Columns: $\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$ / Concentric

Recall: $y = C_1 \sin\sqrt{\frac{P}{EI}}\,x + C_2 \cos\sqrt{\frac{P}{EI}}\,x$

w/ C_1 and C_2 depend on BC's

for $y(x=0) = y(x=\ell) = 0 \Rightarrow C_2 = 0$

$$C_1 \sin\sqrt{\frac{P}{EI}}\,\ell = 0$$

$$\Rightarrow \sqrt{\frac{P}{EI}}\,\ell = n\pi$$

lowest load @ $n=1$ $\therefore$ $\sqrt{\frac{P}{EI}}\,\ell = \pi$

$$P_{critical} = P = \frac{EI\pi^2}{\ell^2} \quad ; \text{ recall } I = AK^2$$

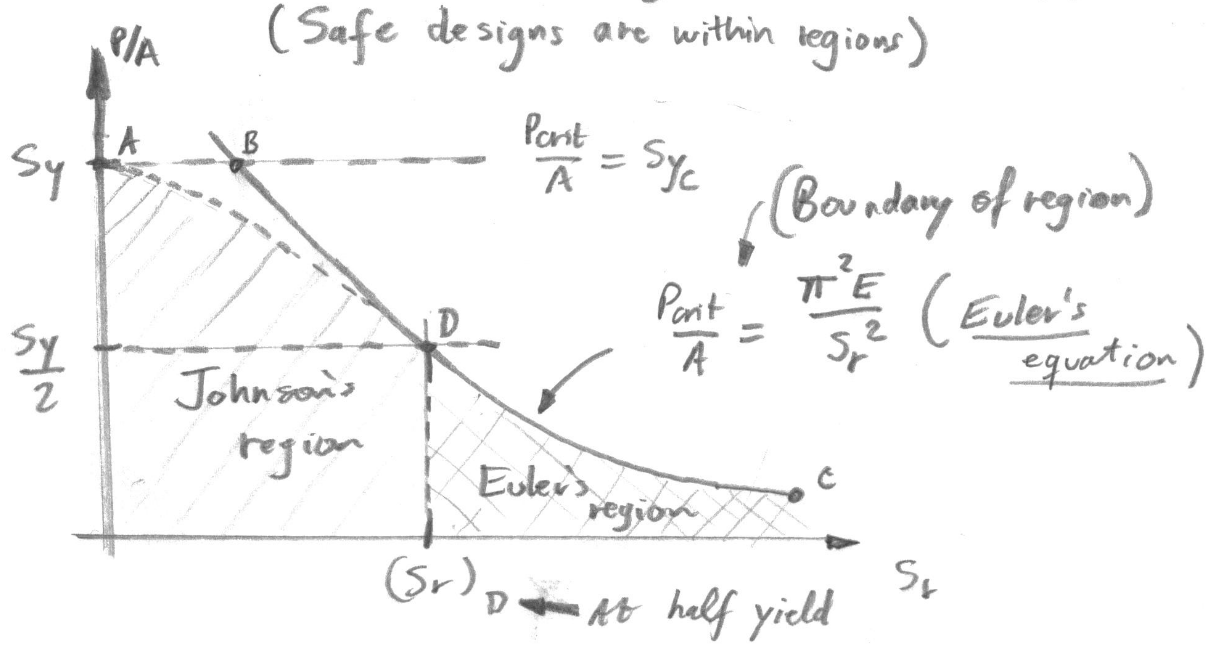
$$P_{crit} = \left(\frac{\pi}{\ell}\right)^2 \cdot EAK^2 = \frac{\pi^2 EA}{\left(\frac{\ell}{K}\right)^2} = \frac{\pi^2 EA}{S_r^2}$$

$$\frac{P_{crit}}{A} = \frac{\pi^2 E}{S_r^2}$$

which depends on BC's. To compensate for $\neq$ BC's:

$$S_r = \frac{\ell_{eff}}{K} \quad ; \ \ell_{eff} \text{ is in Table 4-4, pg 193}$$

Johnson & Euler regions
(Safe designs are within regions)



(1) Need to identify point D — to use Johnson's or Euler's formulations

(2) Set : $\frac{S_{yc}}{2} = \frac{\pi^2 E}{S_r^2} \Rightarrow$ Solve for $S_r \equiv (S_r)_D$

$$(S_r)_D = \pi \sqrt{\frac{2E}{S_{yc}}}$$

$\frac{P}{A} = f(S_r)$

$\frac{P}{A} = C_0 + C_1 S_r^2$

Solve for (C_0, C_1)

(3) Curve fit a parabola between points A and D

Use points A : $(0, S_{yc})$; B : $((S_r)_D, S_{yc}/2)$

Curve fit leads to:

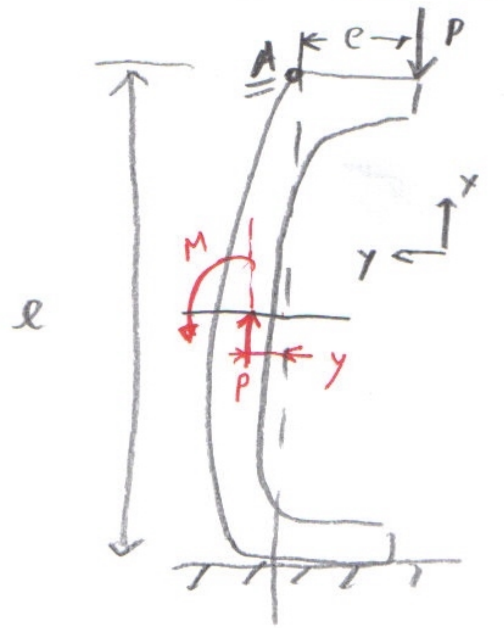
$$\frac{P_{crit}}{A} = S_{yc} - \frac{1}{E}\left(\frac{S_{yc} \cdot S_r}{2\pi}\right)^2 \text{; Johnson's equation}$$

Once P_{crit} is determined, calculate max. allowed forces

$$P_{allowed} = \frac{P_{critical}}{\text{Safety Factor}} = \frac{P_{crit}}{SF}$$

SF > 1 , Need engg. standards (AISC, ASME, SAE, etc.)

Eccentric columns



$$\sum M_A = 0 = -M + P(e+y) = 0$$

Recall

$$M = EI\frac{d^2y}{dx^2}$$

→

ODE

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{Pe}{EI}$$

BC's are:

$$y(x=0)=0 \quad ; \quad \frac{dy}{dx}\left(x=\frac{l}{2}\right)=0$$

$$\Rightarrow \quad y\left(x=\frac{l}{2}\right) = e\left[\sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right) - 1\right]$$

with max. moment

$$M_{max} = -P\left(e + y_{l/2}\right) = -P \cdot e \cdot \sec\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

Comp. stress is

$$\sigma_c = \frac{P}{A} - \frac{Mc}{I} = \frac{P}{A} - \frac{Mc}{AK^2}$$

Using $M = M_{max}$ $\Rightarrow$ $\sigma_c = \frac{P}{A}\left[1 + \left(\frac{e \cdot c}{K^2}\right)\sec\left(\frac{l}{K}\sqrt{\frac{P}{4EA}}\right)\right]$

Using $\sigma_c = S_{yc}$ (Yield strength in compression)

(Note that P is in both sides of Eq.) $$\frac{P}{A} = \frac{S_{yc}}{1 + \left(\frac{e \cdot c}{K^2}\right)\sec\left(\frac{l_{eff}}{K}\sqrt{\frac{P}{4EA}}\right)}$$

PROBLEM 4-51a

Statement: A solid, rectangular column has the dimensions and properties below. Determine if it is a Johnson or an Euler column and find the critical load:

(a) If its boundary conditions are pinned-pinned.
(b) If its boundary conditions are fixed-pinned.
(c) If its boundary conditions are fixed-fixed.
(d) If its boundary conditions are fixed-free.

Units: $N := newton$ $\quad MPa := 10^6 \cdot Pa$ $\quad GPa := 10^9 \cdot Pa$ $\quad kN := 10^3 \cdot N$

Given:

Length of column	$L := 100 \cdot mm$	Material	Steel
Thickness	$t := 10 \cdot mm$	Yield strength	$S_y := 300 \cdot MPa$
Height	$h := 20 \cdot mm$	Modulus of elasticity	$E := 207 \cdot GPa$

Solution: See Mathcad file P0451a.

1. Calculate the slenderness ratio that divides the unit load vs slenderness ratio graph into Johnson and Euler regions.

$$S_{rD} := \pi \cdot \sqrt{\frac{2 \cdot E}{S_y}} \qquad S_{rD} = 116.7$$

2. Calculate the cross-section area, moment of inertia, and the radius of gyration.

Area $\quad A := h \cdot t \quad A = 200.00\,mm^2$

Moment of inertia $\quad I := \frac{h \cdot t^3}{12} \quad I = 1667\,mm^4$

Radius of gyration $\quad k := \sqrt{\frac{I}{A}} \quad k = 2.887\,mm$

3. Define functions to determine column type and critical load.

Type
$$type(S_r) := \begin{cases} \text{"Euler"} & \text{if } S_r > S_{rD} \\ \text{"Johnson"} & \text{otherwise} \end{cases}$$

Critical load
$$P_{cr}(S_r) := \begin{cases} \text{return } A \cdot \frac{\pi^2 \cdot E}{S_r^2} & \text{if } type(S_r) = \text{"Euler"} \\ A \cdot \left[S_y - \frac{1}{E} \cdot \left(\frac{S_y \cdot S_r}{2 \cdot \pi} \right)^2 \right] & \text{otherwise} \end{cases}$$

(a) *pinned-pinned* **ends**

4. Using Table 4-7, calculate the effective column length.

$L_{eff} := 1 \cdot L \qquad L_{eff} = 100\,mm$

5. Calculate the slenderness ratio for the column.

Slenderness ratio $\quad S_r := \frac{L_{eff}}{k} \quad S_r = 34.64$

6. Determine the type and critical load using the functions defined above.

$$type(S_r) = \text{"Johnson"} \qquad P_{cr}(S_r) = 57.36\,kN$$

(b) *fixed-pinned* **ends**

7. Using Table 4-7, calculate the effective column length.

$$L_{eff} := 0.8 \cdot L \qquad L_{eff} = 80\,mm$$

8. Calculate the slenderness ratio for the column.

Slenderness ratio $$S_r := \frac{L_{eff}}{k} \qquad S_r = 27.71$$

9. Determine the type and critical load using the functions defined above.

$$type(S_r) = \text{"Johnson"} \qquad P_{cr}(S_r) = 58.31\,kN$$

(c) *fixed-fixed* **ends**

10. Using Table 4-7, calculate the effective column length.

$$L_{eff} := 0.65 \cdot L \qquad L_{eff} = 65\,mm$$

11. Calculate the slenderness ratio for the column.

Slenderness ratio $$S_r := \frac{L_{eff}}{k} \qquad S_r = 22.52$$

12. Determine the type and critical load using the functions defined above.

$$type(S_r) = \text{"Johnson"} \qquad P_{cr}(S_r) = 58.9\,kN$$

(d) *fixed-free* **ends**

13. Using Table 4-7, calculate the effective column length.

$$L_{eff} := 2.1 \cdot L \qquad L_{eff} = 210\,mm$$

13. Calculate the slenderness ratio for the column.

Slenderness ratio $$S_r := \frac{L_{eff}}{k} \qquad S_r = 72.75$$

14. Determine the type and critical load using the functions defined above.

$$type(S_r) = \text{"Johnson"} \qquad P_{cr}(S_r) = 48.34\,kN$$

PROBLEM 4-52a

Statement: **A solid, circular column, loaded eccentrically, has the dimensions and properties below. Find the critical load:**

(a) If its boundary conditions are pinned-pinned.
(b) If its boundary conditions are fixed-pinned.
(c) If its boundary conditions are fixed-fixed.
(d) If its boundary conditions are fixed-free.

Units: $N := newton$ $MPa := 10^6 \cdot Pa$ $GPa := 10^9 \cdot Pa$ $kN := 10^3 \cdot N$

Given:

Length of column	$L := 100 \cdot mm$	Material	Steel
Outside diameter	$od := 20 \cdot mm$	Yield strength	$S_y := 300 \cdot MPa$
Eccentricity (t)	$e := 10 \cdot mm$	Modulus of elasticity	$E := 207 \cdot GPa$

Solution: See Mathcad file P0452a.

1. Calculate the cross-section area, distance to extreme fiber, and the moment of inertia.

Area	$A := \frac{\pi}{4} \cdot od^2$	$A = 314.16 mm^2$
Distance to extreme fiber	$c := 0.5 \cdot od$	$c = 10 mm$
Moment of inertia	$I := \frac{\pi}{64} \cdot od^4$	$I = 7854 mm^4$

4. Calculate the radius of gyration and eccentricity ratio for the column.

Radius of gyration	$k := \sqrt{\frac{I}{A}}$	$k = 5.00\ mm$
Eccentricity ratio	$E_r := \frac{e \cdot c}{k^2}$	$E_r = 4.0$

(a) ***pinned-pinned*** **ends**

3. Using Table 4-7, calculate the effective column length.

$L_{eff} := 1 \cdot L$ $L_{eff} = 100 mm$

4. Calculate the slenderness ratio for the column.

Slenderness ratio $S_r := \frac{L_{eff}}{k}$ $S_r = 20.00$

5. Calculate the critical load using the Secant equation.

Guess $P := 1 \cdot kN$

Given

$$P = \frac{S_y \cdot A}{1 + E_r \cdot sec\left(S_r \cdot \sqrt{\frac{P}{4 \cdot E \cdot A}}\right)}$$

$P_{cr} := Find(P)$ $P_{cr} = 18.63\ kN$

(b) ***fixed-pinned*** **ends**

6. Using Table 4-7, calculate the effective column length.

$L_{eff} := 0.8 \cdot L$ $\quad L_{eff} = 80\,mm$

7. Calculate the slenderness ratio for the column.

Slenderness ratio $\quad S_r := \frac{L_{eff}}{k} \quad S_r = 16.00$

8. Calculate the critical load using the Secant equation.

Guess $\quad P := 1 \cdot kN$

Given

$$P = \frac{S_y \cdot A}{1 + E_r \cdot sec\left(S_r \cdot \sqrt{\frac{P}{4 \cdot E \cdot A}}\right)}$$

$P_{cr} := Find(P) \quad P_{cr} = 18.71\,kN$

(c) ***fixed-fixed*** **ends**

9. Using Table 4-7, calculate the effective column length.

$L_{eff} := 0.65 \cdot L$ $\quad L_{eff} = 65\,mm$

10. Calculate the slenderness ratio for the column.

Slenderness ratio $\quad S_r := \frac{L_{eff}}{k} \quad S_r = 13.00$

11. Calculate the critical load using the Secant equation.

Guess $\quad P := 1 \cdot kN$

Given

$$P = \frac{S_y \cdot A}{1 + E_r \cdot sec\left(S_r \cdot \sqrt{\frac{P}{4 \cdot E \cdot A}}\right)}$$

$P_{cr} := Find(P) \quad P_{cr} = 18.76\,kN$

(d) ***fixed-free*** **ends**

12. Using Table 4-7, calculate the effective column length.

$L_{eff} := 2.1 \cdot L$ $\quad L_{eff} = 210\,mm$

13. Calculate the slenderness ratio for the column.

Slenderness ratio $\quad S_r := \frac{L_{eff}}{k} \quad S_r = 42$

14. Calculate the critical load using the Secant equation.

Guess $\quad P := 1 \cdot kN$

Given

$$P = \frac{S_y \cdot A}{1 + E_r \cdot sec\left(S_r \cdot \sqrt{\frac{P}{4 \cdot E \cdot A}}\right)}$$

$P_{cr} := Find(P) \quad P_{cr} = 17.93\,kN$