\( \Gamma \) convergence for local minimization

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May 12, 2008
$E_n \rightharpoonup E$ if:

- $u_n \to u \Rightarrow E(u) \leq \liminf_{n \to \infty} E_n(u_n)$
- $\forall u, \exists u_n$ s.t. $u_n \to u$ and $E(u) = \lim_{n \to \infty} E_n(u_n)$

We can then consider $E_n$ in order to understand minimizers of $E$:

$$u \text{ minimizes } E \iff \exists u_n \to u \text{ s.t. } E_n(u_n) = \min E_n + o(1)$$

(assuming some compactness).

A natural question is, what about local minimizers, which are more physically relevant?

Some simple examples show that in general there is little connection between local minimizers of $E_n$ and local minimizers of $E$. 
We consider the *Manhattan metric* function $\phi : \mathbb{Z}^2 \rightarrow \{1, 2\}$

$$
\varphi(x_1, x_2) = \begin{cases} 
1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\
2 & \text{otherwise},
\end{cases}
$$

and the related scaled-perimeter functionals with forcing term $f$

$$
E_n(A) = \int_A f(x) \, dx + \int_{\partial A} \varphi(nx) \, d\mathcal{H}^1
$$
defined on Lipschitz sets $A$. We assume that $\|f\|_\infty \leq 1$, so that the first integral is continuous with respect to the convergence $A_j \rightarrow A$. We then have that $E_n$ $\Gamma$-converge to

$$
E(A) = \int_A f(x) \, dx + \int_{\partial^* A} g(\nu) \, d\mathcal{H}^1
$$
defined on all sets of finite perimeter, where

$$
g(\nu) = \|\nu\|_1 = |\nu_1| + |\nu_2|.
$$
But, it is easy to see that (limits of) local minimizers of \( E_n \) do not correspond to local minimizers of \( E \).

In fact, every set whose boundary lies in the set where \( \phi(n \cdot) = 1 \) is a local minimizer of \( E_n \), so every set of finite perimeter is the limit of local minimizers of \( E_n \).

The question is then, is there some way that, looking only at the \( E_n \) energies, we can deduce the (strict) local minimizers of \( E \)? (Joint work with A. Braides)

**Definition (\( \varepsilon \)-slide and \( \varepsilon \)-stability)**

Let \( F : X \to [0, +\infty] \) and \( \varepsilon > 0 \). An \( \varepsilon \)-slide for \( F \) at \( u \) is a continuous function \( \phi : [0, 1] \to X \) such that \( \phi(0) = u \), \( F(\phi(t)) < F(\phi(s)) + \varepsilon \) if \( 0 \leq s < t \leq 1 \), and \( E(\phi(1)) < E(u) \).

We say that \( u \) is \( \varepsilon \)-stable for \( F \) if no \( \varepsilon \)-slide exists, and stable if it is \( \varepsilon \)-stable for \( \varepsilon > 0 \) small enough.
Definition (stable convergence)

Let $\varepsilon > 0$; we say that $E_n$ converge $\varepsilon$-stably to $E$ if the following hold:

1. If $u$ has an $\varepsilon$-slide for $E$ and $u_n \to u$, then each $u_n$ has an $(\varepsilon + o(1))$-slide for $E_n$.

2. If $u$ is a strict local minimizer of $E$, then there exist $u_n \to u$ such that each $u_n$ is $\varepsilon$-stable for the corresponding $E_n$.

We say that $E_n$ converge stably to $E$ if it converges $\varepsilon$-stably to $E$ for all $\varepsilon > 0$ small enough, and we will write $E_n \overset{s-\Gamma}{\to} E$.

Example

Consider $X = IR$ and set $E_n(x) := 1 + \sin(nx)$. Then $E_n$ $\Gamma$-converges to $E = 0$, but since no $u$ has $\varepsilon$-slides for $E$, and no $u$ is a strict local minimizer of $E$, $E_n$ $\varepsilon$-stably converges to $E$ (as does every other sequence of energies!). But, $E_n(x) + x$ does not $\varepsilon$-stably converge to $E(x) + x$. 
Theorem

For the Manhattan energies $E_n$, we have $E_n \overset{s-\Gamma}{\to} E$, where $E$ is the $\Gamma$-limit of $E_n$.

Proof.

(sketch)

1. is straightforward: given an $A$ and an $\varepsilon$-slide $\psi$, and $A_n \to A$, we modify $\psi$ such that $\psi_n(0) = A_n$, and changes in $\psi_n(\cdot)$ occur within one “cell” at a time, so that the energy never increases by more than $4/n$ (due to the Manhattan perimeter function), plus the increases in the original $\varepsilon$-slide $\psi$.

2. We can choose $\delta > 0$ such that $A$ is a (strict) minimizer of $E$ within $B(A, \delta)$, and $A_n$ a solution of $\min\{E_n(A') : A' \in B(A, \delta)\}$. Then $A_n \to A_0$ and no $\varepsilon$-slide exists for $E_n$ from $A_n$ if $\varepsilon < \min\{E_n(A) : |A \triangle A_0| = \delta\} - E_n(A_n)$. 

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Back to first example:

**Example**

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So, unlike $\Gamma$ convergence, $s - \Gamma$ convergence is not stable under continuous perturbations.

But...
Definition

We say that $E_n \xrightarrow{s-\Gamma} E$ if the following hold:

1. $E_n \xrightarrow{s-\Gamma} E$ and $E_n \Gamma$-converges to $E$
2. If $\phi$ is a path from $u$ and $u_n \to u$, with $E_n(u_n)$ and $E(\phi(\tau))$ bounded, then there exist paths $\psi_n$ and $\phi_n$ such that i) $\psi_n(0) = u_n$ ii) $\tau \mapsto E_n(\psi_n(\tau))$ is decreasing up to $o(1)$ iii) $\psi_n(1) = \phi_n(0)$ iv) $\sup_{\tau \in [0,1]} \text{dist}(\phi_n(\tau), \phi(\tau)) = o(1)$ v) there exist $0 = \tau_1^n < \tau_2^n < ... < \tau_n^n = 1$ with $\tau_i^n - \tau_{i-1}^n = o(1)$ such that $\max |E_n(\phi_n(\tau_i^n)) - E(\phi(\tau_i^n))| = o(1)$ and $E_n(\phi_n(\tau))$ is between $E_n(\phi_n(\tau_i^n))$ and $E_n(\phi_n(\tau_{i+1}^n))$ for $\tau \in (\tau_i^n, \tau_{i+1}^n)$, up to $o(1)$
3. $E_n$ and each $E$ are sequentially lower semicontinuous
4. $\varepsilon$-stability for $E$ implies local minimality for all $\varepsilon$.

Theorem

If $E_n \xrightarrow{\varepsilon-\Gamma} E$, then $(E_n + G) \xrightarrow{s-\Gamma} (E + G)$ for every continuous $G$. 
Finally:

We define a notion of Gamma convergence, so that (with some assumptions on $E_n$ and $E$)

**Theorem**

If $E_n \overset{s-\Gamma}{\to} E$, then if $S$ is the set

$$\left\{ u : \exists \{u_n\}, \varepsilon > 0 \text{ such that } u_n \to u \text{ and } u_n \varepsilon\text{-stable for } E_n \right\},$$

we have

$$\{\text{strict local minimizers of } E\} \subset S \subset \{\text{local minimizers of } E\}.$$