Threshold-based quasi-static brittle damage evolution

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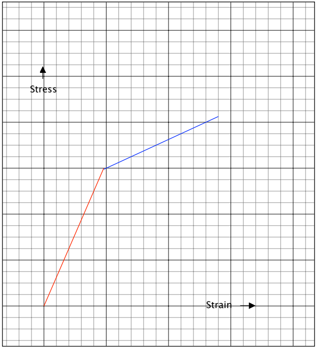
December 11, 2007

Brittle damage



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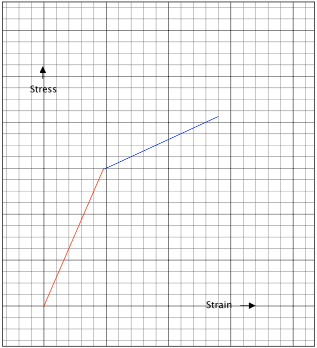
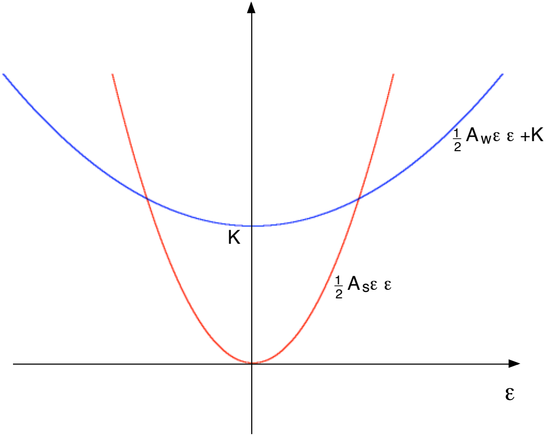


Question: how to model?
Francfort-Marigo ('93): minimize

$$E(u, D) := \frac{1}{2}\int_\Omega (A_s\chi_{D^c} + A_w\chi_D)\varepsilon\,\varepsilon dx + K|D|$$

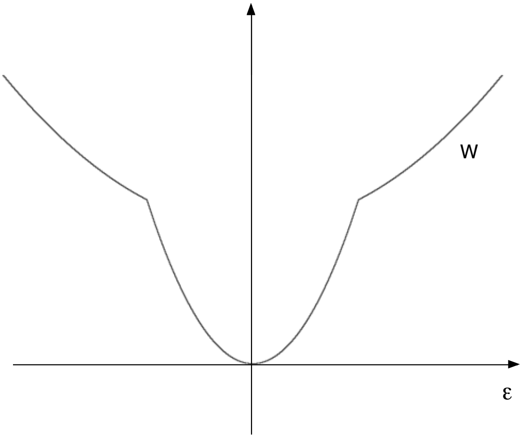
to find the damage set D and the corresponding displacement u.

Relaxation



$\frac{1}{2}A_W\varepsilon\,\varepsilon + K$
K
$\frac{1}{2}A_S\varepsilon\,\varepsilon$
ε

Relaxation



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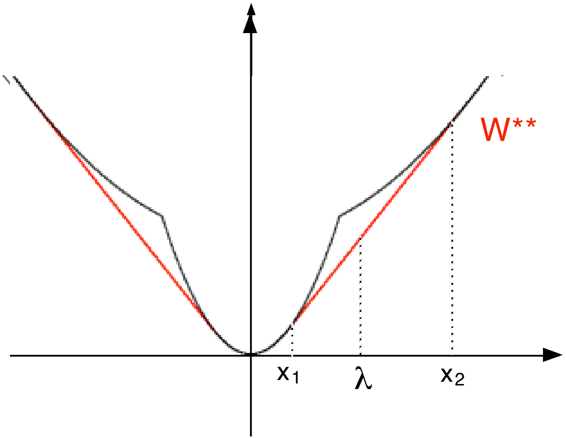

Now minimize

$$QW(\varepsilon) = \min_{\Theta\in[0,1]} \min_{A\in G_\Theta(A_s,A_w)} \left\{ \frac{1}{2} A\varepsilon\,\varepsilon + K\Theta \right\}$$

Quasi-static evolution

Now seek $u(t), D(t)$ corresponding to varying boundary conditions or loads (and always in equilibrium).

New issues:

- Irreversibility of damage
- Relaxation

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- $\Theta(t) \nearrow$
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- at each t there is no lower energy (v, C, θ) with $C \in G(A(t), A_w)$

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- Does this model still correspond to the original threshold criterion for damage? In particular, does this ignore local minimizers?
- Is there more than one way to do the relaxation for the quasi-static problem? If so, which is "right"?
- Can this problem be formulated purely in terms of a threshold criterion for damage?

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Garroni-L. (http://cvgmt.sns.it/papers/), anti-plane setting

First, assuming no relaxation:
$t \mapsto D(t)$ is a theshold-based quasi-static damage evolution with threshold λ if

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3. $D(t)$ is necessary: $\forall E \subset D(t)$ with $|E| > 0$, and all Δt sufficiently small, $\exists \tau < t - \Delta t$ such that if f we consider the displacement v corresponding to damage set $D(\tau + \Delta t) \setminus \Delta E$, where $\Delta E := E \cap [D(\tau + \Delta t) \setminus D(\tau)]$, we have $|\nabla v| > \lambda$ on a set of positive measure in ΔE.

First surprise

Theorem

$t \mapsto D(t)$ is a threshold solution $\iff$ it is an energy-based (F-G) solution

Proof.

$\Leftarrow$

i) If 2. is not satisfied, then a local layering of damage will reduce the energy (note: λ corresponds to K)

ii) If 3. is not satisfied, then at an earlier time, a subset of ΔE had lower energy

$\Rightarrow$

For the discrete version of the problem, going from time t_i to t_{i+1}, minimize the (F-G) energy subject to the constraint that $D \subset D(t_{i+1})$. Then $D = D(t_{i+1})$, since otherwise the threshold would have to be violated in $D(t_{i+1}) \setminus D$. Further, it cannot reduce the energy to then add damage outside $D(t_{i+1})$, since the threshold is not exceeded there. So, $D(t_{i+1})$ is a solution to the discrete (F-G) problem. Then send $t_{i+1} - t_i \longrightarrow 0$.

Relaxation

$(A(t), \Theta(t))$ is a solution of the Threshold Problem (TP) with threshold λ if for every $t \in [0, T]$ there exists a sequence $\{D_n(t)\}$ such that $\sigma_{D_n(t)} \xrightarrow{G} A(t)$ and $\chi_{D_n(t)} \overset{*}{\rightharpoonup} \Theta(t)$ in L^∞, and the following hold

1. Monotonicity: $D_n(\cdot)$ is increasing
2. Threshold: For the equilibrium u_n corresponding to $D_n(t)$, we have that $\forall \delta > 0$, the sets in which there is no damage but the threshold is exceeded by at least δ,

   $$U_n := \{x \notin D_n(t) : |\nabla u_n(x)| > \lambda + \delta\},$$

   satisfy

   $$|U_n| \to 0$$

3. Necessity of the damage: For all $E_n \subset D_n(T)$ with $\liminf |E_n| > 0$, we have that $\forall \delta > 0$ and $\forall\ \Delta t > 0$ small enough, there exists $\tau < T - \Delta t$ such that, setting v_n to be the equilibrium corresponding to $D_n(\tau + \Delta t) \setminus \Delta E_n$, we have that...

$$\Delta E_n^\delta := \{x \in \Delta E_n : \ |\nabla v_n(x)| > \lambda - \delta\},$$

satisfy

$$\liminf_{n\to\infty} |\Delta E_n^\delta| > 0.$$

Question: Does this correspond to the (F-G) problem?

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We want to say that if $D_n(t)$ generates an (F-G) solution $(A(t), \Theta(t))$, then it has a minimality property that implies the threshold cannot be exceeded (too much) outside of $D_n(t)$. But....

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$D_n(t)$ has no (useful) minimality property!

Second Surprise

$A(t)$ is only minimal compared to effective tensors $C \in G(A(t), A_w)$. If D_n generates A and $D_n' \supset D_n$ generates some A', it is not necessarily true that $A' \in G(A, A_w)$.

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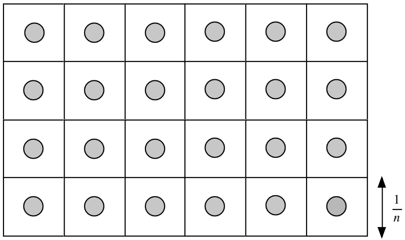

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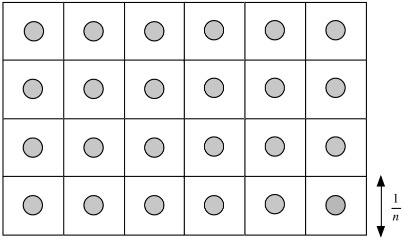

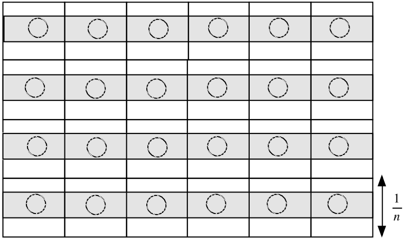

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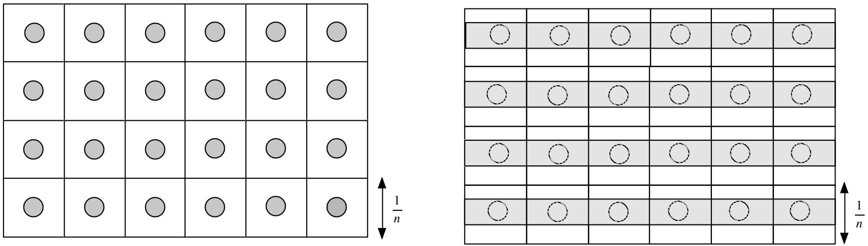

Therefore, we cannot say that $|\nabla u_n|$ (where u_n is the equilibrium corresponding to $D_n(t)$) does not exceed the threshold – again, the argument was that if it did, the energy could be reduced.

Solution

New definition of energy-minimizing solution:

The same as (F-G), except

Minimality: There exists a time-indexed family of sequences of sets $D_n(t)$, monotonically increasing in t, such that for every $t \in [0, T]$,

$$\begin{cases} \chi_{D_n(t)} \overset{*}{\rightharpoonup} \Theta(t) \\ \sigma_{D_n(t)} \xrightarrow{G} A(t)\,, \end{cases} \tag{1}$$

and for every (A', Θ') such that A' is the G-limit of a subsequence of some $\sigma_{D'_n}$ satisfying $\chi_{D'_n} \overset{*}{\rightharpoonup} \Theta'$ (and $D'_n \supset D_n(t)$ if $t > 0$), we have

$$E(A(t), \Theta(t)) \leq E(A', \Theta').$$

Existence

Theorem: there exists such an energy-minimizing solution

Theorem

These energy minimizing solutions are also threshold solutions. Proof: tricky part is 3. If not satisfied, then the triple solves another problem, corresponding to a $\lambda - \delta$ penalty on $\lim_{n\to\infty} E_n$.

and finally, we define stability by: (A, Θ) is stable if

$$\limsup_{\varepsilon\to 0} \sup_{\Theta' \in B(\Theta,\varepsilon)} \frac{E(A,\Theta) - E(A',\Theta')}{\varepsilon} \leq 0,$$

where (A', Θ') must satisfy $D'_n \rightsquigarrow (A', \Theta')$ for some $D'_n \supset D_n$.

Theorem

If $(\{D_n\}, A, \Theta)$ is a local minimizer, then it is a global minimizer. Furthermore, even if the triple is only stable it is a global minimizer.