Γ convergence for local minimization

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$E_n \xrightarrow{\Gamma} E$ if:

- $u_n \to u \Rightarrow E(u) \leq \liminf_{n\to\infty} E_n(u_n)$
- $\forall u$, $\exists u_n$ s.t. $u_n \to u$ and $E(u) = \lim_{n\to\infty} E_n(u_n)$

We can then consider E_n in order to understand minimizers of E:

$$u \text{ minimizes } E \iff \exists u_n \to u \text{ s.t. } E_n(u_n) = \min E_n + o(1)$$

(assuming some compactness).

A natural question is, what about local minimizers, which are more physically relevant?

Some simple examples show that in general there is little connection between local minimizers of E_n and local minimizers of E.

We consider the *Manhattan metric* function $\phi : \mathbb{Z}^2 \to \{1, 2\}$

$$\varphi(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\ 2 & \text{otherwise,} \end{cases}$$

and the related scaled-perimeter functionals with forcing term f

$$E_n(A) = \int_A f(x)\, dx + \int_{\partial A} \varphi(nx)\, d\mathcal{H}^1$$

defined on Lipschitz sets A. We assume that $\|f\|_\infty \leq 1$, so that the first integral is continuous with respect to the convergence $A_j \to A$.
We then have that E_n Γ-converge to

$$E(A) = \int_A f(x)\, dx + \int_{\partial^* A} g(\nu) d\mathcal{H}^1$$

defined on all sets of finite perimeter, where

$$g(\nu) = \|\nu\|_1 = |\nu_1| + |\nu_2|.$$

But, it is easy to see that (limits of) local minimizers of E_n do not correspond to local minimizers of E.

In fact, every set whose boundary lies in the set where $\phi(n\,\cdot) = 1$ is a local minimizer of E_n, so every set of finite perimeter is the limit of local minimizers of E_n.

The question is then, is there some way that, looking only at the E_n energies, we can deduce the (strict) local minimizers of E? (Joint work with A. Braides)

Definition (ε-slide and ε-stability)

Let $F : X \to [0, +\infty]$ and $\varepsilon > 0$. An *ε-slide* for F at u is a continuous function $\phi : [0, 1] \to X$ such that $\phi(0) = u$, $F(\phi(t)) < F(\phi(s)) + \varepsilon$ if $0 \le s < t \le 1$, and $E(\phi(1)) < E(u)$.
We say that u is *ε-stable* for F if no ε-slide exists, and *stable* if it is ε-stable for $\varepsilon > 0$ small enough.

Definition (stable convergence)

Let $\varepsilon > 0$; we say that E_n *converge ε-stably* to E if the following hold:

1. If u has an ε-slide for E and $u_n \to u$, then each u_n has an $(\varepsilon + o(1))$-slide for E_n
2. If u is a strict local minimizer of E, then there exist $u_n \to u$ such that each u_n is ε-stable for the corresponding E_n.

We say that E_n *converge stably* to E if it converges ε-stably to E for all $\varepsilon > 0$ small enough, and we will write $E_n \overset{s-\Gamma}{\to} E$.

Example

Consider $X = IR$ and set $E_n(x) := 1 + \sin(nx)$. Then E_n Γ-converges to $E = 0$, but since no u has ε-slides for E, and no u is a strict local minimizer of E, E_n ε-stably converges to E (as does every other sequence of energies!). But, $E_n(x) + x$ does not ε-stably converge to $E(x) + x$.

Theorem

For the Manhattan energies E_n, we have $E_n \overset{s-\Gamma}{\to} E$, where E is the Γ-limit of E_n.

Proof.

(sketch)

1. is straightforward: given an A and an ε-slide ψ, and $A_n \to A$, we modify ψ such that $\psi_n(0) = A_n$, and changes in $\psi_n(\cdot)$ occur within one "cell" at a time, so that the energy never increases by more than $4/n$ (due to the Manhattan perimeter function), plus the increases in the original ε-slide ψ.
2. We can choose $\delta > 0$ such that A is a (strict) minimizer of E within $B(A, \delta)$, and A_n a solution of $\min\{E_n(A') : A' \in B(A, \delta)\}$. Then $A_n \to A_0$ and no ε-slide exists for E_n from A_n if $\varepsilon < \min\{E_n(A) : |A \triangle A_0| = \delta\} - E_n(A_n)$.

□

Back to first example:

Example

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So, unlike Γ convergence, $s - \Gamma$ convergence is not stable under continuous perturbations.

But...

Definition

We say that $E_n \xrightarrow{\varepsilon\text{-}\Gamma^s} E$ if the following hold:

1. $E_n \overset{s-\Gamma}{\rightarrow} E$ and E_n Γ-converges to E
2. If ϕ is a path from u and $u_n \rightarrow u$, with $E_n(u_n)$ and $E(\phi(\tau))$ bounded, then there exist paths ψ_n and ϕ_n such that i) $\psi_n(0) = u_n$ ii) $\tau \mapsto E_n(\psi_n(\tau))$ is decreasing up to $o(1)$ iii) $\psi_n(1) = \phi_n(0)$ iv) $\sup_{\tau\in[0,1]}\text{dist}(\phi_n(\tau), \phi(\tau)) = o(1)$ v) there exist $0 = \tau_1^n < \tau_2^n < \ldots < \tau_n^n = 1$ with $\tau_i^n - \tau_{i-1}^n = o(1)$ such that $\max|E_n(\phi_n(\tau_i^n)) - E(\phi(\tau_i^n))| = o(1)$ and $E_n(\phi_n(\tau))$ is between $E_n(\phi_n(\tau_i^n))$ and $E_n(\phi_n(\tau_{i+1}^n))$ for $\tau \in (\tau_i^n, \tau_{i+1}^n)$, up to $o(1)$
3. E_n and each E are sequentially lower semicontinuous
4. ε-stability for E implies local minimality for all ε.

Theorem

If $E_n \xrightarrow{\varepsilon\text{-}\Gamma^s} E$, then $(E_n + G) \overset{s-\Gamma}{\rightarrow} (E + G)$ for every continuous G.

Finally:

We define a notion of Gamma convergence, so that (with some assumptions on E_n and E)

Theorem

If $E_n \overset{s-\Gamma}{\to} E$, *then if* S *is the set*

$$\{u : \exists \{u_n\}, \varepsilon > 0 \text{ such that } u_n \to u \text{ and } u_n\ \varepsilon\text{-stable for } E_n\},$$

we have

$$\{\text{strict local minimizers of } E\} \subset S \subset \{\text{local minimizers of } E\}.$$