Cohesive fracture without relaxation

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Fracture energies

Energy functional:

$$E(u) := \int_{\Omega} W(\nabla u)dx + \int_{S(u)} \varphi([u])\, d\mathcal{H}^{N-1}$$

where $\Omega \subset \mathcal{R}^3$,
elastic energy density $W(\nabla u)$,
fracture energy density $\varphi([u])$, $[u]$ is the jump of u,
$S(u)$ is the jump set of u.

Brittle fracture: $\varphi \equiv G_c$, cohesive: $\varphi(x) \to 0$ as $x \to 0$.

Plan: minimize over $u \in SBV(\Omega)$.

Usual assumptions: W quasiconvex, φ concave with infinite slope at zero ("necessary" for compactness in SBV).

Cohesive fracture

Some advantages for φ with finite slope at zero, e.g., stress threshold for fracture initiation.

Problem: if we take a sequence u_n with bounded energy, then it can happen that $u_n \to u$, with $u \in BV \setminus SBV$. For lower semicontinuity, we "need" to extend E to BV by

$$E(u) := \int_\Omega W^*(\nabla u) + c|D^c u| + \int_{S(u)} \varphi([u]) d\mathcal{H}^{N-1}$$

where $W^*(A) :=\min\{W(A), c|A|\}$, $c :=$slope of φ at zero.

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Goal: (Dal Maso, Garroni, L.) remove smoothness assumption, show all local minimizers (in BV) are actually SBV with gradient bound.

Partial result: consider the energy functional:

$$E(u) := |D^r u|(\Omega) + \int_{S(u)} \varphi([u])\, d\mathcal{H}^{N-1}$$

where $\Omega \subset \mathcal{R}^2$,
$D^r u := D^a u + D^c u$,
$\varphi : [0, \infty) \rightarrow [0, \infty)$ strictly concave and such that

$$\lim_{t \to 0+} \frac{\varphi(t)}{t} = 1$$

$$\lim_{t \to 0+} \frac{t - \varphi(t)}{t^2} = c_0 > 0\,.$$

Theorem

Let Ω be a bounded open set in $\mathcal{R}^2$ and let u be a minimizer of a Dirichlet problem in Ω. If $S(u) = \emptyset$, then $Du = 0$ on Ω.

The reason, as in DM-G, is that the strict concavity of φ causes variation to concentrate in $S(u)$.

More specifically, the idea is to consider level sets of u, and note that by coarea

$$|Du|(B) = \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}(\partial^* \{u > t\} \cap B)\, dt$$

for any Borel set B.
If we can shift level sets and combine them, creating a new function v with jump, then the energy reduction will be

$$\int_{S(v)} [v]^2 d\mathcal{H}^{N-1}$$

while the cost will be the increase in total variation, which is the same as the increases in lengths of the level sets, due to coarea.

Preliminaries

Lemma

Assume that Ω *is a bounded domain in* $\mathcal{R}^N$*. Let* $u \in BV(\Omega)$ *be a local minimizer for the total variation in* Ω*. Then for almost every* $t \in \mathcal{R}$ *the set* $E_t := \{u > t\}$ *is a local minimizer of* $P(\cdot, \Omega)$*.*

The only issue in the proof is to show that if one replaces $\{u > t\}$ with minimizers of perimeter satisfying the same boundary conditions, then these new sets are the t-level sets of some other function. Coarea then says this function has lower variation than u, if u's level sets were not already minimal.

Remark

Of course if $N = 2$ *this implies that almost every level set of a local minimizer for the total variation is locally a straight line.*

Proof of theorem