Fracture evolution and locality

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Quasi-static evolution

The problem that we consider is to predict crack paths in a quasi-static setting. More precisely, we have an elastic material occupying a domain Ω and we suppose that the material is in equilibrium subject to a boundary condition $f(t)$. Then, if there is no crack, the displacement $u(t)$ minimizes

$$E_{el}(v) = \int_\Omega |\nabla v|^2 \quad \text{subject to } v = f(t) \text{ on } \partial\Omega$$

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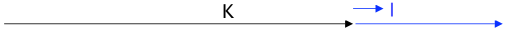
If there is a fixed crack K, each displacement would solve the same Dirichlet problem, but in the space $H^1(\Omega \setminus K)$ instead of $H^1(\Omega)$, which implies that the stored elastic energy can only be lower if there is a crack.

Griffith's criterion

The starting point for predicting crack growth is Griffith's criterion (1920). Griffith considered a pre-existing crack K with a specified future path (here in blue).



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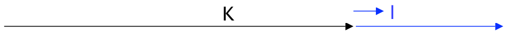

For a crack increment of length l, $E(l)$ is the elastic energy of the corresponding elastic equilibrium. The criterion states that the crack can only grow if the rate of decrease of elastic energy as l increases is large enough, i.e., if

$$-\frac{dE(l)}{dl} \quad \begin{array}{ll} < G_c & \text{the crack can not run} \\ = G_c & \text{the crack can run} \\ > G_c & \text{the crack is unstable.} \end{array}$$

The static problem

Formulated by Ambrosio and Braides (1995): If u minimizes

$$v \mapsto \int_\Omega |\nabla v|^2 + \mathcal{H}^1(S_v)$$

over $v \in SBV_f(\Omega)$, then the crack $K := S_u$ is stable (taking $G_c = 1$). The reason is that each increment in length l cannot reduce the energy, i.e.,

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Note: This and all that follows is in any dimension, but we will assume 2-D throughout.

Quasi-static formulation

This led to the following quasi-static formulation (Francfort-Marigo, 1998), where $f = f(t)$.

For discrete times $\{t_i\}$, $u(t_i)$ minimizes

$$v \mapsto \int_\Omega |\nabla v|^2 + \mathcal{H}^1(S_v \setminus \bigcup_{j<i} S_{u(t_j)})$$

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The plan was then to take a sequence of discretizations $\{t_i^n\}$ with, e.g., $t_i^n - t_{i-1}^n = \frac{1}{n}$, resulting in a sequence $\{u_n\}$ that hopefully converges to a u that is a solution to a corresponding continuous-time problem. This has been carried out (Dal Maso, Francfort, L., Toader).

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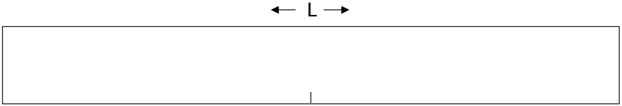

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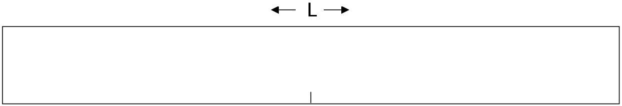

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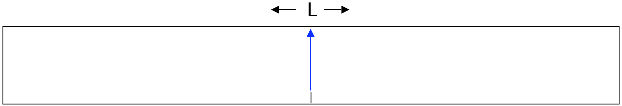

At the pre-existing crack, the energy release rate can be made arbitrarily small by choosing a suitable boundary condition, independent of L, but if L is large enough, global minimization will result in the crack growing. This violates Griffith.

Note the connection to local vs. global minimality – the initial crack was a local minimizer and was stable in the sense of Griffith.

Quasi-static evolution with local minimization

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We want to have that if u is a strict local minimizer, then there is no accessible state with lower energy $\iff$ strict local minimizers are stable.

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Unfortunately, there are technical difficulties in proving that, when $u_n(t) \to u(t)$, we have the properties we want for $u(t)$. In particular, local minimality is a problem.

In fact, the same problem occurs with accessibility – we would need to show that since the u_n were stable, so is u, i.e., if there is a v that is accessible from u and has lower energy, then there are v_n that are accessible from u_n and have lower energy.

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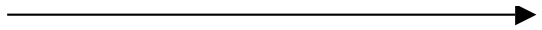
We illustrate part of the problem in terms of stability.

Consider a crack K that is unstable, i.e., the total energy decreases as the crack grows:



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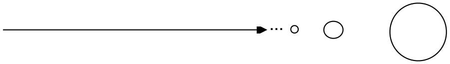

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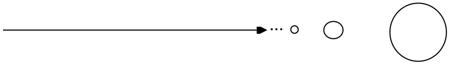

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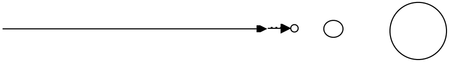
Our view:



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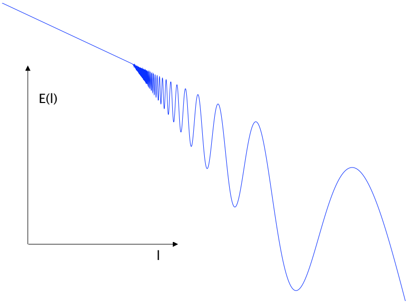

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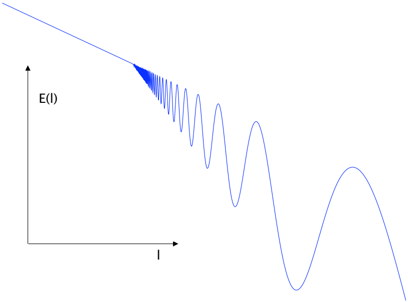

so that we cannot move to the state that we want without the energy initially increasing by a small amount. Allowing small energy increases in our definitions of accessibility and stability overcomes all mathematical issues.

Definition (ε-accessible)

v is ε-accessible from u if there exists a continuous function $\phi : [0,1] \to SBV(\Omega)$ such that $\phi(0) = u$, $\phi(1) = v$, and

$$\sup_{\tau_1 < \tau_2} [E(\phi(\tau_2)) - E(\phi(\tau_1))] < \varepsilon.$$

We then have the corresponding definition of stability:

Definition (ε-stability)

u is ε-stable if there does not exist an ε-accessible v with strictly lower energy. The path to such a v is called an ε-slide.

We also define $\bar{\varepsilon}$-accessibility, where the inequality is not strict.

Existence theorem

Theorem

Given $f(t)$ with sufficient regularity, there exists a quasi-static evolution $u(t)$ with the properties of a globally minimizing evolution, modified as follows:

- *$u(t)$ is a local minimizer at every t (coming from being $\bar{\varepsilon}$-stable)*
- *Energy inequality:*

$$E(u(t_2)) - E(u(t_1)) \leq \int_{t_1}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \dot{f} \, dxdt$$

  for every $t_1 \leq t_2$.
- *If $u^-(t) \neq u^+(t)$, then $u^+(t)$ is $\bar{\varepsilon}$-accessible from $u^-(t)$ and has lower energy than all states that are ε-accessible from $u^-(t)$.*

Proof.

- First issue: show that if $u_n \rightharpoonup u$ and there exists an ε-slide for u, then for n sufficiently large, there exists an ε-slide for u_n.

  Strategy: For an ε-slide (ϕ, K), "transfer" $K(\tau) \cap S_u$ to S_{u_n}, leaving the rest alone. Precisely,

  $$K_n(\tau) := \cup_i (K(\tau_i) \setminus K(\tau_i')) \cup (K(\tau) \setminus S_u) \cup \mathcal{T}_n\Big(K(\tau)\Big)$$

  and define $\phi_n(\tau)$ to be the elastic minimizer subject to $S_{\phi_n(\tau)} \subset K_n(\tau)$.
- Second: show that all drops in energy for u come from drops in energy for u_n.
- Smaller issue: show that ε-stability implies local minimality (quick for fracture).

□