_w, and so there exist ε slides ϕ_n for $U_n(\tau_n)$ with respect to S_{w_n} for n sufficiently large. Since $S_{w_n} \subset \mathcal{C}_n(\tau_n)$ for n sufficiently large, these ϕ_n are also ε-slides with respect to $\mathcal{C}_n(\tau_n)$, a contradiction.

□

Remark 5.4. *The same argument shows that $U_n \overset{SBV}{\rightharpoonup} U$ a.e. as follows. Let τ be a continuity point of f, F and let w be such that $U_n(\tau) \overset{SBV}{\rightharpoonup} w$ (for a subsequence). The aim is to show that $w = U(\tau)$ (defined to be the limit from below of U on the dense, countable set used in first defining U). By the above proof, w is ε-stable with respect to $\mathcal{C}(\tau)$, but so is $U(\tau)$, since it is the limit of $U(\tau_n)$, $\tau_n \nearrow \tau$, and these functions are also ε-stable stable with respect to $\mathcal{C}(\tau_n)$. Since τ is a continuity point of f, $\mathcal{H}^{N-1}(\mathcal{C}(\tau) \setminus \mathcal{C}(\tau_n)) \to 0$. The same argument shows also that $(u(t), C(t))^-$ exists, as does $(u(t), C(t))^+$.*

Lemma 5.5. *Each $u(t)$ is ε-stable with respect to $C(t)$, for $t \notin I_\infty$.*

Proof. Just as above, we suppose that $u(t)$ does not have this stability. Then there exists an ε-slide ϕ with respect to $C(t)$ from $u(t)$. In particular, ϕ is also an $(\varepsilon - \delta)$-slide, for $\delta > 0$ sufficiently small. By definition,

$$C(t) = \bigcup_{\tau_i \in \mathcal{D}(t)} S_{U(\tau_i)}$$

with $\mathcal{D}(t) := [0, f(t^-)] \cap (D \cup \mathcal{T})$. By the definition of $f(t^-)$, we can choose $t' \in I_\infty$ with $t' \le t$ such that $\mathcal{H}^{N-1}(C(t) \setminus C(t')) < \frac{\delta}{4}$. Then, ϕ is an $(\varepsilon - \frac{\delta}{4})$-slide for $u(t)$ with respect to $C(t')$. We next claim that $\bar{\phi}$ defined by

$$\bar{\phi}(\tau) := \phi(\tau) + g(t') - g(t)$$

is an ε-slide for $u(t')$ with respect to $C(t')$, for δ small enough, contradicting the stability of $u(t')$.

We have

$$\frac{1}{2}\int_\Omega |\nabla\bar{\phi}(\tau)|^2 = \frac{1}{2}\int_\Omega |\nabla\phi(\tau)|^2 + \int_\Omega \nabla\phi(\tau)\cdot\nabla[g(t') - g(t)] + \frac{1}{2}\int_\Omega |\nabla[g(t') - g(t)]|^2,$$

so that we can get

$$\frac{1}{2}\int_\Omega |\nabla\bar{\phi}(\tau)|^2 < \frac{1}{2}\int_\Omega |\nabla\phi(\tau)|^2 + \frac{\delta}{2}$$

since

$$\left| \int_\Omega \nabla\phi(\tau)\cdot\nabla[g(t')-g(t)] + \frac{1}{2}\int_\Omega |\nabla[g(t')-g(t)]|^2 \right| \\ \leq \|\nabla\phi(\tau)\|\|\nabla[g(t')-g(t)]\| + \frac{1}{2}\|\nabla[g(t')-g(t)]\|^2,$$

which is less than $\frac{\delta}{2}$ if t' is close enough to t so that

$$\|\nabla[g(t')-g(t)]\| < \min\left\{\frac{\delta}{2(\|\nabla u(t)\|+1)}, 1\right\}.$$

Therefore, $\bar\phi$ is an ε-slide for $u(t')$ with respect to $C(t')$ if $E_{\bar\phi}(\bar\phi(1)) < E(u(t'),C(t'))$. But, just as above we also have that

$$\frac{1}{2}\int_\Omega |\nabla u(t')|^2 \geq \frac{1}{2}\int_\Omega |\nabla u(t)|^2 + \int_\Omega \nabla u(t')\cdot\nabla[g(t')-g(t)] \geq \frac{1}{2}\int_\Omega |\nabla u(t)|^2 - \frac{\delta}{2}.$$

Hence,

$$\begin{aligned} E(u(t'),C(t')) - E_{\bar\phi}(\bar\phi(1)) &\geq E_{el}(u(t)) - \frac{\delta}{2} - E_{el}(\bar\phi(1)) - \mathcal{H}^{N-1}([C(t)\cup C_\phi(1)]\setminus C(t')) \\ &\geq E_{el}(u(t)) - \frac{\delta}{2} - E_{el}(\phi(1)) - \frac{\delta}{4} - \mathcal{H}^{N-1}(C_\phi(1)\setminus C(t)) \\ &\quad - \mathcal{H}^{N-1}(C(t)\setminus C(t')) \\ &\geq E(u(t)) - E_\phi(1) - \frac{\delta}{2} - \frac{\delta}{4} - \frac{\delta}{4} > 0 \end{aligned}$$

if δ is originally chosen less than $E(u(t),C(t)) - E_\phi(1)$. □

Lemma 5.6. *$\nabla u_n(t_n) \to \nabla u(t)$ strongly in $L^2(\Omega)$ if $t_n \to t$ and $\mathcal{H}^{N-1}(C(\cdot))$ is continuous at t.*

Proof. We first note, from lemmas 5.2 and 5.5, that each $u(t)$ is ε-stable with respect to $C(t)$. Let v be the weak limit of a subsequence of $u_n(t_n)$. We claim that this v must be $u(t)$, and the convergence is strong (i.e, $\nabla u_n(t_n) \to \nabla u(t)$ in $L^2(\Omega)$). To show that v must be $u(t)$, it is enough to show that v is ε-stable with respect to $C(t)$, and therefore the minimizer of the Dirichlet energy, over functions with jump set in $C(t)$.

Suppose it isn't. Then there exists $w \in SBV$ and an ε-slide with respect to $C(t)$ from v to w. Repeating the proof of Lemma 5.2 with $u_n(t_n)$ replacing $u_n(t)$ and using the continuity of $C(t)$, we contradict the ε-stability of $u_n(t_n)$. The strong convergence then follows from Lemma 4.5. □

6. Energy inequality and accessibility of jumps

Lemma 6.1. *For (u,C) defined as above, we have that for every $t_1 \leq t_2$,*

$$E(u(t_2),C(t_2)) \leq E(u(t_1),C(t_1)) + \int_{t_1}^{t_2}\int_\Omega \nabla u\cdot\nabla\dot g\,dx\,dt. \tag{6.17}$$

Furthermore, if $t \mapsto \mathcal{H}^{N-1}(C(t))$ has only jumps of size less than ε between t_1 and t_2, then the above inequality is an equality.

Proof. We first note that for each $n \in \mathbb{N}$, the number of times $t_i^n \in T_n$ such that $\mathcal{H}^{N-1}(C_n(t_{i+1}^n) \setminus C_n(t_i^n)) \geq \varepsilon$ is bounded, uniformly in n since the total energy is bounded. Taking a further subsequence if necessary, these sets of times converge as $n \to \infty$ to some finite set J.

Let $t_1 < t_2$ be given such that $[t_1, t_2] \cap J = \emptyset$. As usual, the method is to prove a discrete version of the theorem for $u_n(t_i)$, and then pass to the limit $n \to \infty$. The discrete inequality comes from summing

$$\begin{aligned} E(u_n(t_{i+1}^n), C_n(t_{i+1}^n)) \leq & E(u_n(t_i^n), C_n(t_i^n)) + \int_\Omega \nabla u_n(t_i^n) \cdot \Big(\nabla g(t_{i+1}^n) - \nabla g(t_i^n)\Big) \\ & + \frac{1}{2} \int_\Omega |\nabla g(t_{i+1}^n) - \nabla g(t_i^n)|^2, \end{aligned}$$

which follows from the $\bar{\varepsilon}$-accessibility of $u_n(t_i^n) + g(t_{i+1}^n) - g(t_i^n)$ from $v_n(t_{i+1}^n)$, together with the $\bar{\varepsilon}$-stability of $u_n(t_{i+1}^n)$, for $t_{i+1}^n - t_i^n$ small enough. Note that the corresponding inequality in the opposite direction,

$$\begin{aligned} E(u_n(t_{i+1}^n), C_n(t_{i+1}^n)) \geq & E(u_n(t_i^n), C_n(t_i^n)) + \int_\Omega \nabla u_n(t_i^n) \cdot \Big(\nabla g(t_{i+1}^n) - \nabla g(t_i^n)\Big) \\ & - \frac{1}{2} \int_\Omega |\nabla g(t_{i+1}^n) - \nabla g(t_i^n)|^2, \end{aligned}$$

is only guaranteed to hold since the increment in C_n is less than ε, so that $u_n(t_{i+1}^n) - g(t_{i+1}^n) + g(t_i^n)$ is $\bar{\varepsilon}$-accessible from $u_n(t_i^n)$.

We now iterate these inequalities from $i = B(n)$ to $F(n)$, where $t_{B(n)}^n$ is the closest time, in the nth discretization, to t_1, and similarly for $F(n)$ and t_2, to obtain first

$$\begin{aligned} E(u_n(t_{F(n)}^n), & C_n(t_{F(n)}^n)) \leq E(u_n(t_{B(n)}^n), C_n(t_{B(n)}^n)) \\ & + \sum_{i=B(n)}^{F(n)-1} \int_\Omega \Big[\nabla u_n(t_i^n) \cdot \nabla\Big(g(t_{i+1}^n) - g(t_i^n)\Big) + \frac{1}{2}\Big|\nabla\Big(g(t_{i+1}^n) - g(t_i^n)\Big)\Big|^2\Big]. \end{aligned} \tag{6.18}$$

This sum can then be rewritten:

$$\int_{t_{B(n)}^n}^{t_{F(n)}^n} \int_\Omega \Big[\nabla \bar{u}_n(t) \cdot \nabla \dot{\bar{g}}_n(t) + \frac{1}{2}|\nabla \dot{\bar{g}}_n(t)|^2\Big] dx dt,$$

where $\bar{g}_n$ is the piecewise-affine extension of g^n restricted to $\mathcal{T}_n$, and $\bar{u}_n$ is the piecewise-constant extension of u_n restricted to $\mathcal{T}_n$. By Lemma 5.6, $\nabla \bar{u}_n(t) \to \nabla u(t)$ strongly in L^2 for a.e. t. Since $g \in C^1(0, T)$, we also have that $\nabla \dot{\bar{g}}_n(t) \to \nabla \dot{g}(t)$ strongly in L^2 for every t. By the bounds on $\nabla \dot{\bar{g}}_n(t)$ and $\nabla \bar{u}_n(t)$, the bounded convergence theorem gives

$$\int_{t_{B(n)}^n}^{t_{F(n)}^n} \int_\Omega \Big[\nabla \bar{u}_n(t) \cdot \nabla \dot{\bar{g}}_n(t) + \frac{1}{2}|\nabla \dot{\bar{g}}_n(t)|^2\Big] dx dt \to \int_{t_1}^{t_2} \int_\Omega \nabla u(t) \cdot \nabla \dot{g}(t) dx dt.$$

In addition, the strong convergence in L^2 of $\nabla u_n(t_{B_n}^n)$ to $\nabla u(t_1)$ implies $E_{el}(u_n(t_{B_n}^n)) \to E_{el}(t_1)$ (and similarly for t_2). Hence

$$\begin{aligned} E_{el}(u(t_2)) + \lim_{n\to\infty} \mathcal{H}^{N-1}(C_n(t_2) \setminus C_n(t_1)) \leq & E_{el}(u(t_1)) \\ & + \int_{t_1}^{t_2} \int_\Omega \nabla u(t) \cdot \nabla \dot{g}(t) dx dt. \end{aligned} \tag{6.19}$$

Using the other inequality and a similar argument, we get

$$\begin{aligned} E_{el}(u(t_2)) + \lim_{n\to\infty} \mathcal{H}^{N-1}(C_n(t_2)\setminus C_n(t_1)) \geq & E_{el}(u(t_1)) \\ & + \int_{t_1}^{t_2}\int_\Omega \nabla u(t)\cdot\nabla\dot{g}(t)dxdt. \end{aligned} \tag{6.20}$$

Suppose first the $0\notin J$. Then, taking $t_1 = 0$, we note that $\mathcal{H}^{N-1}(C(t_1)) = \mathcal{H}^{N-1}(C_n(t_1))$, and so for any $t\in[0,t_2]$, we get from the above inequalities together with the proof of Lemma 3.6 in [8], that

$$E(u(t),C(t)) = E(u(0),C(0)) + \int_{t_1}^{t_2}\int_\Omega \nabla u(t)\cdot\nabla\dot{g}(t)dxdt,$$

which further implies that $\mathcal{H}^{N-1}(C(t)) = \lim_{n\to\infty}\mathcal{H}^{N-1}(C_n(t))$. At the first point $p\in J$, it follows from lemma 5.1 to that $\mathcal{H}^{N-1}(C(p^+)) = \lim_{n\to\infty}\mathcal{H}^{N-1}(C_n(p^+))$. Repeating these arguments gives

$$\mathcal{H}^{N-1}(C(t)) = \lim_{n\to\infty}\mathcal{H}^{N-1}(C_n(t))$$

for all $t\in[0,T]$, as well as (6.1).

□

Finally, we have the following accessibility and optimality:

Lemma 6.2. *If $u^-(t) := \lim_{s\to t^-} u(s) \neq u^+(t) := \lim_{s\to t^+} u(s)$, then there exists an $\bar{\varepsilon}$-slide from $u^-(t)$ to $u^+(t)$ with respect to $C^-(t)$. Furthermore, $E(u^+(t),C^+(t)) \leq E(v, C^-(t)\cup C_\psi(1))$ for all $v\in SBV$ that are ε-accessible from $u^-(t)$ with respect to $C^-(t)$ (with ψ an ε-slide for v).*

Proof. We first set $\phi(\tau) := U(\tau - \min f(t))$ for $\tau\in[0,\tau']$ and $\tau' := \max f(t) - \min f(t)$, and show that it has the properties that, according to Lemma 4.1, imply the existence of an $\bar{\varepsilon}$-slide from $u(t^-)$ to $u(t^+)$. We choose $t^n_{l(n)}$ such that $f_n(t^n_{l(n)}) \to f(t^-)$, and similarly choose $t^n_{u(n)}$ for $f(t^+)$. Then set

$$\phi_n(\tau) := U_n\Big(f_n(t^n_{l(n)}) + \tau(f_n(t^n_{u(n)}) - f_n(t^n_{l(n)}))\Big)$$

so we now have that

$$\begin{aligned} \phi(0) &= u^-(t) \\ \phi(\tau') &= u^+(t) \\ E_{el}(\phi(\tau)) &\leq \liminf_{n\to\infty} E_{el}(\phi_n(\tau)) \\ \mathcal{H}^{N-1}(C_\phi(\tau)) &= \liminf_{n\to\infty}\mathcal{H}^{N-1}(C_{\phi_n}(\tau)) \end{aligned} \tag{6.21}$$

for $\tau\in[0,\tau']$, where the inequality follows from lower semicontinuity. Now, lemma 4.1 implies that there exists an $\bar{\varepsilon}$-slide from $u^-(t)$ to $u^+(t)$.

Finally, we note that the optimality of $u(t^+)$ follows just as in the proof that $u(t)$ is ε-stable. □

Acknowledgments. This work began on Ipanema and Leblon, during a visit to Instituto Nacional de Matemática Pura e Aplicada in Rio de Janeiro, and at the

University of Warwick. I am happy to thank them both for hosting me. This material is based on work supported by the National Science Foundation under Grant No. DMS-0505660.

References

[1] L. Ambrosio. A compactness theorem for a new class of functions of bounded variation. Boll. Un. Mat. Ital., **3-B**, 857-881, 1989.

[2] L. Ambrosio, E. Fusco, Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[3] A. Chambolle, A. Giacomini, M. Ponsiglione. Crack initiation in brittle materials. Arch. Ration. Mech. Anal., **188**, 309-349, 2008.

[4] G. Dal Maso, G. A. Francfort, R. Toader. Quasistatic crack growth in nonlinear elasticity. Arch. Rat. Mech. Anal. **176**, 165-225, 2005.

[5] G. Dal Maso, R. Toader. A model for the quasi-static growth of brittle fractures: existence and approximation results. Arch. Rat. Mech. Anal. **162**, 101-135, 2002.

[6] G. Dal Maso, R. Toader. A model for the quasi-static growth of brittle fractures based on local minimization. Math. Models Methods Appl. Sci. **12**, 1773-1800, 2002.

[7] L.C. Evans, R.F. Gariepy. *Measure theory and fine properties of functions.* Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[8] G.A. Francfort, C.J. Larsen. Existence and convergence for quasi-static evolution in brittle fracture. Comm. Pure Appl. Math. **56** 1465-1500, 2003.

[9] G.A. Francfort, J.J. Marigo. Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids, **46-8**, 1319-1342, 1998.

[10] A. Griffith. The phenomena of rupture and flow in solids. Phil. Trans. Roy. Soc. London, **CCXXI-A**, 163–198, 1920.

[11] Stefan Hildebrandt and Anthony Tromba. The Parsimonious Universe, Springer-Verlag, 1996 (page 148).

[12] Dorothee Kneesm, Alexander Mielke, and Chiara Zanini. On the inviscid limit of a model for crack propagation, Math. Models Methods Appl. Sci., **18**, 1529-1569, 2008.

[13] C. J. Larsen, M. Ortiz, and C. L. Richardson. Fracture Paths from Front Kinetics: Relaxation and Rate Independence. Arch. Rat. Mech. Anal., to appear.

[14] M. Negri and C. Ortner. Quasi-static crack propagation by Griffith's criterion, Math. Models Methods Appl. Sci., **18**, 1895-1925, 2008.

[15] W.P. Ziemer. *Weakly Differentiable functions.* Springer-Verlag, Berlin, 1989.