Fracture evolution and locality

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May 11, 2008
Quasi-static evolution

The problem that we consider is to predict crack paths in a quasi-static setting. More precisely, we have an elastic material occupying a domain $\Omega$ and we suppose that the material is in equilibrium subject to a boundary condition $f(t)$. Then, if there is no crack, the displacement $u(t)$ minimizes

$$E_{el}(v) = \int_{\Omega} |\nabla v|^2$$

subject to $v = f(t)$ on $\partial \Omega$ for every time $t$, where we consider the simplified elastic energy $E_{el}$ and we suppose that $f$ varies slowly compared to the speed with which the material reaches equilibrium.

If there is a fixed crack $K$, each displacement would solve the same Dirichlet problem, but in the space $H^1(\Omega \setminus K)$ instead of $H^1(\Omega)$, which implies that the stored elastic energy can only be lower if there is a crack.

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Griffith’s criterion

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For a crack increment of length $l$, $E(l)$ is the elastic energy of the corresponding elastic equilibrium. The criterion states that the crack can only grow if the rate of decrease of elastic energy as $l$ increases is large enough, i.e., if

$$ -\frac{dE(l)}{dl} < G_c \quad \text{the crack can not run} $$
$$ = G_c \quad \text{the crack can run} $$
$$ > G_c \quad \text{the crack is unstable}. $$
The static problem

Formulated by Ambrosio and Braides (1995): If $u$ minimizes

$$v \mapsto \int_{\Omega} |\nabla v|^2 + \mathcal{H}^1(S_v)$$

over $v \in SBV_f(\Omega)$, then the crack $K := S_u$ is stable (taking $G_c = 1$). The reason is that each increment in length $l$ cannot reduce the energy, i.e.,

$$E(l) + l \geq E(0),$$

or

$$-\frac{E(l) - E(0)}{l} \leq 1.$$
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Note: This and all that follows is in any dimension, but we will assume 2-D throughout.
Quasi-static formulation

This led to the following quasi-static formulation (Francfort-Marigo, 1998), where \( f = f(t) \).

For discrete times \( \{t_i\} \), \( u(t_i) \) minimizes

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\nu \mapsto \int_{\Omega} |\nabla \nu|^2 + \mathcal{H}^1(S_\nu \setminus \bigcup_{j<i} S_{u(t_j)})
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over \( \nu \in SBV_{f(t_i)}(\Omega) \). The technicality in the last term models irreversibility of fracture, so that only the new crack at \( t_i \) is penalized.
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The plan was then to take a sequence of discretizations \( \{t^n_i\} \) with, e.g., \( t^n_i - t^n_{i-1} = \frac{1}{n} \), resulting in a sequence \( \{u_n\} \) that hopefully converges to a \( u \) that is a solution to a corresponding continuous-time problem. This has been carried out (Dal Maso, Francfort, L., Toader).
Connection to Griffith

The resulting solution \( u(t) \) with \( K(t) := \bigcup_{\tau \leq t} S_{u(\tau)} \) satisfies Griffith’s criterion if \( t \mapsto \mathcal{H}^1(K(t)) \) is continuous.
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Problem:

At the pre-existing crack, the energy release rate can be made arbitrarily small by choosing a suitable boundary condition, independent of $L$, but if $L$ is large enough, global minimization will result in the crack growing. This violates Griffith.

Note the connection to local vs. global minimality – the initial crack was a local minimizer and was stable in the sense of Griffith.
Quasi-static evolution with local minimization

Big question: how do we do quasi-static evolution based on local minimization?

Our view: the real issue is accessibility ⇔ stability.
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At each time step $t_i$, instead of minimizing the total energy over all functions, we should minimize only over those that are accessible from $u(t_{i-1})$.

With global minimization, every state is accessible from every other state $\iff$ the only stable states are global minimizers (resulting in a Griffith violation).

We want to have that if $u$ is a strict local minimizer, then there is no accessible state with lower energy $\iff$ strict local minimizers are stable.
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This suggests a definition of accessibility:
\( v \) is accessible from \( u \) \iff \( \) there exists a continuous path \( \phi \) from \( u \) to \( v \) along which the total energy is nonincreasing.
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An idea like this has been tried by Dal Maso and Toader (2002), based on following gradient flows from \( u(t_{i-1}) \) to find \( u(t_i) \). Unfortunately, there are technical difficulties in proving that, when \( u_n(t) \to u(t) \), we have the properties we want for \( u(t) \). In particular, local minimality is a problem.

In fact, the same problem occurs with accessibility – we would need to show that since the \( u_n \) were stable, so is \( u \), i.e., if there is a \( \nu \) that is accessible from \( u \) and has lower energy, then there are \( \nu_n \) that are accessible from \( u_n \) and have lower energy.
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We illustrate part of the problem in terms of stability.

Consider a crack $K$ that is unstable, i.e., the total energy decreases as the crack grows:
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Our view:
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so that we cannot move to the state that we want without the energy initially increasing by a small amount. Allowing small energy increases in our definitions of accessibility and stability overcomes all mathematical issues.
**Definition (ε-accessible)**

ν is ε-accessible from u if there exists a continuous function φ : [0, 1] → SBV(Ω) such that φ(0) = u, φ(1) = ν, and

$$\sup_{\tau_1 < \tau_2} [E(\phi(\tau_2)) - E(\phi(\tau_1))] < \varepsilon.$$ 

We then have the corresponding definition of stability:

**Definition (ε-stability)**

u is ε-stable if there does not exist an ε-accessible ν with strictly lower energy. The path to such a ν is called an ε-slide.

We also define ¯ε-accessibility, where the inequality is not strict.
Existence theorem

Theorem

Given $f(t)$ with sufficient regularity, there exists a quasi-static evolution $u(t)$ with the properties of a globally minimizing evolution, modified as follows:

- $u(t)$ is a local minimizer at every $t$ (coming from being $\bar{\varepsilon}$-stable)
- Energy inequality:

$$E(u(t_2)) - E(u(t_1)) \leq \int_{t_1}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \dot{f} \, dx \, dt$$

for every $t_1 \leq t_2$.
- If $u^-(t) \neq u^+(t)$, then $u^+(t)$ is $\bar{\varepsilon}$-accessible from $u^-(t)$ and has lower energy than all states that are $\varepsilon$-accessible from $u^-(t)$. 

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Proof.

- First issue: show that if $u_n \rightharpoonup u$ and there exists an $\varepsilon$-slide for $u$, then for $n$ sufficiently large, there exists an $\varepsilon$-slide for $u_n$.

Strategy: For an $\varepsilon$-slide $(\phi, K)$, “transfer” $K(\tau) \cap S_u$ to $S_{u_n}$, leaving the rest alone. Precisely,

$$K_n(\tau) := \bigcup_i (K(\tau_i) \setminus K(\tau_i')) \cup (K(\tau) \setminus S_u) \cup \mathcal{T}_n\left(K(\tau)\right)$$

and define $\phi_n(\tau)$ to be the elastic minimizer subject to $S_{\phi_n(\tau)} \subset K_n(\tau)$.

- Second: show that all drops in energy for $u$ come from drops in energy for $u_n$.

- Smaller issue: show that $\varepsilon$-stability implies local minimality (quick for fracture).