Crack fronts and dissipation

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joint work with
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Quasi-static evolution

Suppose that we have an elastic material initially occupying a domain Ω and for each time t the material is in equilibrium subject to a boundary condition $f(t)$. Without the possibility of fracture, we have that the displacement $u(t)$ minimizes

$$E_{el}(v) = \int_\Omega W(\nabla v)dx \quad \text{subject to } v = f(t) \text{ on } \partial\Omega$$

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If there is a fixed crack C, the displacement at each time would solve the same Dirichlet problem as above, but in the space $H^1(\Omega \setminus C)$ instead of $H^1(\Omega)$, which implies that the stored elastic energy can only be lower if there is a crack.

Griffith's criterion

The essence of Griffith's criterion for crack growth is that a crack grows only when the reduction in elastic energy equals or exceeds the increase in surface area of the crack (which can be viewed as energy dissipation).

This led to modeling quasi-static evolution by time-incremental minimization problems, first in the engineering community and then more mathematically, using SBV, by Francfort and Marigo:

For discrete times $\{t_i\}$, $u(t_i)$ minimizes

$$v \mapsto \int_\Omega W(\nabla v) + \mathcal{H}^{N-1}(S_v \setminus C(t_{i-1}))$$

over $v \in SBV_{f(t_i)}(\Omega)$, where

$$C(t_{i-1}) := \bigcup_{j<i} S_{u(t_j)}.$$

The technicality in the last term models irreversibility of fracture, so that only the new crack at t_i is penalized.

Front speeds and dissipation

Now suppose that for a given crack path $t \mapsto C(t)$ we have defined a crack "front" at time t, $F(t) \subset \mathcal{R}^N$, which we expect to be $N-2$-dimensional, with front speed $v(x,t)$. Then the above dissipation, over a time interval $[t, t+\Delta t]$, is also given by

$$\int_t^{t+\Delta t} \int_{F(\tau)} v \; d\mathcal{H}^{N-2} d\tau,$$

since this is the same as

$$\mathcal{H}^{N-1}\Big(C(t+\Delta t) \setminus C(t)\Big).$$

We note that this dissipation is rate-independent, since all that matters is the size of the increment, and not the speed, for example. Here, the dissipation rate at time t is then

$$\int_{F(t)} v \; d\mathcal{H}^{N-2}.$$

Crack derivatives: fronts and front speeds

Question: can we consider general dissipations that are local to "where" the crack is growing and depend on the local speed (possibly rate dependent)? If so, maybe we will see some features that are experimentally observed, but do not occur in rate-independent fracture (branching, mother-daughter cracking).

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We need to define the crack front (or "tip" in 2-dimensions). This is a plus, since fracture mechanics relies heavily on the idea of a crack tip (e.g., stress intensity factors, J-integrals, etc.) without there being a mathematical definition.

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$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x) d\mathcal{H}^{N-1}(x) dt = -\int_0^T \varphi(t) \int_{F(t)} f(x) v(x,t) d\mathcal{H}^{N-2}(x) dt$$

$\forall \varphi \in C_0^1([0,T]), \forall f \in C_0(\Omega)$, then F is the front and v is the front speed.

General dissipations

We can then consider general dissipations of the form

$$\int_0^T \int_{F(t)} \psi(v) d\mathcal{H}^{N-2} dt$$

e.g., instead of $\mathcal{H}^{N-1}\big(S_v \setminus C(t_{i-1})\big)$ in the discrete quasi-static formulation, take

$$\inf_C \int_{t_{i-1}}^{t_i} \int_{F(t)} \psi(v) d\mathcal{H}^{N-2} dt \qquad (1)$$

over C satisfying $C(t_{i-1}) = \emptyset$, $C(t_i) = S_v \setminus C(t_{i-1})$.

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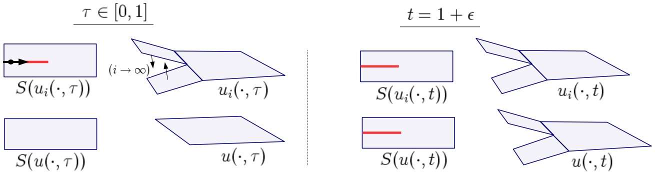
over C satisfying $C(t_{i-1}) = \emptyset$, $C(t_i) = S_v \setminus C(t_{i-1})$.
We can also consider functionals of the form introduced by Mielke and Ortiz:

$$I(u, C) := \int_0^T e^{-\frac{t}{\varepsilon}} \left\{ \frac{1}{\varepsilon} \int_\Omega W(\nabla u) dx + \int_{F(t)} \psi(v) d\mathcal{H}^{N-2} \right\} dt.$$

Existence

With the Mielke-Ortiz functional, we have serious existence issues. For example, we cannot take $C(t) := \cup_{\tau \leq t} S_{u(\tau)}$ as is done in the discrete-time problem. That is, if we take a minimizing sequence u_i, and we have $u_i \longrightarrow u$, then u might not have the front representation:



with $t \mapsto \mathcal{H}^{N-1}(C(t))$ not absolutely continuous in time.

Existence

We have existence in 2D of a minimizer for

$$I(p) := \int_{[0,T]} e^{-\frac{t}{\varepsilon}} \left\{ \frac{1}{\varepsilon} \int_\Omega W(\nabla u) dx + \int_{F(t)} v^p d\mathcal{H}^0 \right\} dt,$$

over the class $p = (u, C, F) \in \mathcal{P}$:

- $C : [0, T] \to \{K : K \subset \Omega, \mathcal{H}^1(K) < \infty\}$ such that:
  - C nondecreasing
  - $\forall \tau \leq t$, $S(u(\tau)) \subset C(t)$
  - (C, F) satisfy the front representation
- Plus constraints on how the crack can grow (e.g., a bound on the number of points in the front (1!), $t \mapsto F(t)$ needs to be continuous)

Idea of Proof

1. $p_i = (u_i, C_i, F_i)$ minimizing
2. F_i bounded in $W^{1,p}$, $F_i \to F$ uniformly ($F \in W^{1,p}$)
3. $C(t) := \bigcup_{\tau < t} F(t)$
4. $u(t)$ defined as minimizer of

$$v \mapsto \int_\Omega W(\nabla v) : v \in SBV(\Omega), S_v \subset C(t)$$

5. The triple $p = (u, C, F)$ is a minimizer (real issue: show $p \in \mathcal{P}$).

More General Models

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with, e.g., $\psi(v) = \alpha + v^p$. Features:

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where $\mathcal{C} = \inf \frac{\psi(v)}{v}$.

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where $\mathcal{C} = \inf \frac{\psi(v)}{v}$. This is rate independent!

Relaxation

For the Mielke-Ortiz functional:

Convergence for admissible u_i, u:

$$u_i \to u \iff u_i(\cdot, t) \stackrel{SBV}{\to} u(\cdot, t) \; \forall t \in [0, T].$$

We then try to find the relaxation of I (with the above convergence):

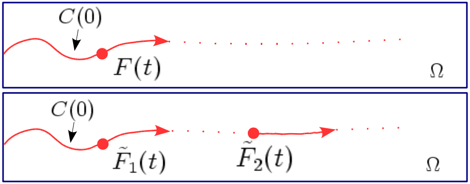
$$I^*(u) := \inf_{\substack{u_i \in \mathcal{P} \\ u_i \to u}} \left\{ \liminf_{i \to \infty} I(u_i) \right\}. \tag{2}$$

Relaxation Result

The relaxed energy is:

$$I^*(u) = \int_0^T e^{-t/\varepsilon} \left\{ \frac{1}{\varepsilon} \int_\Omega W(\nabla u) dx + \mathcal{C} \int_{F(t)} v \, d\mathcal{H}^{N-2} \right\} dt.$$

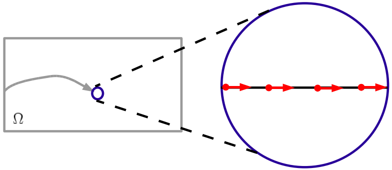
Reason: use multiple fronts ahead of the crack tip (daughter cracks), and always use the optimal speed (achieving $\mathcal{C}$).



Idea of Proof

This generalizes by a covering argument:

- Take an arbitrary admissible u.
- Break up $[0, T]$ into increments like $\frac{1}{i}$.
- Using Lebesgue points and Besicovitch covering, we get a countable disjoint collection of balls in which $C(t_i) \setminus C(t_{i-1})$ is close to a hyperplane
- In each ball use an optimal front configuration to build the crack increment



N-dimensions