Stress at a point
Uniaxial load

Uniaxial load

Section

Stress cube for uniaxial stress loading

\[ \sigma_x = \frac{P}{A} \]
Stress at a point

General load case. Stress cube in 2D

2D projection: x-y plane

Notation

- $\sigma$: Normal Stress
- $\tau$: Shear Stress

Equilibrium conditions require that

$$\tau_{xy} = \tau_{yx}$$

Why?
Stress at a point

General load case. Stress cube in 3D

There are 9 components of stress.

Equilibrium conditions are used to reduce the number of stress components to 6:

\[
\begin{align*}
\tau_{xy} &= \tau_{yx} \\
\tau_{xz} &= \tau_{zx} \\
\tau_{yz} &= \tau_{zy}
\end{align*}
\]
Stress tensor

Cauchy stress tensor

\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}
\]

Tensors are quantities that are invariant to coordinate transformations.
Determination of principal stresses
Principal \textit{normal} stresses

- This problem involves performing coordinate transformation, which can provide a stress tensor that does NOT contain shear stresses.
- In 2D, this can be illustrated as:

\begin{itemize}
  \item Stress cube in original coordinate system \((x,y)\)
  \item Stress cube in transformed coordinate system \((x',y')\) -- only normal stresses exist: \(\sigma_1\) and \(\sigma_3\), in this 2D case.
\end{itemize}
Determination of principal stresses
Principal **normal** stresses

This problem involves performing coordinate transformation, which can provide a stress tensor that does NOT contain shear stresses, that is

\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz}
\end{bmatrix}
\begin{bmatrix}
\hat{n} = \\
\end{bmatrix}
\begin{bmatrix}
\sigma & 0 & 0 \\
0 & \sigma & 0 \\
0 & 0 & \sigma
\end{bmatrix}
\]

*Initial stress tensor*

*Unit vector, normal to principal plane*

*Transformed stress tensor*

*Unit vector, normal to principal plane*

*Same vectors*
Determination of principal stresses

Principal normal stresses

Previous equation can be written as

\[
\begin{bmatrix}
\sigma_{xx} - \sigma & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz} - \sigma
\end{bmatrix}
\hat{n} = \begin{bmatrix}
\sigma_{xx} - \sigma & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz} - \sigma
\end{bmatrix}
\begin{bmatrix}
n_x \\
n_y \\
n_z
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}
\]

implying that the determinant

\[
\begin{vmatrix}
\sigma_{xx} - \sigma & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz} - \sigma
\end{vmatrix} = 0
\]

(this is an Eigenvalue problem)
Determination of principal stresses

Principal normal stresses

Expanding determinant and setting it to zero yields

$$\sigma^3 - C_2\sigma^2 + C_1\sigma - C_0 = 0$$

in which

$$C_2 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$C_1 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$C_0 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

are stress invariants (have the same magnitudes for all choices of coordinate axes \((x,y,z)\) in which the applied stresses are measured or calculated)

The principal normal stresses, \(\sigma_1, \sigma_2, \sigma_3\), are the three roots of the cubic polynomial -- always real and typically ordered as: \(\sigma_1 > \sigma_2 > \sigma_3\)
Determination of principal stresses

Principal shear stresses

Principal shear stresses can be found from values of the principal normal stresses as

\[ \tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2} \]

\[ \tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} \]

\[ \tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2} \]
Mohr's circle: principal normal and shear stresses

Graphical representation of previous equations: 3D

Equivalent nomenclature:

\[ \sigma_x = \sigma_{xx} \]
\[ \sigma_y = \sigma_{yy} \]
\[ \sigma_{xy} = \tau_{xy} \]
Determination of principal stresses
Principal normal and shear stresses: 2D case

We will use these equations extensively

Principal normal stresses:

$$\sigma_1, \sigma_3 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Maximum shear stress:

$$\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$
Mohr's circle: principal normal and shear stresses

Graphical representation of previous equations: 2D

Principal stresses:
\[ \sigma_{1,3} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]

Applied normal stress
\[ \sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} \]

Applied shear stress
\[ \tau_{\text{max in-plane}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]

Maximum shear
Determination of principal stresses
Review author’s textbook

- Examples: 4-1, 4-2, and 4-3
- Review and master Section 4.6
Determination of principal stresses
Example: solve in class / assignment

A piece of chalk is subjected to combined loading consisting of a tensile load $P$ and a torque $T$, see figure. The chalk has an ultimate strength $\sigma_u$ as determined by a tensile test. The load $P$ remains constant at such a value that it produces a tensile test of $0.51 \cdot \sigma_u$ on any cross-section. The torque $T$ is increased gradually until fracture occurs on some inclined surface.

Assuming that fracture takes place when the maximum principal stress $\sigma_1$ reaches the ultimate strength $\sigma_u$, determine the magnitude of the torsional shearing stress produced by the torque $T$ at fracture and determine the orientation of the fracture surface.
Determination of principal stresses

Example: solve in class / assignment

Stress element

\[ \sigma_{xx} = 0.51\sigma_u \]

\[ \sigma_{xy} = -0.70\sigma_u \]
Reading assignment

- Chapters 4 of textbook: Sections 4.0 to 4.6
- Review notes and text: ES2501, ES2502

Homework assignment

- Author’s: 4-2, 4-3
- Solve: 4-1(a,c,f,i), 4-4