LECTURE: 21 - M, 26 November 2001

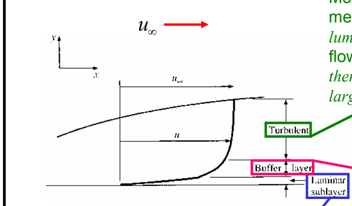
ES-3003 Heat Transfer

Today's topics

- Turbulent boundary layer heat transfer
  - Eddy viscosity
  - Eddy thermal conductivity
  - Fluctuations
  - Fluid-friction analogy
- Turbulent boundary layer thickness
- Heat transfer in tube flow
  - Bulk temperature
- Turbulent flow in a tube
- Heat transfer in a high-speed flow
  - Mach number

Term B'2001

Turbulent boundary layer heat transfer

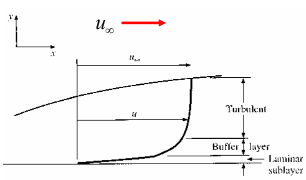


Momentum and heat exchange mechanism involves *macroscopic lumps* of fluid moving about in the flow. *Eddy viscosity* and *eddy thermal conductivity* may be *much larger than the molecular values.*

Some turbulent action is experienced, but the *molecular* viscous action and heat conduction are still important

A very thin region near the plate surface that has a *laminar character* and the viscous action and heat transfer take place under circumstances like those in a laminar flow

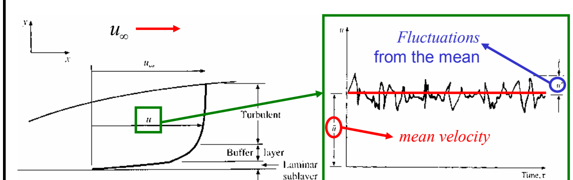
Heat transfer in turbulent flow



- The physical mechanism of *heat transfer in turbulent flow* is quite *similar to that in laminar flow*
- The primary difference is that *one must deal with eddy properties* instead of the ordinary thermal conductivity and viscosity

- The main difficulty in an analytical treatment is that *the eddy properties vary across the boundary layer*
- The difficulties are compounded because *the variations can only be determined experimentally*
- All *analyses of turbulent flow rely on experimental data* because *there is no adequate theory to predict turbulent flow*, at this time

Laser anemometry measurement of fluctuations



The *instantaneous velocity u* in the *x*-direction is defined as

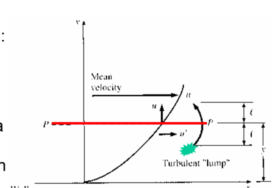
$$u = \bar{u} + u'$$

and similarly, the instantaneous velocity *v* in the *y*-direction is defined as

$$v = \bar{v} + v'$$

Turbulent shear stress

Fluctuations give rise to a *turbulent shear stress*, which may be analyzed as follows:



- The instantaneous turbulent mass-transport rate per unit area across the plane *P-P* is $\rho v'$
- Associated with this mass-transport is a change in the *x*-component of velocity u'
- The net momentum flux per unit area, in the *x*-direction, represents the turbulent shear stress at the plane *P-P*, or $\rho v' u'$
- When the turbulent lump moves upward (i.e., $v'>0$), it enters a region of higher $\bar{u}$ and is therefore likely to effect a slowing-down fluctuation u', i.e., $u'<0$
- A similar argument can be made for $v'<0$, so that the *average turbulent-shear stress* τ_t will be given as

$$\tau_t = -(\rho v' u')_{avg} \quad \text{where} \quad \bar{v}' = \bar{u}' = 0 \quad \text{and} \quad (v' u')_{avg} \neq 0$$

Eddy viscosity

Eddy viscosity, or *eddy diffusivity for momentum* ε_M is defined as

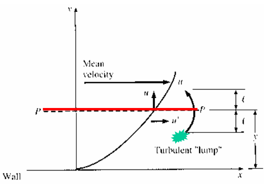
$$\tau_t = -(\rho v' u')_{avg} = \rho \varepsilon_M \frac{du}{dy}$$

Molecular-transport problems are analyzed using the concept of *mean free path*, or the average distance a particle travels between collisions.

A similar concept was introduced by Prandtl for describing turbulent-flow phenomena.

The *Prandtl mixing length* is the distance traveled, on the average, by the turbulent lumps of fluid in the direction normal to the mean flow.

Prandtl mixing length



If a turbulent lump is located a distance l above or below the plane P-P, it can move back and forth across P-P and give rise to the eddy or turbulent shear-stress effect. At $y+l$ and $y-l$, the velocities would be, respectively,

$$u(y+l) \approx u(y) + l\frac{du}{dy}$$

$$u(y-l) \approx u(y) - l\frac{du}{dy}$$

and lead to turbulent fluctuations

$$u' \approx l\frac{du}{dy} \quad \text{and} \quad v' \approx l\frac{du}{dy} \longrightarrow l \text{ is the } \textit{Prandtl mixing length}$$

Velocity profile in a turbulent boundary layer

$$\tau_t = -\left(\rho v' u'\right)_{avg} = \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2 = \rho \varepsilon_M \frac{\partial u}{\partial y} \longrightarrow \varepsilon_M = l^2 \frac{\partial u}{\partial y}$$

where the mixing length is proportional to the distance from the wall, i.e.,

$$l = Ky$$

Since near the wall region the shear stress is approximately zero

$$\tau_t \approx \tau_w$$

$$\tau_w = \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2 = \rho K^2 y^2 \left(\frac{\partial u}{\partial y}\right)^2 \longrightarrow \frac{\partial u}{\partial y} = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}}\frac{1}{y}$$

Upon integration we obtain $u = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}} \ln y + C$

Velocity profile in a turbulent boundary layer in good agreement with experiments except in the region very close to the wall

Universal velocity profile

$$\frac{\tau}{\rho} = (\nu + \varepsilon_M)\frac{\partial u}{\partial y} \longrightarrow du^+ = \frac{dy^+}{1 + \frac{\varepsilon_M}{\nu}}$$

where

$$u^+ = \frac{u}{\sqrt{\frac{\tau_w}{\rho}}} \quad \text{and} \quad y^+ = \frac{\sqrt{\frac{\tau_w}{\rho}}\, y}{\nu}$$

In the laminar sublayer, $\varepsilon_M = 0$ and

$$du^+ = \frac{dy^+}{1 + \frac{\varepsilon_M}{\nu}} \longrightarrow du^+ = dy^+ \longrightarrow u^+ = y^+ + C$$

At the wall, $u^+ = 0$ for $y^+ = 0$ so that $C = 0$ $\longrightarrow$ $u^+ = y^+$ Linear velocity profile in the laminar sublayer

Universal velocity profile, cont'd

In the turbulent region $\frac{\varepsilon_M}{\nu} >> 1$

$$\varepsilon_M = l^2 \frac{\partial u}{\partial y} = (Ky)^2 \left(\frac{1}{K}\sqrt{\frac{\tau_w}{\rho}}\frac{1}{y}\right) = K\sqrt{\frac{\tau_w}{\rho}}\, y = Ky^+ \nu \longrightarrow \frac{\varepsilon_M}{\nu} = Ky^+$$

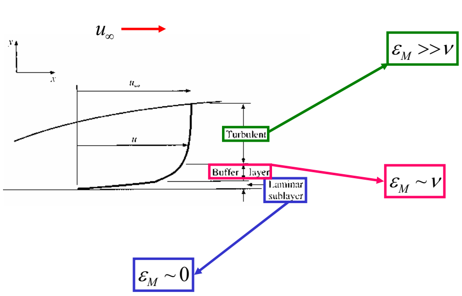
$$du^+ = \frac{dy^+}{1 + \frac{\varepsilon_M}{\nu}} \longrightarrow du^+ = \frac{dy^+}{\left(\frac{\varepsilon_M}{\nu}\right)} = \frac{dy^+}{Ky^+} \longrightarrow u^+ = \frac{1}{K}\ln y + C$$

A similar equation is also obtained for the buffer zone. In summary:

Laminar sublayer:	$0 < y^+ < 5$	$u^+ = y^+$
Buffer layer:	$5 < y^+ < 30$	$u^+ = 5.0 \ln y^+ - 3.05$
Turbulent layer:	$30 < y^+ < 400$	$u^+ = 2.5 \ln y^+ + 5.5$

} Universal velocity profile

Turbulent boundary layer heat transfer



Turbulent heat transfer

BASED ON FLUID-FRICTION ANALOGY

In the turbulent-flow region, where

$\varepsilon_M >> \nu$ and $\varepsilon_H >> \alpha$ $\qquad$ $\varepsilon_M \approx 100\nu$ and $\varepsilon_H \approx 100\alpha$

eddy viscosity; molecular viscosity; eddy diffusivity; molecular diffusivity

The *turbulent Prandtl number* is defined as

$$\Pr = \frac{\varepsilon_M}{\varepsilon_H}$$

The *local skin friction coefficient* is defined as

$$C_{fx} = 0.0592 \mathrm{Re}_x^{-1/5} \quad \text{for} \quad 5\times10^5 < \mathrm{Re}_x < 10^7$$

and

$$C_{fx} = 0.370 (\log \mathrm{Re}_x)^{-2.584} \quad \text{for} \quad 10^7 < \mathrm{Re}_x < 10^9$$

Turbulent heat transfer, cont'd

BASED ON FLUID-FRICTION ANALOGY

The *average-friction coefficient* for a flat plate is

$$\bar{C}_f = \frac{0.455}{(\log \mathrm{Re}_L)^{2.584}} - \frac{A}{\mathrm{Re}_L} \quad \text{for} \quad \mathrm{Re}_L < 10^9$$

or

$$\bar{C}_f = \frac{0.074}{\mathrm{Re}_L^{1/5}} - \frac{A}{\mathrm{Re}_L} \quad \text{for} \quad \mathrm{Re}_L < 10^7$$

where A is obtained from

Table 5-1

Re_{crit}	3×10^5	5×10^5	10^6	3×10^6
A	1055	1742	3340	8940

Turbulent heat transfer, cont'd

BASED ON FLUID-FRICTION ANALOGY

Applying the *fluid-friction analogy* → $St\,\mathrm{Pr}^{2/3} = \frac{C_f}{2}$

The local turbulent heat transfer is

$$St_x\,\mathrm{Pr}^{2/3} = 0.0296\,\mathrm{Re}_x^{-1/5} \quad \text{for} \quad 5\times10^5 < \mathrm{Re}_x < 10^7$$

and

$$St_x\,\mathrm{Pr}^{2/3} = 0.185(\log \mathrm{Re}_x)^{-2.584} \quad \text{for} \quad 10^7 < \mathrm{Re}_x < 10^9$$

The average heat transfer over the *entire laminar-turbulent boundary layer* is → $\bar{St}\,\mathrm{Pr}^{2/3} = \frac{\bar{C}_f}{2}$

For $\mathrm{Re}_{crit} = 5\times10^5$ and $\mathrm{Re}_L < 10^7$

$$\bar{St}\,\mathrm{Pr}^{2/3} = \frac{\bar{C}_f}{2} = \frac{1}{2}\left(\frac{0.074}{\mathrm{Re}_L^{1/5}} - \frac{1742}{\mathrm{Re}_L}\right) = \frac{0.037}{\mathrm{Re}_L^{1/5}} - \frac{871}{\mathrm{Re}_L}$$

Turbulent heat transfer, cont'd

BASED ON FLUID-FRICTION ANALOGY

$$\bar{St}\,\mathrm{Pr}^{2/3} = \frac{0.037}{\mathrm{Re}_L^{1/5}} - \frac{871}{\mathrm{Re}_L}$$

However, $\bar{St} = \frac{\bar{Nu}}{\mathrm{Re}_L\,\mathrm{Pr}}$ → $\bar{St}\,\mathrm{Pr}^{2/3} = \frac{\bar{Nu}_L}{\mathrm{Re}_L\,\mathrm{Pr}^{1/3}}$

$$\bar{St}\,\mathrm{Pr}^{2/3} = \bar{Nu}_L = \mathrm{Re}_L\,\mathrm{Pr}^{1/3}\left(\frac{0.037}{\mathrm{Re}_L^{1/5}} - \frac{871}{\mathrm{Re}_L}\right)$$

$$\bar{Nu}_L = \mathrm{Pr}^{1/3}\left(0.037\,\mathrm{Re}_L^{0.8} - 871\right)$$ Hwrk P-5-105

The average heat transfer coefficient can also be obtained by integrating the local values over the entire length of the plate

$$\bar{h} = \frac{1}{L}\left(\int_0^{x_{crit}} h_{lam}\,dx + \int_{x_{crit}}^{L} h_{turb}\,dx\right)$$

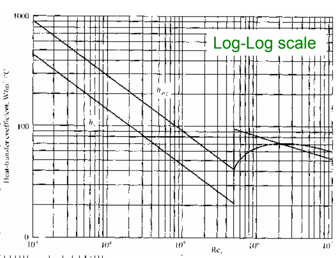
Local and average heat-transfer coefficient

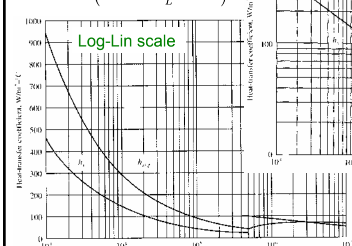
ATMOSPHERIC AIR FLOWING OVER ISOTHERMAL PLATE AT 30 m/s

$$\bar{St}\,\mathrm{Pr}^{2/3} = 0.332\,\mathrm{Re}_x^{-1/2}$$

$$\bar{St}\,\mathrm{Pr}^{2/3} = 0.0296\,\mathrm{Re}_x^{-1/5}$$

$$\bar{Nu} = \mathrm{Pr}^{1/3}\left(0.037\,\mathrm{Re}_L^{0.8} - 871\right)$$





$$\mathrm{Re}_{crit} = 5\times10^5$$

$$x_{crit} = 0.2615\,m$$

Velocity and thickness of turbulent boundary layer

The velocity profile in a turbulent boundary layer, outside the laminar sublayer, can be described by a *one-sevenths-power relation*

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

The thickness of the laminar sublayer can be computed from

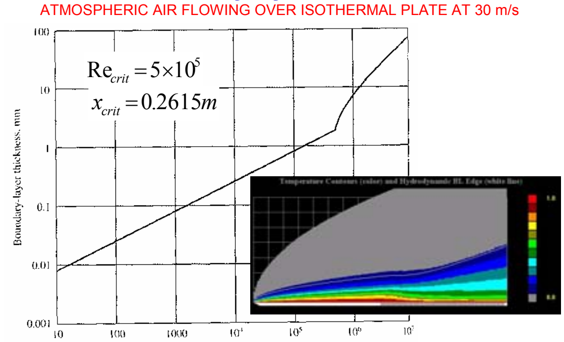
$$\frac{\delta}{x} = \frac{5.0}{\mathrm{Re}_x^{1/2}} \rightarrow \frac{\delta}{x_{crit}} = \frac{5.0}{\mathrm{Re}_{crit}^{1/2}} \quad \text{where} \quad \mathrm{Re}_{crit} = 5\times10^5$$

The thickness of the turbulent layer can be computed from

$$\frac{\delta}{x} = 0.381\,\mathrm{Re}_x^{-1/5} - 10.256\,\mathrm{Re}_x^{-1} \quad \text{for} \quad 5\times10^5 < \mathrm{Re}_x < 10^7$$

Boundary layer thickness

ATMOSPHERIC AIR FLOWING OVER ISOTHERMAL PLATE AT 30 m/s



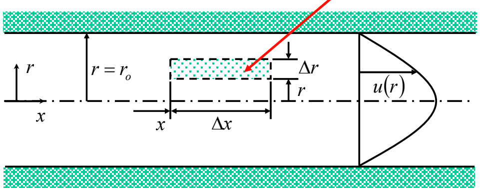
$$\mathrm{Re}_{crit} = 5\times10^5$$

$$x_{crit} = 0.2615\,m$$

Flow in a tube

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Apply the principle of conservation of momentum and energy to a small control volume

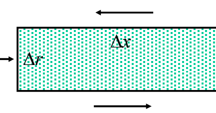


Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Force balance

$$-\mu \frac{du}{dr} 2\pi r \Delta x \Big|_{r+\Delta r} = -\mu \frac{du}{dr} 2\pi r \Delta x \Big|_{r} + \frac{d}{dr}\left(\mu \frac{du}{dr} 2\pi r \Delta x\right)\Delta r$$



$$2\pi r \Delta r P\big|_x$$

$$2\pi r \Delta r P\big|_{x+\Delta x} = 2\pi r \Delta r P\big|_x + 2\pi r \Delta r \frac{dP}{dx}\Delta x$$

$$-\mu \frac{du}{dr} 2\pi r \Delta x \Big|_{r}$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Pressure forces = viscous forces

$$2\pi \Delta r P\big|_x - \left(2\pi \Delta r P\big|_x + 2\pi r \Delta r \frac{dP}{dx}\Delta x\right) =$$

$$= -\mu \frac{du}{dr} 2\pi r \Delta x \Big|_{r} - \left[-\mu \frac{du}{dr} 2\pi r \Delta x \Big|_{r} + \frac{d}{dr}\left(\mu \frac{du}{dr} 2\pi r \Delta x\right)\Delta r\right]$$

$$r\frac{dP}{dx} - \frac{d}{dr}\left(\mu \frac{du}{dr} r\right) = 0$$

$$\frac{d}{dr}\left(r\mu \frac{du}{dr}\right) = r\frac{dP}{dx}$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

$$\frac{d}{dr}\left(r\mu \frac{du}{dr}\right) = r\frac{dP}{dx}$$

Subject to boundary conditions

$\frac{du}{dr} = 0$ at $r = 0$ because of symmetry

$u = 0$ at $r = r_o$ no slip at the wall

First integration yields

$$r\mu \frac{du}{dr} = \frac{r^2}{2}\frac{dP}{dx} + C_1 \qquad \frac{du}{dr} = \frac{r}{2\mu}\frac{dP}{dx} + \frac{C_1}{r\mu} \rightarrow C_1 = 0$$

Second integration yields

$$u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_2 \rightarrow C_2 = -\frac{r_o^2}{4\mu}\frac{dP}{dx}$$

$$u = \frac{r_o^2}{4\mu}\left(-\frac{dP}{dx}\right)\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

The velocity is given by:

$$u = \frac{r_o^2}{4\mu}\left(-\frac{dP}{dx}\right)\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

The maximum velocity is at the center $r = 0$

$$u_o = \frac{r_o^2}{4\mu}\left(-\frac{dP}{dx}\right)$$

The velocity can therefore be written as:

$$u = u_o\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

Eq. 5-98, p.243

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Bulk velocity is defined by

$$u_b = \frac{1}{\pi r_o^2}\int_0^{r_o} u 2\pi r dr$$

Carrying out the integration

$$u_b = \frac{1}{\pi r_o^2}\int_0^{r_o}\left\{u_o\left[1 - \left(\frac{r}{r_o}\right)^2\right]\right\}2\pi r dr = \frac{1}{2}u_o = \frac{r_o^2}{8\mu}\left(-\frac{dP}{dx}\right)$$

The velocity profile can then be written as

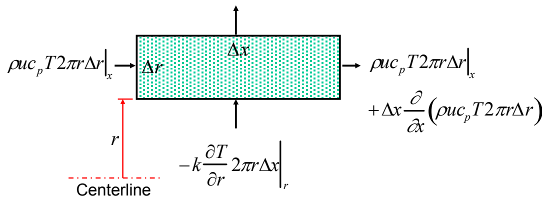
$$u = 2u_b\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Energy balance

$$-k\frac{\partial T}{\partial r}2\pi r\Delta x\Big|_r + \Delta r\frac{\partial}{\partial r}\left(-k\frac{\partial T}{\partial r}2\pi r\Delta x\right)$$



Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

$$\underbrace{\rho u c_p T 2\pi r\Delta r\big|_x}_{x} - \underbrace{\Delta x\frac{\partial}{\partial x}\left(\rho u c_p T 2\pi r\Delta r\right) - \rho u c_p T 2\pi r\Delta r\big|_x}_{x+\Delta x}$$

$$= \underbrace{-k\frac{\partial T}{\partial r}2\pi r\Delta x\Big|_r}_{r} + \underbrace{k\frac{\partial T}{\partial r}2\pi r\Delta x\Big|_r - \Delta r\frac{\partial}{\partial r}\left(-k\frac{\partial T}{\partial r}2\pi r\Delta x\right)}_{r+\Delta r}$$

The remaining terms are:

$$\Delta x\frac{\partial}{\partial x}\left(\rho u c_p T 2\pi r\Delta r\right) = \Delta r\frac{\partial}{\partial r}\left(-k\frac{\partial T}{\partial r}2\pi r\Delta x\right)$$

or

$$\rho u c_p r\frac{\partial T}{\partial x} = k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

The governing equation is

$$\rho u c_p r\frac{\partial T}{\partial x} = k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Rearranging and introducing $\alpha = k/\rho c_p$

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Substitute the velocity profile:

$$2u_b\left[1-\left(\frac{r}{r_o}\right)^2\right]\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

This is a partial differential equation for $T(r,x)$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

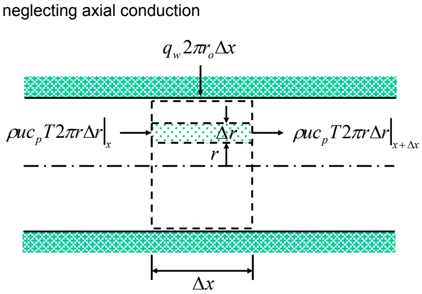
$$2u_b\left[1-\left(\frac{r}{r_o}\right)^2\right]\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

In general, the partial differential equation is not easily solved. However, for *fully developed flow* and *uniform heating*, $\partial T/\partial x$ is constant and it is possible to solve the equation analytically

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

To determine $\partial T/\partial x$, do an energy balance for a small slice of the pipe, neglecting axial conduction



Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Enthalpy flowing out - enthalpy flowing in = heat conducted in

$$\int_0^{r_o}\rho u c_p T 2\pi r dr\Big|_{x+\Delta x} - \int_0^{r_o}\rho u c_p T 2\pi r dr\Big|_x = q_w 2\pi r_o\Delta x$$

Use the definition of the bulk temperature

$$T_b = \frac{\int_0^{r_o}\rho u T 2\pi r dr}{\dot{m}}$$

to rewrite the integrals:

$$\int_0^{r_o}\rho u c_p T 2\pi r dr = \dot{m} c_p T_b = \rho u_b \pi r_o^2 c_p T_b$$

The energy balance is then

$$\rho u_b c_p \pi r_o^2 T_b\big|_{x+\Delta x} - \rho u_b c_p \pi r_o^2 T_b\big|_x = q_w 2\pi r_o\Delta x$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

$$\rho u_b c_p \pi r_o^2 T_b\big|_{x+\Delta x} - \rho u_b c_p \pi r_o^2 T_b\big|_x = q_w 2\pi r_o \Delta x$$

Noting that

$$\rho u_b c_p \pi r_o^2 T_b\big|_{x+\Delta x} = \rho u_b c_p \pi r_o^2 T_b\big|_x + \Delta x \frac{d}{dx}\left(\rho u_b c_p \pi r_o^2 T_b\right)$$

The energy balance is then

$$\Delta x \frac{d}{dx}\left(\rho u_b c_p \pi r_o^2 T_b\right) = q_w 2\pi r_o \Delta x$$

or

$$\frac{dT_b}{dx} = \frac{2q_w}{\rho u_b c_p r_o} = \text{Constant}$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Although the temperature in the tube increases, the shape of the temperature profile does not change, once the flow and the temperature is fully developed. Hence, the difference between the temperature and the bulk temperature is a function of radius only:

Thus,

$$T - T_b = f(r)$$

or

$$\frac{\partial T}{\partial x} - \frac{dT_b}{dx} = 0$$

and

$$\frac{\partial T}{\partial x} = \frac{dT_b}{dx} = \frac{2q_w}{\rho u_b c_p r_o}$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

The partial differential equation for the temperature

$$2u_b\left[1-\left(\frac{r}{r_o}\right)^2\right]\frac{\partial T_b}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{df}{dr}\right)$$

can now be written as an ordinary differential equation

$$\left[1-\left(\frac{r}{r_o}\right)^2\right]\frac{4q_w}{kr_o} = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)$$

The boundary conditions are

$$\frac{dT}{dr} = \frac{df}{dr} = 0 \quad \text{at} \quad r = 0 \quad \text{because of symmetry}$$

$$T = T_w \quad \text{at} \quad r = r_o \quad \text{or} \quad f = T_w - T_b$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

$$\left[1-\left(\frac{r}{r_o}\right)^2\right]\frac{4q_w}{kr_o} = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)$$

Rearrange the equation

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) = r\left[1-\left(\frac{r}{r_o}\right)^2\right]\frac{4q_w}{kr_o}$$

and integrate once to obtain

$$r\frac{df}{dr} = \frac{4q_w}{kr_o}\left(\frac{r^2}{2} - \frac{r^4}{4r_o^2}\right) + C_1$$

Flow in a tube, cont'd

DETERMINE VELOCITY AND TEMPERATURE PROFILES

Divide both sides by r:

$$\frac{df}{dr} = \frac{4q_w}{kr_o}\left(\frac{r}{2} - \frac{r^3}{4r_o^2}\right) + \frac{C_1}{r} \quad \longrightarrow \quad C_1 = 0$$

Integrating again we get

$$f = \frac{4q_w}{kr_o}\left(\frac{r^2}{4} - \frac{r^4}{16r_o^2}\right) + C_2 \quad \longrightarrow \quad C_2 = (T_w - T_b) - \frac{3}{4}\frac{q_w r_o}{k}$$

The temperature is therefore:

$$T = T_w - \frac{4q_w}{kr_o}\left(\frac{3r_o^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_o^2}\right)$$

Flow in a tube, cont'd

DETERMINE NUSSELT NUMBER

The bulk temperature is defined by

$$T_b = \frac{\int_0^{r_o} uT2\pi r dr}{\pi r_o^2 u_b}$$

Substituting for the velocity and the temperature:

$$u = 2u_b\left[1-\left(\frac{r}{r_o}\right)^2\right] \quad \text{and} \quad T = T_w - \frac{4q_w}{kr_o}\left(\frac{3r_o^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_o^2}\right)$$

we obtain

$$T_b = \frac{4}{r_o^2}\int_0^{r_o}\left[1-\left(\frac{r}{r_o}\right)^2\right]\left[T_s - \frac{4q_s}{kr_o}\left(\frac{3r_o^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_o^2}\right)\right]rdr = T_w - \frac{11}{24}\frac{q_w r_o}{k}$$

Therefore

$$T_w - T_b = \frac{11}{24}\frac{q_w r_o}{k}$$

Flow in a tube, cont'd

DETERMINE NUSSELT NUMBER

Using $$T_w - T_b = \frac{11}{24}\frac{q_w r_o}{k}$$

and the definition of the heat transfer coefficient

$$h_c = \frac{q_W}{T_w - T_b}$$

we obtain

$$h_c = \frac{24k}{11 r_o} = \frac{48k}{11 d_o} \quad \text{where} \quad d_o = 2r_o$$

The Nusselt number is

$$\text{Nu}_d = \frac{h_c d_o}{k} = \frac{48}{11} = 4.364$$

Eq. 5-107, p.245

Example 21-1

AN OIL HEATER

Oil flows through a heater at 1.81 x 10^{-2} kg/s inside a 1 cm diameter tube that is heated electrically at a rate of 76 W/m. At a specific location where the flow and heat transfer are fully developed, the wall temperature is 370 K. Determine (i) the oil bulk temperature, (ii) the centerline temperature, (iii) the axial gradient in bulk temperature, and (iv) the heat transfer coefficient. For properties, take k = 0.139 W/m, ρ = 854 kg/m³, c_p = 2120 J/kg-K, and ν = 41 x 10^{-6} m²/s.

Given: Oil flowing inside a uniformly heated tube.

Required:
(i) T_b
(ii) T_o
(iii) dT_b/dx
(iv) h_c.

Assumptions: Fully developed flow and heat transfer.

Example 21-1, cont'd

AN OIL HEATER

First check the Reynolds number → $\text{Re}_d = \frac{u_b d_o}{\nu}$

where the bulk velocity is

$$u_b = \frac{\dot{m}}{\rho \pi r_o^2} = \frac{\left(1.81\times10^{-2}\,\frac{kg}{s}\right)}{\left(854\,\frac{kg}{m^3}\right)(\pi)(0.005\,m)^2} = 0.270\,\frac{m}{s}$$

giving $$\text{Re}_d = \frac{u_b d_o}{\nu} = \frac{\left(0.270\,\frac{m}{s}\right)(0.01\,m)}{\left(41\times10^{-6}\,\frac{m^2}{s}\right)} = 66$$

The flow is therefore laminar and the bulk temperature is

$$T_b = T_w - \frac{11}{24}\frac{q_w r_o}{k}$$

Example 21-1, cont'd

AN OIL HEATER

To obtain the wall heat flux q_w, the heat input per unit length must be divided by the tube perimeter πd_o, i.e.,

$$q_w = \frac{q_w'}{P} = \frac{q_w'}{\pi d_o} = \frac{\left(76\,\frac{W}{m}\right)}{(\pi)(0.01\,m)} = 2{,}419\,\frac{W}{m^2}$$

Hence,

$$T_b = T_w - \frac{11}{24}\frac{q_w r_o}{k} = 370K - \frac{11}{24}\frac{\left(2{,}419\,\frac{W}{m^2}\right)(0.005\,m)}{\left(0.139\,\frac{W}{mK}\right)} = 330.1K$$

Substituting $r = 0$ into

$$T = T_w - \frac{4q_w}{kr_o}\left(\frac{3r_o^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_o^2}\right) \rightarrow T_o = T_w - \frac{3}{4}\frac{q_w r_o}{k}$$

Centerline temperature

Example 21-1, cont'd

AN OIL HEATER

$$T_o = T_w - \frac{3}{4}\frac{q_w r_o}{k} = 370K - \frac{3}{4}\frac{\left(2{,}419\,\frac{W}{m^2}\right)(0.005\,m)}{\left(0.139\,\frac{W}{mK}\right)} = 304.7K$$

$$\frac{dT_b}{dx} = \frac{2q_w}{\rho u_b c_p r_o} = \frac{2\left(2{,}419\,\frac{W}{m^2}\right)}{\left(854\,\frac{kg}{m^3}\right)\left(0.270\,\frac{m}{s}\right)\left(2{,}120\,\frac{J}{kgK}\right)(0.005\,m)} = 1.98\,\frac{K}{m}$$

Example 21-1, cont'd

AN OIL HEATER

The Nusselt number is given by

$$\text{Nu}_d = \frac{h_c d_o}{k} = 4.364 \rightarrow h_c = \frac{\text{Nu}_d k}{d_o} = \frac{4.364k}{d_o}$$

$$h_c = \frac{\text{Nu}_d k}{d_o} = \frac{(4.364)\left(0.139\,\frac{W}{mK}\right)}{(0.01\,m)} = 60.7\,\frac{W}{m^2K}$$

Heat transfer in high-speed flow

- When the free stream velocity is very high, as in high-speed aircraft, viscous dissipation effects must be considered
- Consider an adiabatic case, i.e, a perfectly insulated wall
- In this case, the wall temperature may be considerably higher than the free stream temperature even though no heat transfer takes place
- This high temperature results from two situations:
  - increase in temperature as the fluid is brought to rest at the plate while KE of the fluid is converted into internal TE
  - the heating effect due to viscous dissipation
- Conversion of KE to TE as the gas is brought to rest can be determined using FLOT (i.e., by doing an EB on the fluid) under steady-state conditions:

$$\sum_i E_{\infty i} = \sum_j E_{oj} \longrightarrow KE_\infty + PE_\infty + TE_\infty = KE_o + PE_o + TE_o$$

$$m\frac{u_\infty^2}{2g_c} + m\frac{gz_\infty}{g_c} + mi_\infty = m\frac{u_o^2}{2g_c} + m\frac{gz_o}{g_c} + mi_o$$

Heat transfer in high-speed flow, cont'd

$$\frac{u_\infty^2}{2g_c} + \cancel{\frac{gz_\infty}{g_c}} + i_\infty = \frac{u_o^2}{2g_c} + \cancel{\frac{gz_o}{g_c}} + i_o$$

At *stagnation*, $u_o = 0$

$$i_o = \frac{u_\infty^2}{2g_c} + i_\infty \longrightarrow i_o - i_\infty = \frac{u_\infty^2}{2g_c} \longrightarrow$$ i_o is the *stagnation enthalpy*

Assuming that the flowing gas satisfies the Ideal Gas Law (IGL)

$$di = c_p dT \longrightarrow \int_{i_\infty}^{i_o} di = \int_{T_\infty}^{T_o} c_p dT \longrightarrow i_o - i_\infty = c_p(T_o - T_\infty)$$

where $c_p = Constant$

$$c_p(T_o - T_\infty) = \frac{u_\infty^2}{2g_c}$$

Stagnation temperature (T_o)

Free-stream temperature (T_∞)

Heat transfer in high-speed flow, cont'd

$$c_p(T_o - T_\infty) = \frac{u_\infty^2}{2g_c} \longrightarrow T_o = T_\infty + \frac{u_\infty^2}{2g_c c_p}$$

$$\frac{T_o}{T_\infty} = 1 + \frac{u_\infty^2}{2g_c c_p T_\infty} = 1 + \frac{u_\infty^2}{2g_c c_p\left(\frac{c_v}{c_v}\right)\left(\frac{R}{R}\right)T_\infty} = 1 + \frac{u_\infty^2\left(\frac{R}{c_v}\right)}{2g_c\left(\frac{c_p}{c_v}\right)RT_\infty}$$

Defining *ratio of specific heats* γ as $\longrightarrow \gamma = \frac{c_p}{c_v}$

And noting that for ideal gas the specific gas constant is $\longrightarrow R = c_p - c_v$

$$\frac{T_o}{T_\infty} = 1 + \frac{u_\infty^2\left(\frac{c_p - c_v}{c_v}\right)}{2g_c\gamma RT_\infty} = 1 + \frac{(\gamma-1)u_\infty^2}{2(g_c\gamma RT_\infty)} = 1 + \frac{\gamma-1}{2}\frac{u_\infty^2}{a^2}$$

Heat transfer in high-speed flow, cont'd

$$\frac{T_o}{T_\infty} = 1 + \frac{\gamma-1}{2}\frac{u_\infty^2}{a^2}$$

where $a = \sqrt{g_c\gamma RT_\infty} \longrightarrow$ *Acoustic speed* in the fluid at T_∞ and characterized by γ and R

Defining $\frac{u_\infty}{a} = M_\infty \longrightarrow$ *Mach number*

We obtain

$$\frac{T_o}{T_\infty} = 1 + \frac{\gamma-1}{2}M_\infty^2$$

Hwrk P-5-72

- In the actual case of a boundary layer flow problem, the fluid is not brought to rest reversibly because the viscous action is irreversible
- Also, not all the free-stream KE is converted to TE - part is lost as heat and part is dissipated in the form of viscous work

Heat transfer in high-speed flow, cont'd

- To take into account irreversibilities in the boundary-layer flow system, a *recovery factor r* is defined as

$$r = \frac{T_{aw} - T_\infty}{T_o - T_\infty}$$

Hwrk P-5-72

where T_{aw} is the *actual adiabatic temperature*

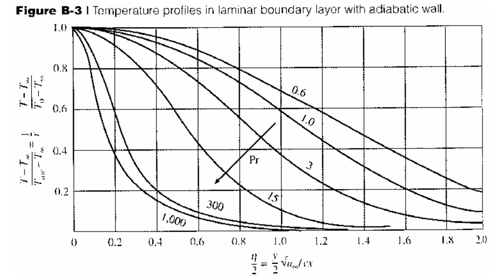
- The boundary layer equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

was solved for the high-speed-flow taking into account the viscous-heating

Heat transfer in high-speed flow, cont'd

Figure B-3 | Temperature profiles in laminar boundary layer with adiabatic wall.



Heat transfer in high-speed flow, cont'd

- The heat transfer rate in high-speed flow may be computer from

$$Q = qA = \bar{h}A(T_w - T_{aw})$$ Hwrk P-5-72

- For gasses with Prandtl numbers near unity, the recovery factors are

$r = \Pr^{1/2}$ for laminar flow Hwrk P-5-72

$r = \Pr^{1/3}$ for turbulent flow Hwrk P-5-72

The above recovery factors may be used to obtain the T_{aw}

- In high-speed boundary layers substantial thermal gradients may occur, which will lead to large property variations across the boundary layer
- The constant-property heat-transfer equations may still be used if the properties are evaluated at a *reference temperature* T^*

$$T^* = T_\infty + 0.50(T_w - T_\infty) + 0.22(T_{aw} - T_\infty)$$ Hwrk P-5-72

Heat transfer in high-speed flow, cont'd

SUMMARY OF RELATIONS

- Laminar boundary layer ($Re_x < 5\times10^5$)

$$St_x^* \Pr^{*2/3} = 0.332 \operatorname{Re}_x^{*-1/2}$$

- Turbulent boundary layer ($5\times10^5 < Re_x < 10^7$)

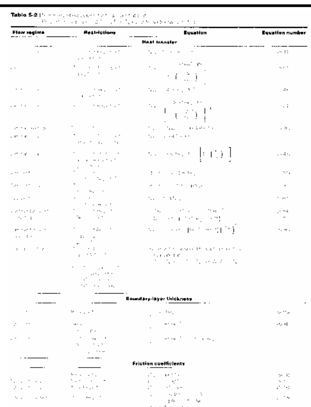
$$St_x^* \Pr^{*2/3} = 0.0296 \operatorname{Re}_x^{*-1/5}$$ Hwrk P-5-72

- Turbulent boundary layer ($10^7 < Re_x < 10^9$)

$$St_x^* \Pr^{*2/3} = 0.185 \left(\log \operatorname{Re}_x^*\right)^{-2.584}$$

In the above equations, the superscript * indicates that the properties are evaluated at the *reference temperature* T^*

- To obtain an average heat transfer coefficient, the above equations must be integrated over the length of the plate



Procedure for solution of convection problems

- Establish the geometry of the solution
- Determine the fluid(s) involved and evaluate its (their) properties
- Establish the boundary conditions
- Establish the flow regime as determined by the Reynolds number
- Develop/select appropriate equation(s), taking into account the flow regime and/or any fluid property restrictions that may apply
- Calculate the value(s) of the convective heat-transfer coefficient and/or heat transfer

Figure 7-15 | Summary of convection calculation procedure