

WORCESTER POLYTECHNIC INSTITUTE MECHANICAL ENGINEERING DEPARTMENT

Optical Metrology and NDT
ME-593L, B'2011

Introduction: Wave Optics
07 November 2011



Wave optics: light waves

Wave equation

An optical wave -- monochromatic -- can be described mathematically by the complex wavefunction

$$U(x, y, z, t) = a(x, y, z) \cdot \exp[j\phi(x, y, z)] \cdot \exp[j2\pi\nu t] \quad (1)$$

where

x, y, z are the components of the position vector r

t is time

$\phi(x, y, z)$ is the optical phase

$a(x, y, z)$ is the amplitude

ν is the frequency [Hz]

$(\omega = 2\pi\nu = \text{angular frequency [rad/sec]})$

j is the complex quantity $\sqrt{-1}$

Wave optics: light waves

Wave equation

Equation (1) can be written as

$$U(x, y, z, t) = U(x, y, z) \cdot \exp[j2\pi\nu t] \quad (2)$$

where the time-independent term,

$$U(x, y, z) = a(x, y, z) \cdot \exp[j\phi(x, y, z)] \quad (3)$$

is the **complex amplitude** of the optical wave $U(x, y, z, t)$



Wave optics: light waves

Wave equation

Function $U(x, y, z, t)$ must satisfy the *wave equation* (in order to represent a valid wave function), therefore,

$$\nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0 \quad (4)$$

where

$$c = \frac{c_0}{n} \quad \text{for } n \geq 1 \quad (5)$$

in which c_0 is the speed of light in free-space and the wave propagates in a medium with index of refraction n .



Wave optics: light waves

Wave equation

By substituting Eq. 2 into the wave equation, Eq. 4, the following equation is obtained -- exercise in class/homework

$$(\nabla^2 + k^2)U(x, y, z) = 0 \quad (6)$$

which is called the *Helmholtz equation*, where

$$k = \frac{2\pi\nu}{c} = \frac{\omega}{c} = \frac{2\pi}{\lambda} \quad (7)$$

in the *wave number*, and λ is the spatial wavelength.

Note that:

$$\lambda = \frac{c}{\nu} \quad (8)$$



Wave optics: light waves

Elementary waves

The two canonical solutions of the Helmholtz equation in a homogenous medium are: (1) *the plane wave*, and (2) *the spherical wave*.

(1) The plane wave

The plane wave has the complex amplitude:

$$U(x, y, z) = A \exp(-j \mathbf{k} \cdot \mathbf{r}) \quad (9)$$
$$= A \exp[-j(k_x \cdot x + k_y \cdot y + k_z \cdot z)]$$

where

A is the **amplitude**, or *complex envelope*

$\mathbf{k} = k_x \hat{\mathbf{i}} + k_y \hat{\mathbf{j}} + k_z \hat{\mathbf{k}} = (k_x, k_y, k_z)$ is the **propagation direction vector**

$\mathbf{r} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}} = (x, y, z)$ is the **position vector**

j is the complex quantity $\sqrt{-1}$

Elementary waves

Plane waves

For Eq. 9 to satisfy the Helmholtz equation, Eq. 6, it is necessary that

$$k_x^2 + k_y^2 + k_z^2 = k^2 \quad (10.1)$$

that is, the magnitude of the propagation direction vector, \mathbf{k} , is equal to the wave number, k ,

$$|\mathbf{k}| = k \quad (10.2)$$



Elementary waves

Plane waves

Since the phase, or $\arg[U(x, y, z)] = \arg[A] - \mathbf{k} \cdot \mathbf{r}$, the wavefronts are

$$k_x \cdot x + k_y \cdot y + k_z \cdot z = 2\pi q + \arg[A] \quad (11)$$

for $q = \text{integer}$

Equation 11 describes the family of parallel planes that is perpendicular to the propagation direction vector, \mathbf{k} . These planes are called: wavefronts.

Planes are separated by the distance

$$\lambda = \frac{2\pi}{k} \quad (12)$$



Elementary waves

Plane waves

If the z -axis is taken in the direction of the propagation vector, k , then

$$U(z) = A \exp(-j k z) \quad (13)$$

using Eqs 13 and 2,

$$U(z, t) = A(z) \cdot \exp[-j k z] \cdot \exp[j 2\pi \nu t] \quad (14)$$

$$= A(z) \cdot \exp[j(2\pi \nu t - k z)]$$

$$= |A| \cdot \exp\{j(2\pi \nu t - k z + \arg[A])\} \quad (15)$$

and by separating the real component of Eq. 15, it is obtained

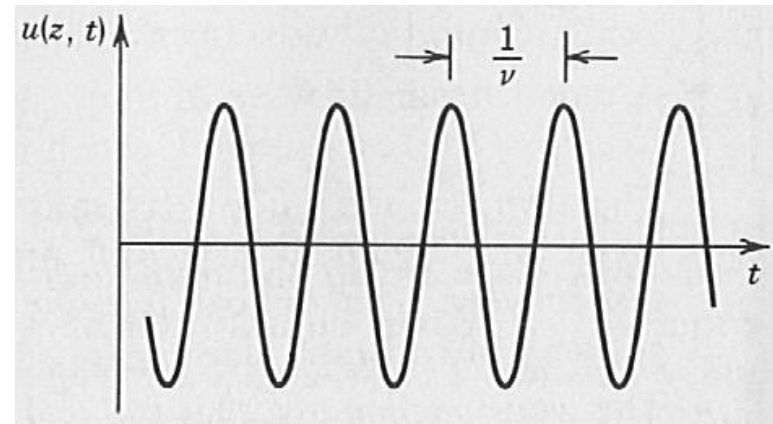
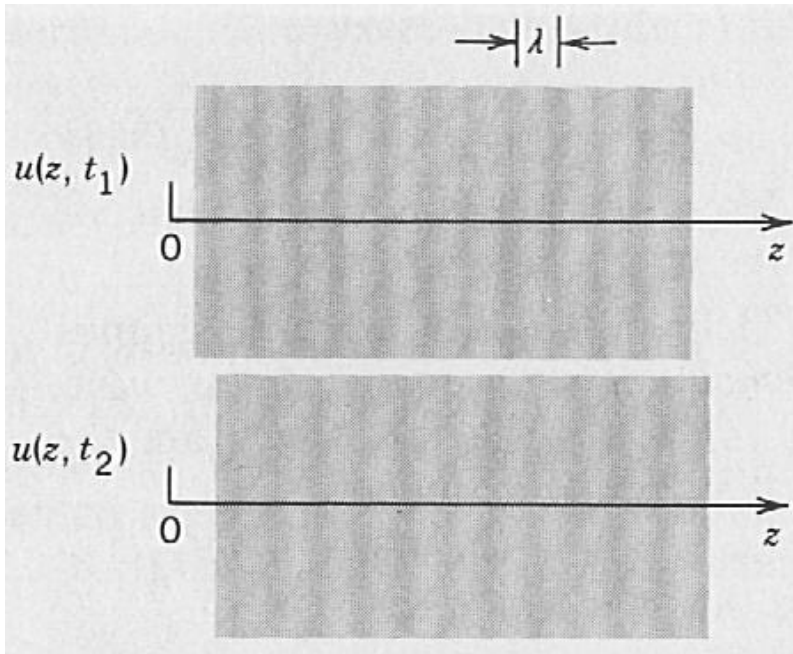
$$u(z, t) = \text{Re}\{U(z, y)\} = |A| \cdot \cos(2\pi \nu t - k z + \arg[A]) \quad (16)$$

$$= |A| \cdot \cos\left\{2\pi \nu \left(t - \frac{z}{c}\right) + \arg[A]\right\}$$



Elementary waves

Plane waves



A plane wave traveling in the z -direction is a periodic function of z with spatial period λ and a periodic function of t with temporal period $1/\nu$.

Elementary waves

Observations

- Optical phase, obtained from Eq. 16,

$$\arg[\operatorname{Re}\{U(z, t)\}] = 2\pi \nu(t - \frac{z}{c}) + \arg[A] \quad (17)$$

varies as a function of time and position

- Optical intensity is determined as

$$I = |U|^2 = U \cdot U^* \quad (18)$$

where

U^* is the **complex conjugate** of U



Wave optics: light waves

Elementary waves

(2) The spherical wave

The spherical wave is another canonical solution of the Helmholtz equation. Its complex amplitude is

$$U(r) = \frac{A}{r} \exp(-j k r) \quad (19)$$

where r is the distance from the propagation origin.

$k = \frac{2\pi}{\lambda}$ is the wave number, and

$I = U \cdot U^* = \frac{|A|^2}{r^2}$ (proportional to the square of the distance from the origin)



Elementary waves

Spherical waves

Taking $\arg[A] = 0$, for simplicity,

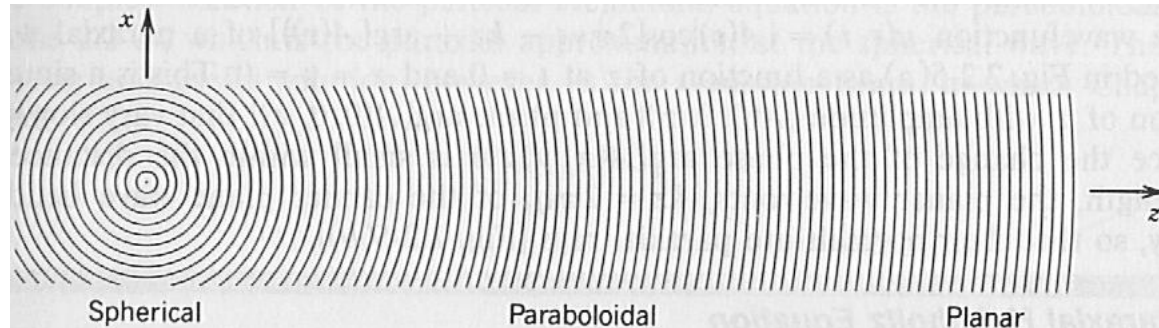
$$k \cdot r = k \sqrt{x^2 + y^2 + z^2} = 2\pi q + \arg[A] \quad (20)$$

for $q = \text{integer}$

Equation 20 describes **the family of concentric spheres**: spherical wavefronts.

Spheres are separated by the distance

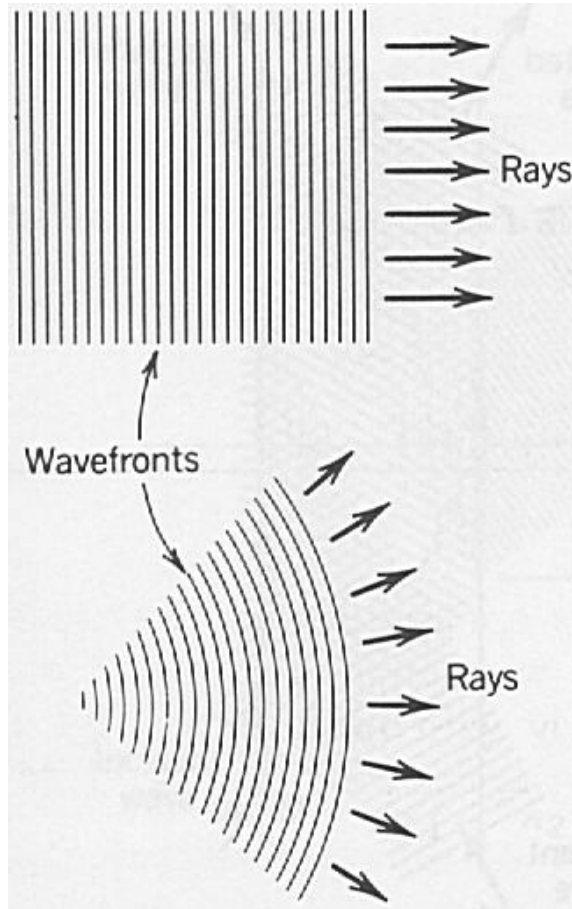
$$\lambda = \frac{2\pi}{k} \quad (21)$$



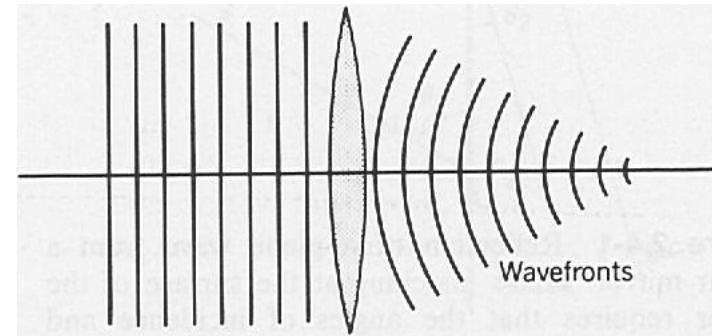
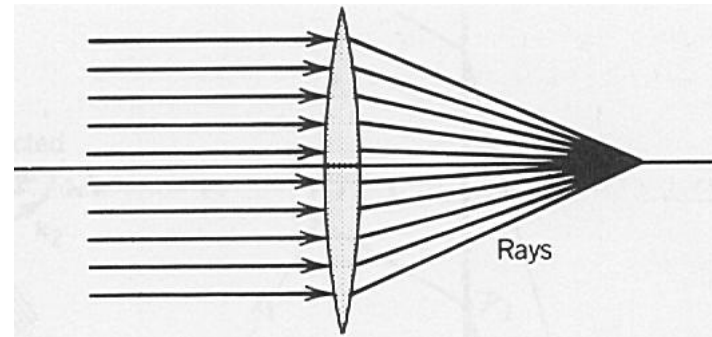
Cross-section of the wavefronts of a spherical wave

Elementary waves

Wavefronts



Effect of adding a lens

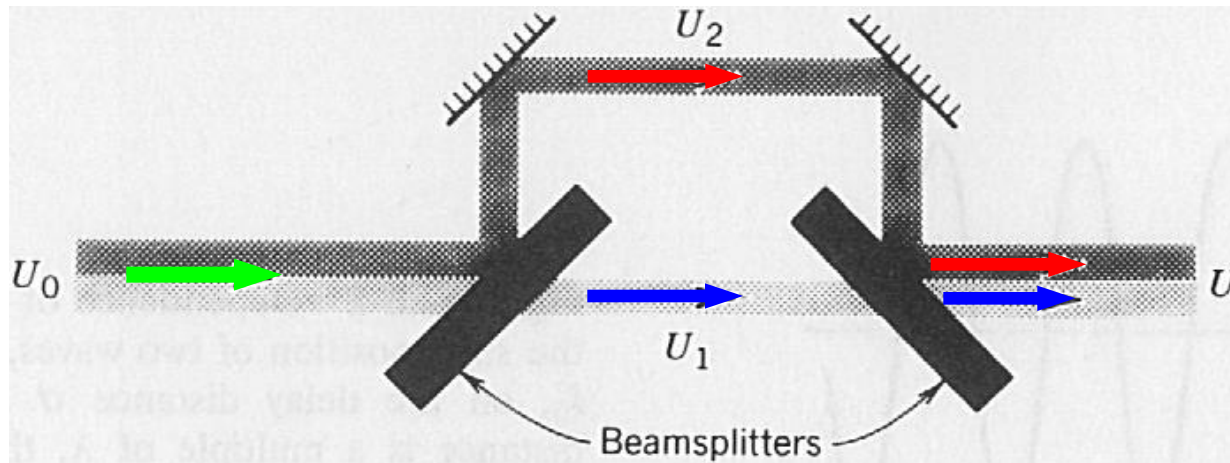


The rays of ray optics are orthogonal to the wavefronts of wave optics. Note the effect of a lens on rays and wavefronts.

Optical interference

Interferometers

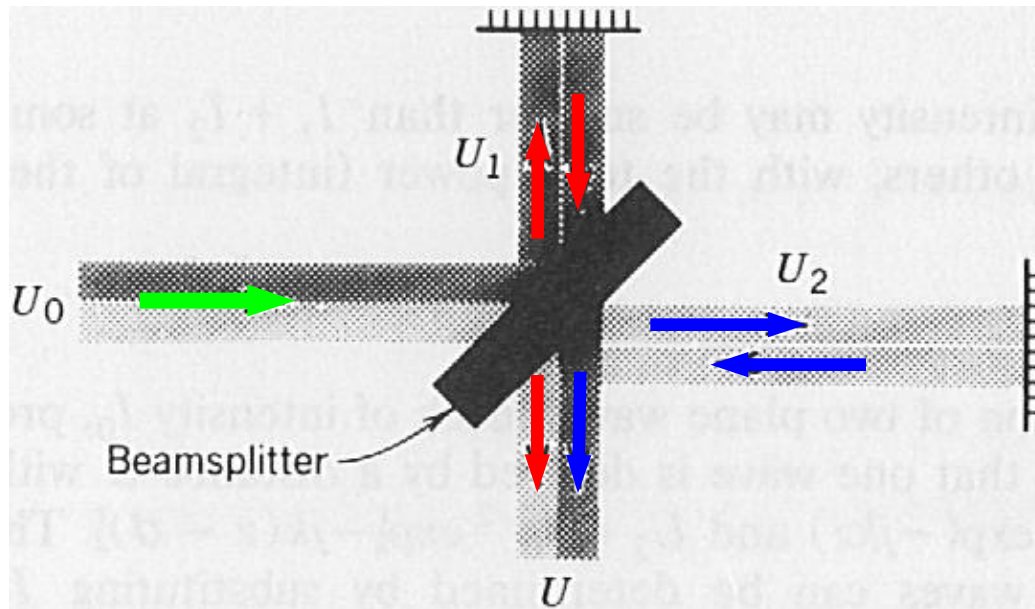
Mach-Zehnder



Optical interference

Interferometers

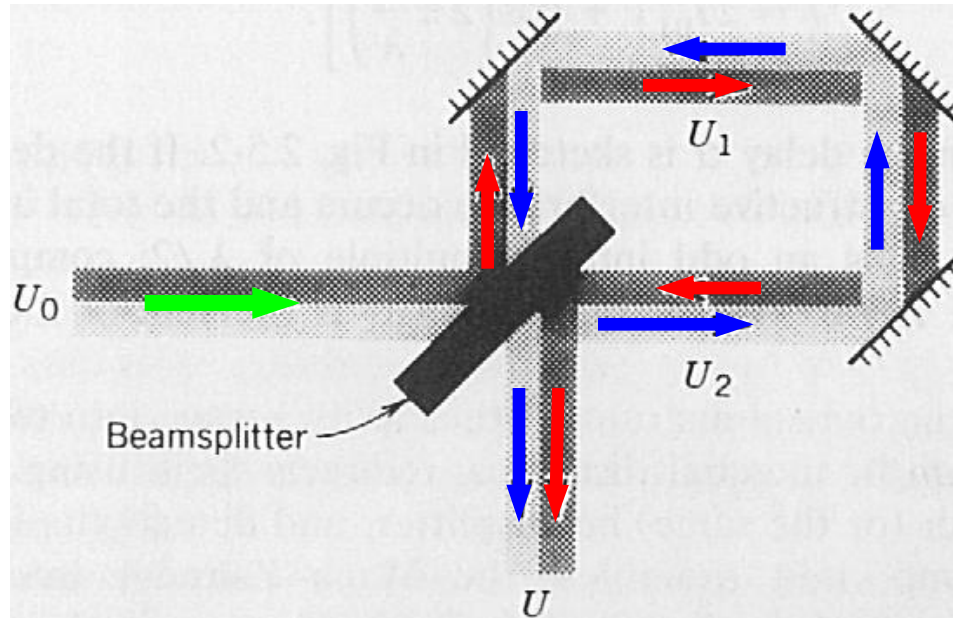
Michelson



Optical interference

Interferometers

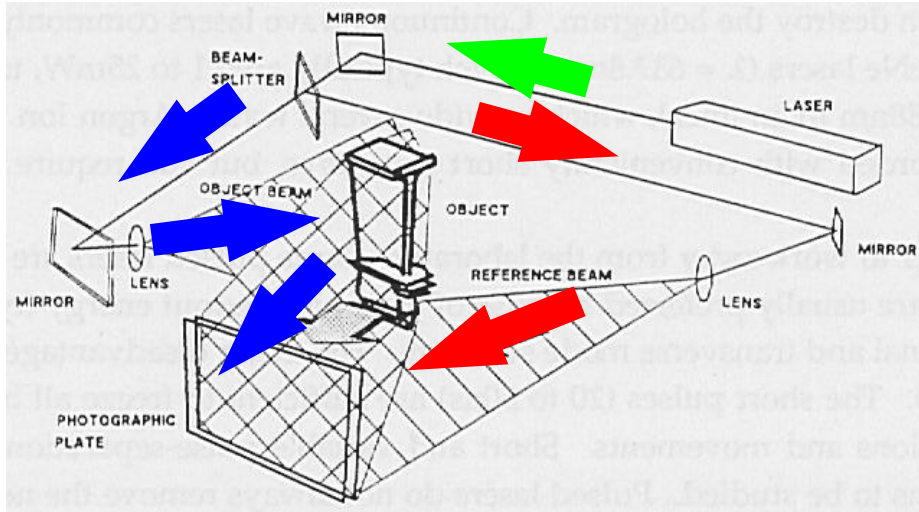
Sagnac



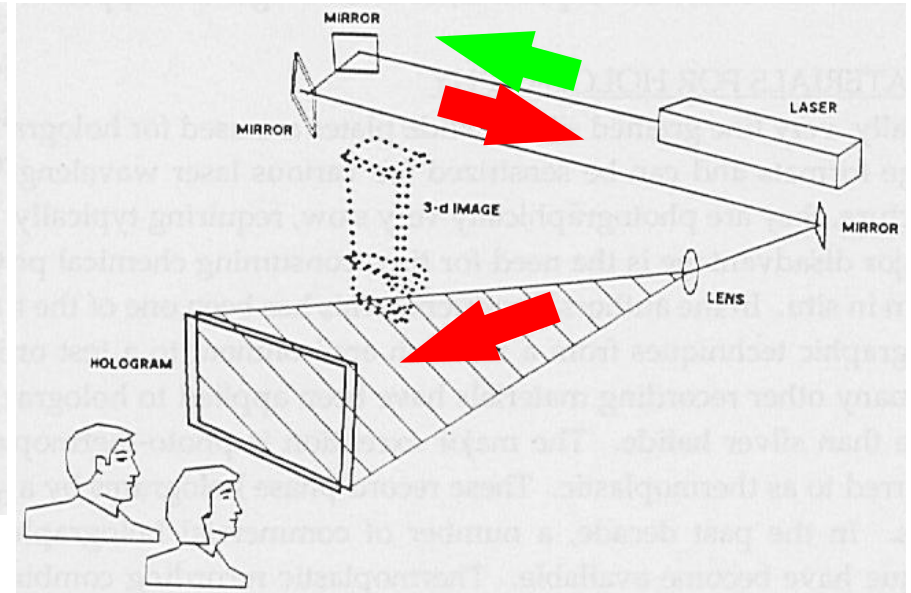
Optical interference

Holographic interferometry

Recording



Reconstruction



Interference equation

Consider the superposition of two monochromatic plane waves U_1 and U_2 from the same light source

$$U = U_1 + U_2 \quad (22)$$

The corresponding intensity is,

$$I = |U|^2 = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1 U_2^* + U_1^* U_2 \quad (23)$$

$$\text{if } U_1 = A_1 \exp(-j \mathbf{k}_1 \cdot \mathbf{r}) = A_1 \exp(-j \phi_1),$$

$$U_2 = A_2 \exp(-j \mathbf{k}_2 \cdot \mathbf{r}) = A_2 \exp(-j \phi_2)$$

The observed intensity, measured, is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\phi_2 - \phi_1) \quad (24)$$



Interference equation, cont'd

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\phi_2 - \phi_1) \quad (25)$$

Defining: $I_B = I_1 + I_2 =$ Background intensity
 $I_M = 2\sqrt{I_1 I_2} =$ Modulation intensity

Intensity becomes:

$$I = I_B + I_M \cos(\Delta\phi) \quad (26)$$

