WORCESTER POLYTECHNIC INSTITUTE

MECHANICAL ENGINEERING DEPARTMENT

Engineering Experimentation
ME-3901, A'2010

Lecture 11

04 October 2010

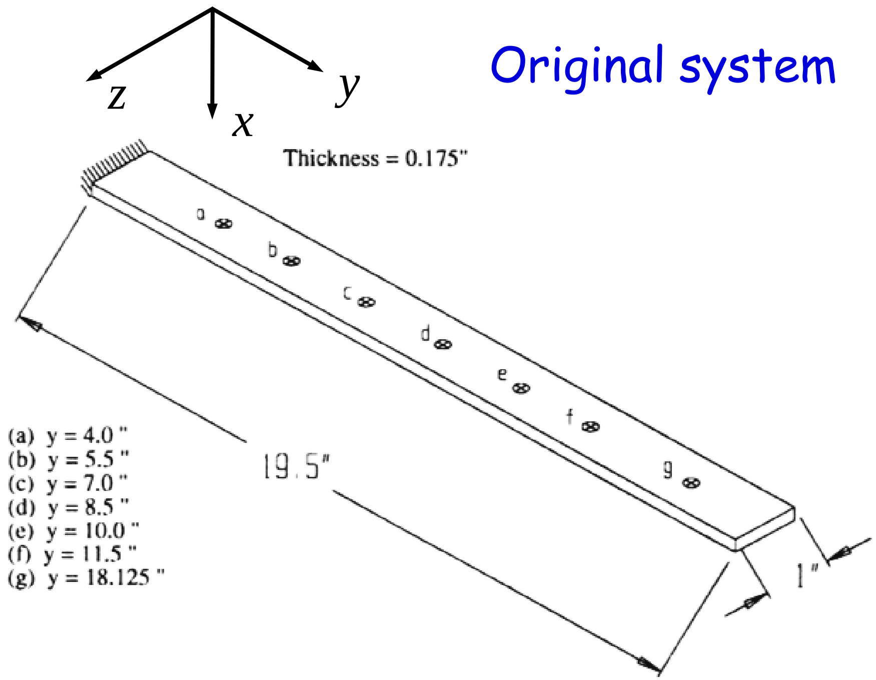
General information

Office hours

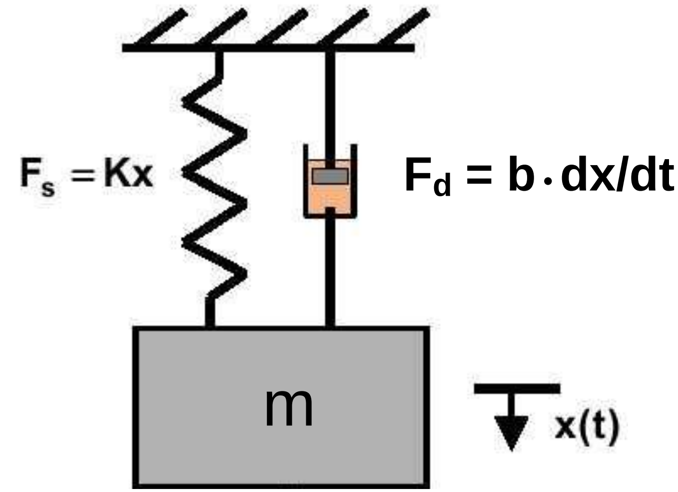
Instructor: Cosme Furlong; cfurlong@wpi.edu
Everyday from 11:00 to 11:50 am
or by appointment

Teaching Assistant: Jeffrey Laut & Kazim Naqvi;
During Lab Sessions

Equivalent systems

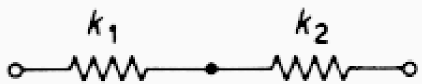


Original system ⟶ Equivalent system

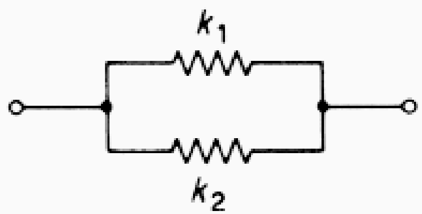


Equivalent systems

Table of Spring Stiffness



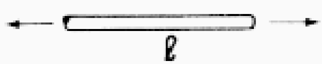
$$k = \frac{1}{1/k_1 + 1/k_2}$$



$$k = k_1 + k_2$$

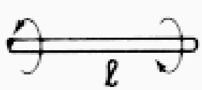
$$k = \frac{EI}{l}$$

I = moment of inertia of cross-sectional area
l = total length



$$k = \frac{EA}{l}$$

A = cross-sectional area



$$k = \frac{GJ}{l}$$

J = torsion constant of cross section

$$k = \frac{Gd^4}{64nR^3}$$

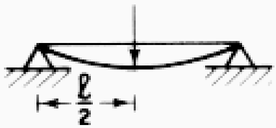
n = number of turns

From: W. T. Thomson et al., "Theory of vibrations with applications," Prentice-Hall, 1993

Equivalent systems

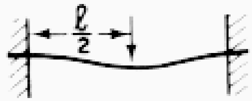
$$k = \frac{3EI}{l^3}$$

k at position of load

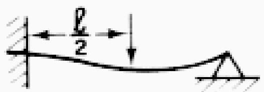


$$k = \frac{48EI}{l^3}$$

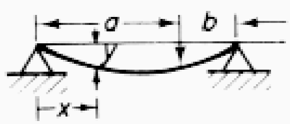
(Use the deflection of a beam equations to obtain these results)



$$k = \frac{192EI}{l^3}$$

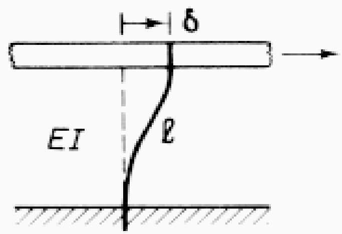


$$k = \frac{768EI}{7l^3}$$

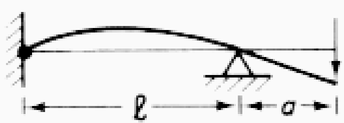


$$k = \frac{3EIl}{a^2b^2}$$

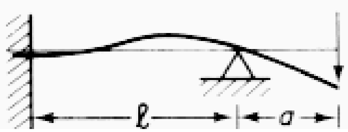
$$y_x = \frac{Pbx}{6EIl}(l^2 - x^2 - b^2)$$



$$k = \frac{12EI}{l^3}$$



$$k = \frac{3EI}{(l + a)a^2}$$



$$k = \frac{24EI}{a^2(3l + 8a)}$$

Shear and bending-moment diagrams

Singularity functions

Singularity functions:

•Definitions:

•$n<0^*$: $f_n(x) \equiv \langle x-a \rangle_n = \begin{cases} \infty & x=a \\ 0 & x \neq a \end{cases}$

•$n \geq 0$: $f_n(x) \equiv \langle x-a \rangle^n = \begin{cases} (x-a)^n & x \geq a \\ 0 & x < a \end{cases}$

•Integration rules:

•$n<0$: $\int_{-\infty}^{x} \langle x-a \rangle_n \, dx = \langle x-a \rangle_{n+1}$

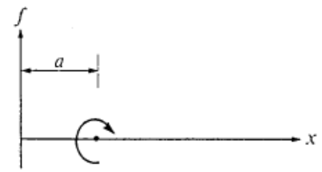
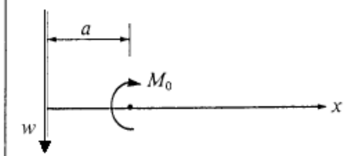
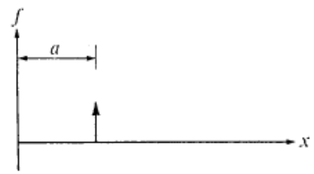
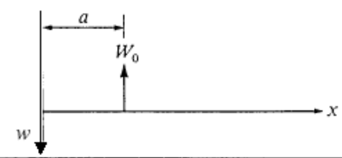
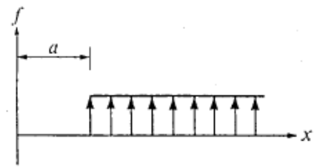
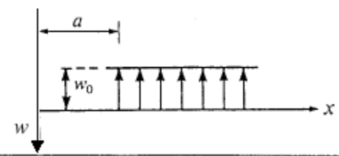
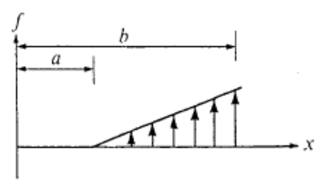
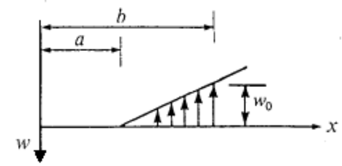
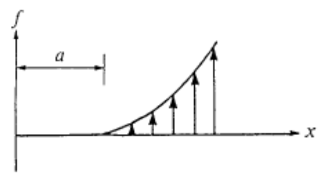
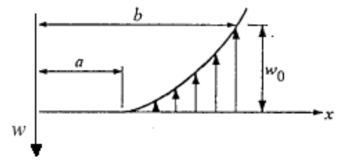
•$n \geq 0$: $\int_{-\infty}^{x} \langle x-a \rangle^n \, dx = \frac{1}{n+1} \langle x-a \rangle^{n+1}$

*Remark: the subscript positioning of n when $n < 0$ is sometimes used to emphasize the fact that the singularity function behaves differently from $n \geq 0$

Shear and bending-moment diagrams

Singularity functions

Main singularity functions and their use

Singularity function	Graphical representation	Loading	
$f_{-2}(x) = \langle x - a \rangle_{-2}$ (couple)		$w(x) = -M_0 \langle x-a \rangle_{-2}$	
$f_{-1}(x) = \langle x - a \rangle_{-1}$ (concentrated load)		$w(x) = -W_0 \langle x-a \rangle_{-1}$	
$f_0(x) = \langle x - a \rangle^0$ (uniformly distributed load)		$w(x) = -w_0 \langle x-a \rangle^0$	
$f_1(x) = \langle x - a \rangle^1$ (linearly distributed load)		$w(x) = -\dfrac{w_0}{b-a} \langle x-a \rangle^1$	
$f_2(x) = \langle x - a \rangle^2$ (quadratic distributed load)		$w(x) = -\dfrac{w_0}{(b-a)^2} \langle x-a \rangle^2$	

Shear and bending-moment diagrams
Singularity functions

Loading function: $q(x)$

Shear function: $V(x) = \int q(x)\, dx$

Moment function: $M(x) = \int V(x)\, dx$

Deflection in beams

Recall:

$$\frac{q}{EI} = \frac{d^4 y}{dx^4}$$ Load function - deflection

$$\frac{V}{EI} = \frac{d^3 y}{dx^3}$$ Shear function - deflection

$$\frac{M}{EI} = \frac{d^2 y}{dx^2}$$ Moment function - deflection

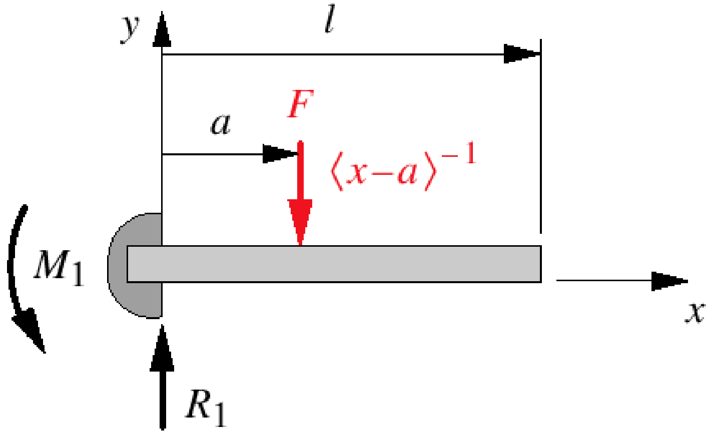
$$\theta = \frac{dy}{dx}$$ Slope - deflection

$$y = f(x)$$ Deflection

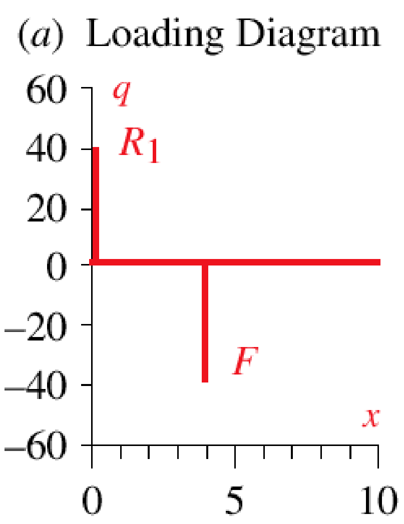
Deflection in beams

Example: use of singularity functions

Determine and plot the shear, moment, slope, and deflection functions for the cantilever beam shown:



(b) Cantilever beam with concentrated loading



Deflection in beams

Example: use of singularity functions

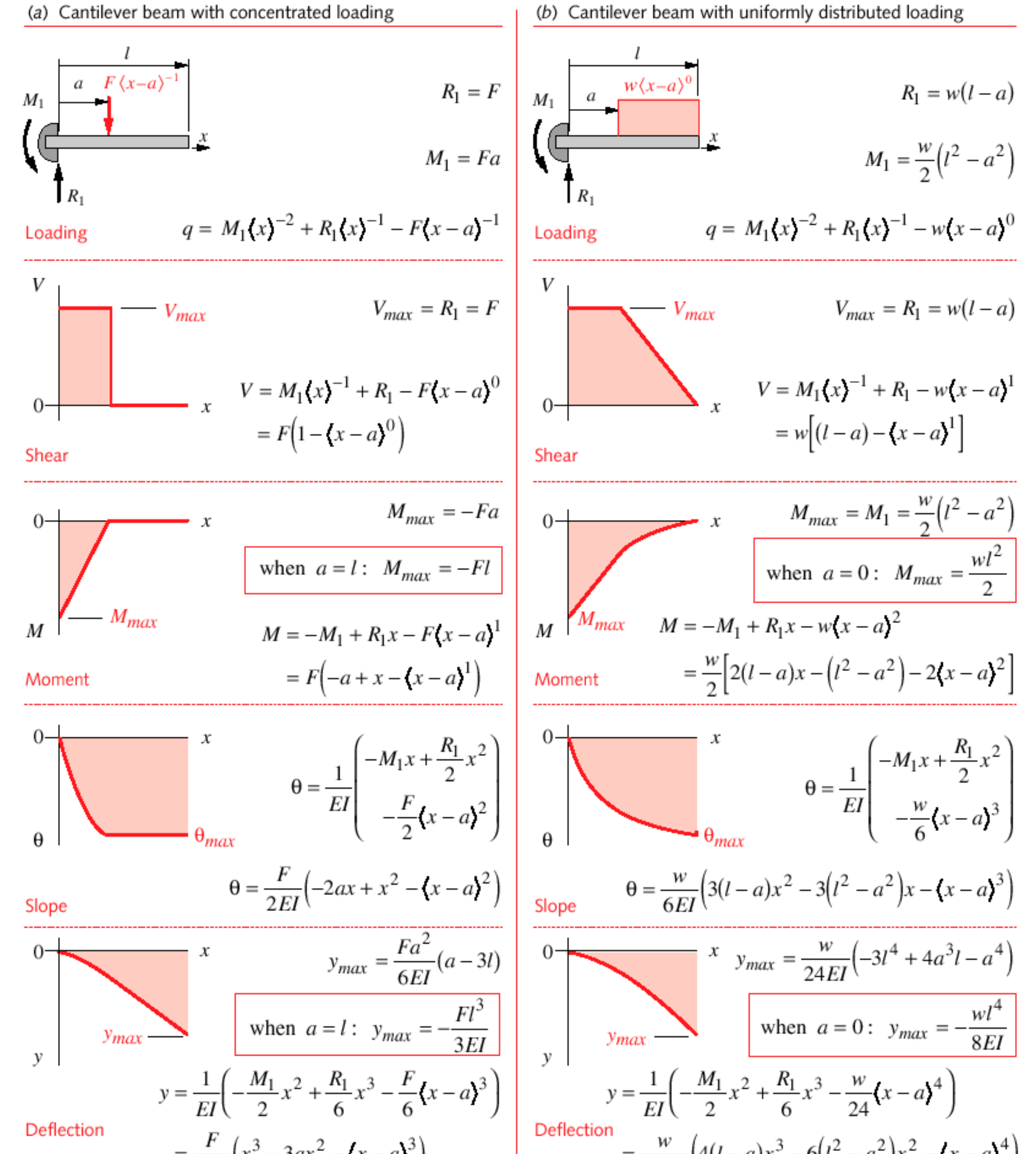
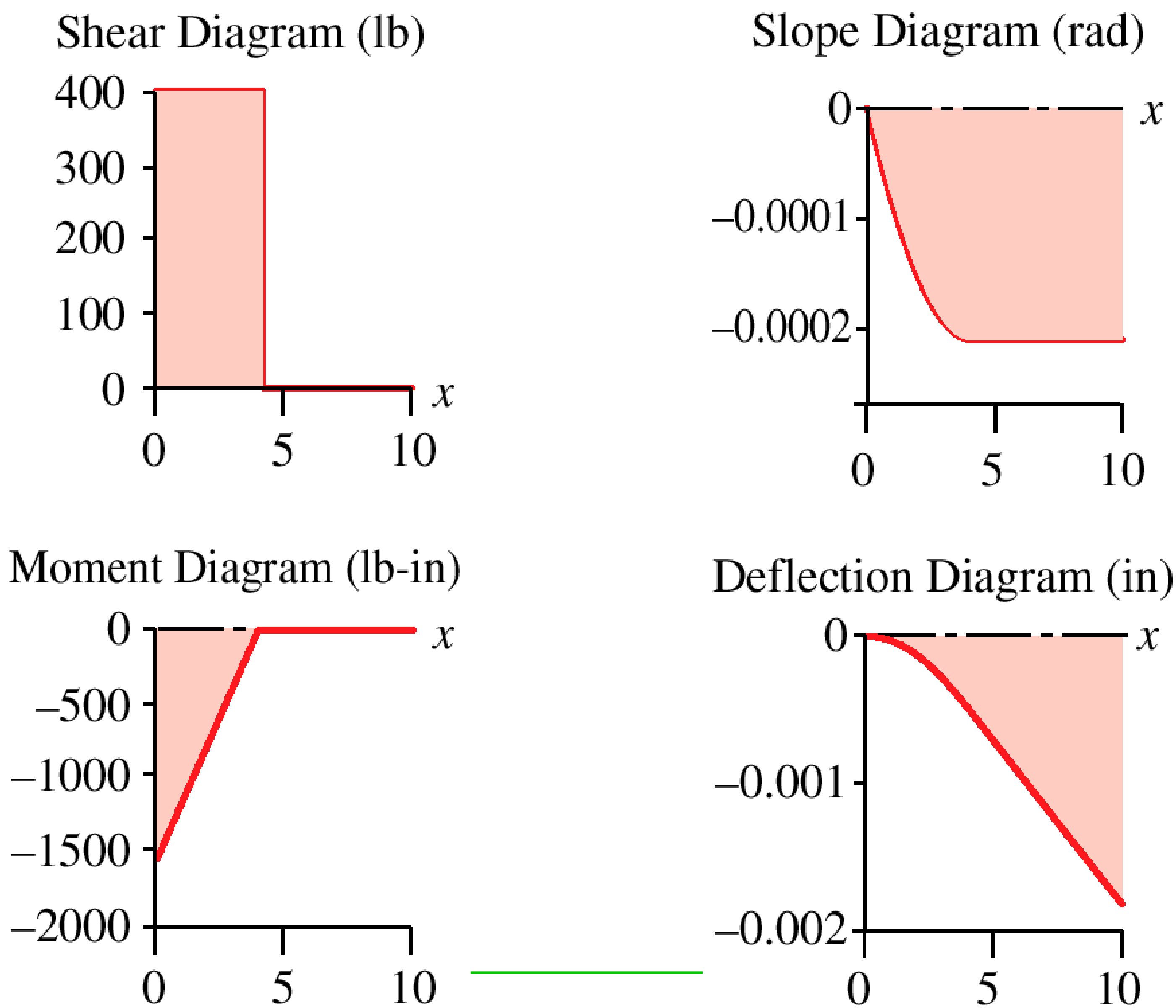


FIGURE D-1

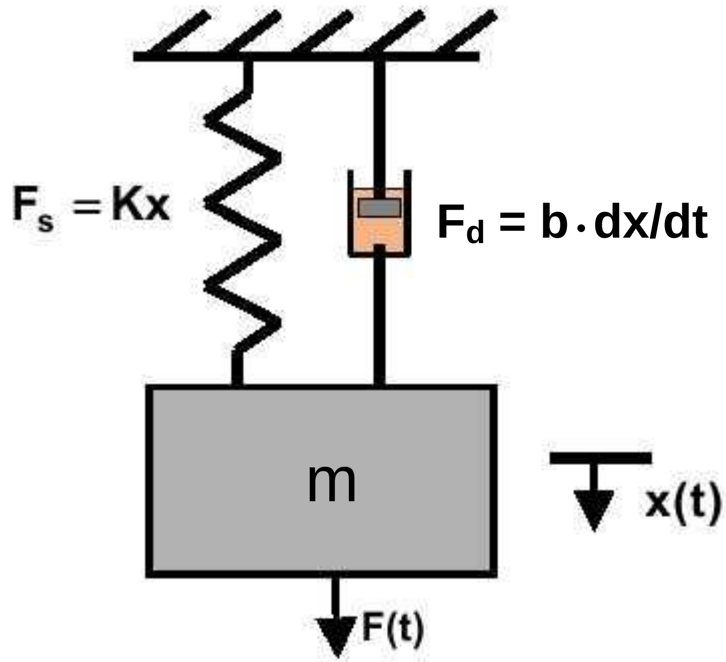
Cantilever Beams with Concentrated or Distributed Loading. Note: < > Denotes a Singularity Function

Deflection in beams

Example: use of singularity functions



Analysis of a single degree of freedom system



Governing equation: $$m\frac{d^2x}{dt^2} = \sum_i F_i$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F(t)$$

External force

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

Consider governing equation:
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Second order, ordinary, differential equation with *constant coefficients*.

Governing equation can be written as:
$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

Possible solution has the form:
$$x(t) = e^{m_i t}$$

So the **characteristic equation** is:
$$m_i^2 + 2\lambda m_i + \omega^2 = 0$$

Roots of characteristic equation:

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$$
$$m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 = 0$$ Critically damped system

$$\lambda^2 - \omega^2 > 0$$ Over-damped system

$$\lambda^2 - \omega^2 < 0$$ Under-damped system

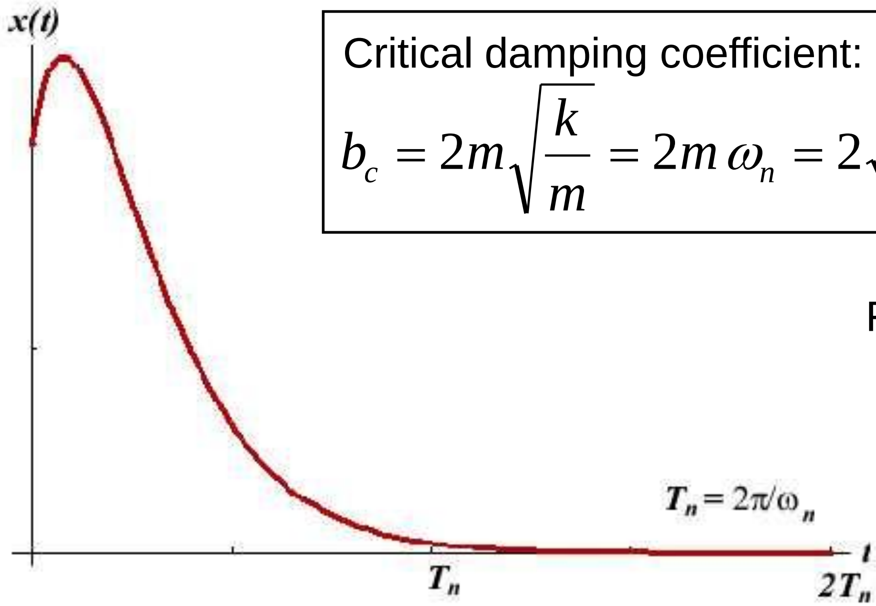
Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$\lambda^2 - \omega^2 = 0$ → Critically damped system

Solution to the governing differential equation:

$$x(t) = C_1 e^{m_1 t} + C_2 t e^{m_1 t}$$



Critical damping coefficient:

$$b_c = 2m\sqrt{\frac{k}{m}} = 2m\,\omega_n = 2\sqrt{k\,m}$$

Fundamental frequency:

$$\omega = \omega_n = \sqrt{\frac{k}{m}}$$

Critical damping factor b_c is the minimum damping that results in non-periodic motions

Analysis of a single degree of freedom system

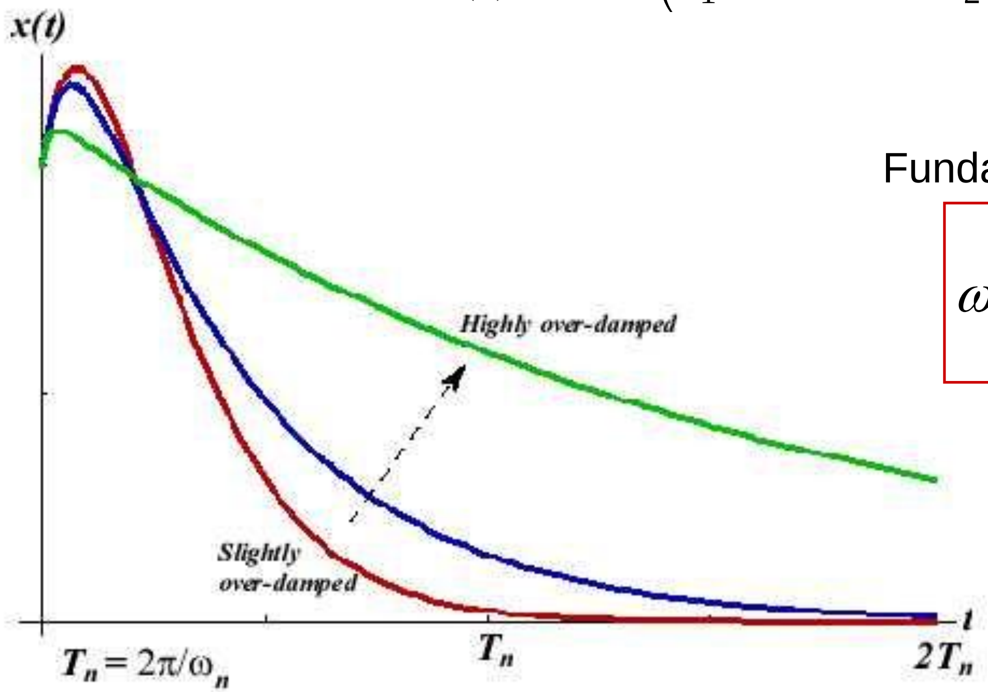
First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 > 0$$ → Over-damped system

Solution to the governing differential equation:

$$x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

$$x(t) = e^{-\lambda t}\left(C_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + C_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$



Fundamental frequency:

$$\omega = \omega_n = \sqrt{\frac{k}{m}}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

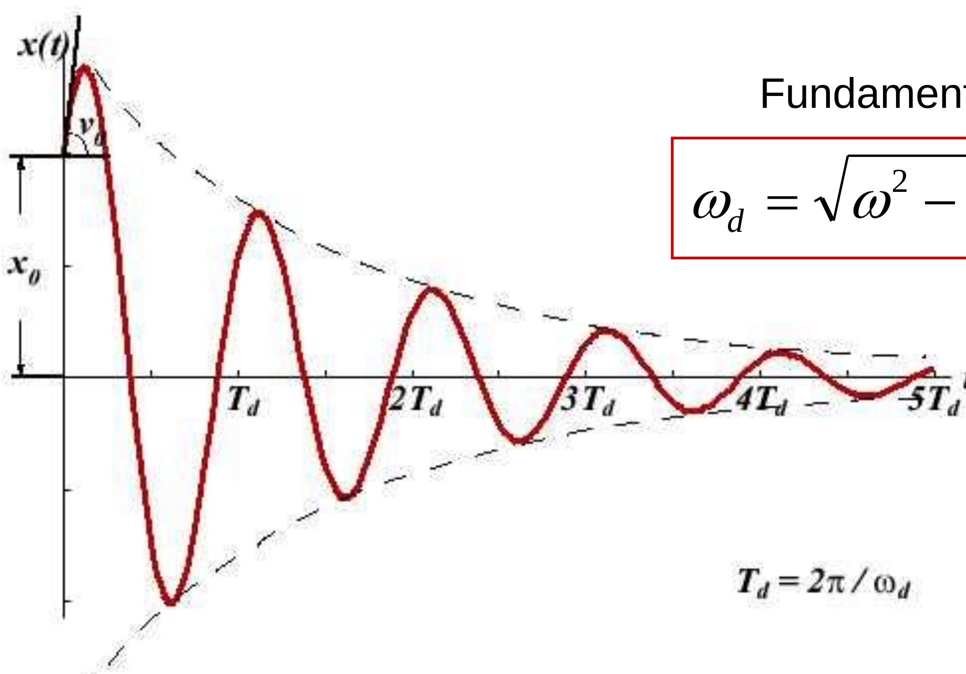
$$\lambda^2 - \omega^2 < 0 \quad \longrightarrow \quad \text{Under-damped system}$$

Solution to the governing differential equation:

$$x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

(m_1 and m_2 are complex numbers, why?)

$$x(t) = e^{-\lambda t}[C_1 \cos(\sqrt{\omega^2 - \lambda^2}\, t) + C_2 \sin(\sqrt{\omega^2 - \lambda^2}\, t)]$$



Fundamental frequency:

$$\omega_d = \sqrt{\omega^2 - \lambda^2} = \sqrt{\omega_n^2 - \lambda^2}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$\lambda^2 - \omega^2 < 0$ → Under-damped system

Fundamental frequency: $\omega_d = \sqrt{\omega_n^2 - \lambda^2}$ -- see previous equation for $x(t)$

$$\omega_d = \sqrt{\omega_n^2 - \lambda^2} = \omega_n \sqrt{1 - \frac{\lambda^2}{\omega_n^2}} = \omega_n \sqrt{1 - \frac{\left(\frac{b}{2m}\right)^2}{\frac{k}{m}}}$$

Recall: critical damping coefficient:

$$b_c = 2m\sqrt{\frac{k}{m}} = 2m\,\omega_n = 2\sqrt{k\,m}$$

Un-damped fundamental frequency

$$\omega_d = \omega_n \sqrt{1 - \left(\frac{b}{b_c}\right)^2} = \omega_n \sqrt{1 - \zeta^2}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 < 0$$ → Under-damped system

Note that it is possible to write: $\lambda = \zeta\,\omega_n$ (Demonstrate in-class)

Solution of the governing differential equation can be written as:

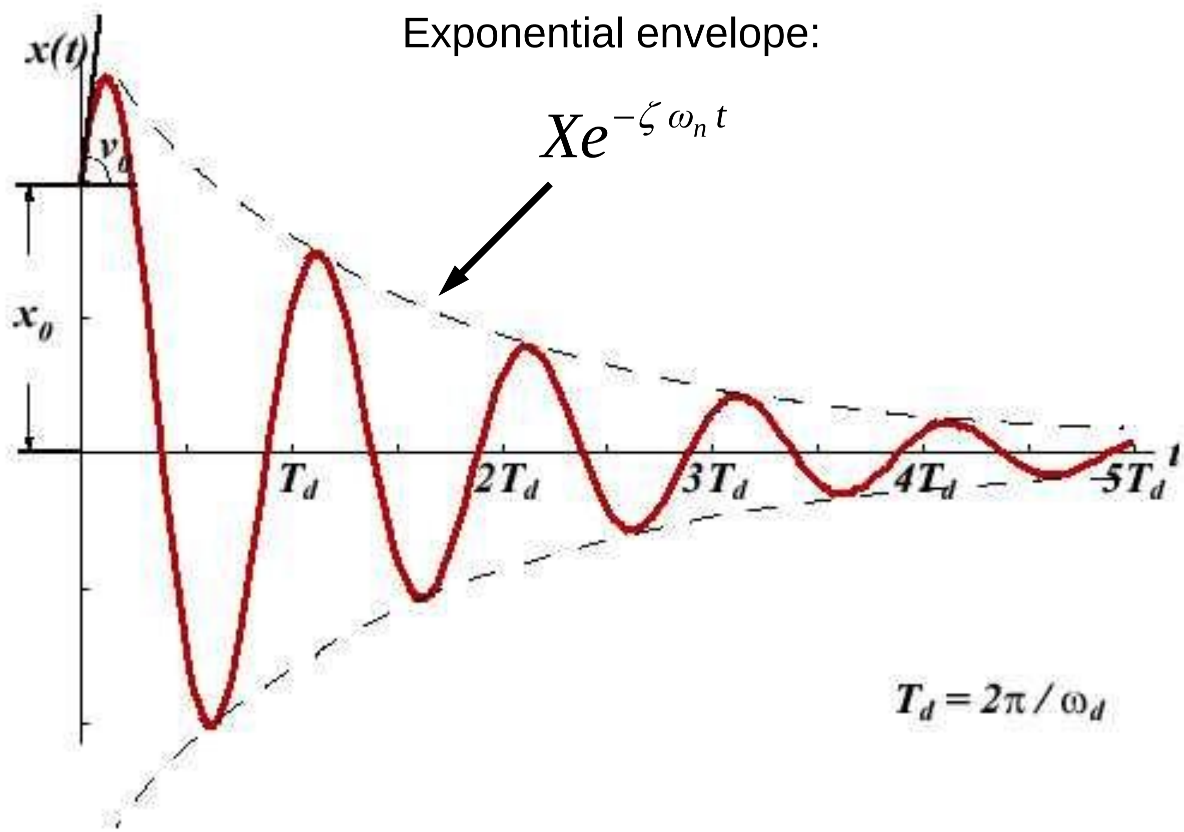
$$x(t) = e^{-\zeta\,\omega_n t}[C_1 \cos(\sqrt{1-\zeta^2}\,\omega_n\,t) + C_2 \sin(\sqrt{1-\zeta^2}\,\omega_n\,t)]$$

$$= Xe^{-\zeta\,\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n\,t + \phi)$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 < 0 \longrightarrow \text{Under-damped system}$$

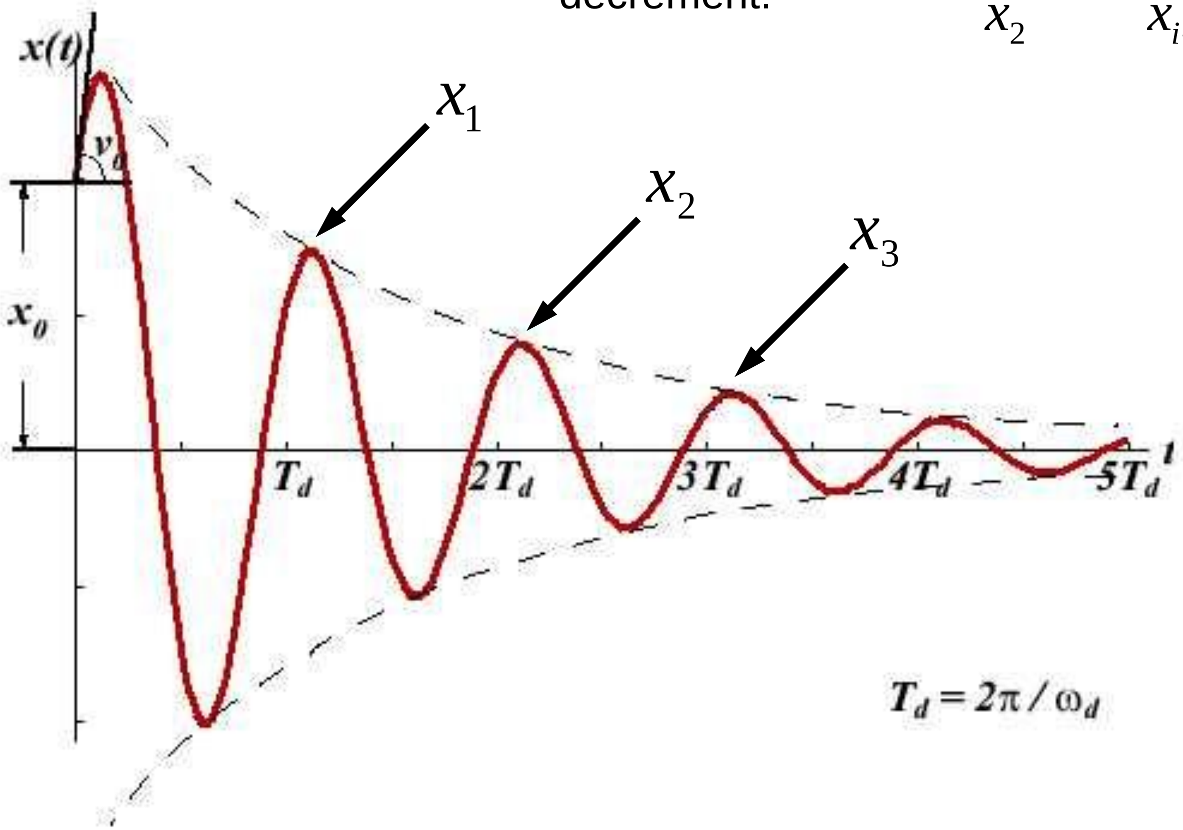


Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 < 0$$ → Under-damped system

Logarithmic decrement: $$\delta = \ln \frac{x_1}{x_2} = \ln \frac{x_i}{x_{i+1}}$$



Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 < 0 \longrightarrow \text{Under-damped system}$$

Logarithmic decrement:

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{e^{-\zeta\,\omega_n t_1}\sin(\sqrt{1-\zeta^2}\,\omega_n\, t_1 + \phi)}{e^{-\zeta\,\omega_n (t_1+T_d)}\sin[\sqrt{1-\zeta^2}\,\omega_n\,(t_1+T_d)+\phi]}$$

$$= \ln\frac{x_1}{x_2} = \ln\frac{e^{-\zeta\,\omega_n t_1}}{e^{-\zeta\,\omega_n (t_1+T_d)}} = \ln e^{\zeta\,\omega_n T_d} = \zeta\,\omega_n\,T_d$$

Recall that: $T_d = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$ $\Rightarrow$ $\boxed{\delta = \dfrac{2\pi\,\zeta}{\sqrt{1-\zeta^2}}}$

Reading assignment

- **Beckwith**: Ch. 6, 17
- **Bishop:** Ch. 11

References:

- J.P.Holman, *Experimental methods for engineers*, McGraw-Hill, 1989
- T. G. Beckwith, R. D. Marangoni, and J. H. Lienhard, *Mechanical Measurements*, 5th ed., Addison-Wesley, 1995
- C. Furlong, *MEMS: introduction and applications*, Course notes on MEMS, ISTFA, 2004, Worcester, MA

Homework assignment: Handout-K

- **Beckwith**:

K1.- The following data are given for a vibrating system with viscous damping: weight = 10 lbf; k = 30 lbf/in; b = 0.12 lbf/in/s. Determine the logarithmic decrement and the ratio of any two successive amplitudes

K2.- A 0.453-Kg mass attached to a light spring elongates it 7.87 mm. Determine the natural frequency of the system

K3.- A spring-mass system, k_1 and m, has a natural frequency of f_1. If a second spring k_2 is added in series with the first spring, it is observed that the natural frequency is lowered to $0.5\, f_1$. Determine k_2 in terms of k_1.

- **Bishop**: Section 11.6.2