Rigidity, connectivity and graph decompositions

Fragments

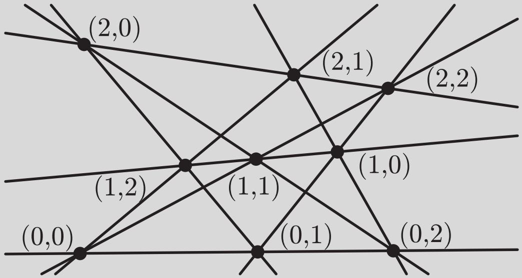
April 4, 2007

1. The Pappus Configuration

Pappus's Theorem says that if you are given two ordered triples of collinear points, say (A, B, C), and (A', B', C'), then the three pairwise intersections $A'' = BC' \wedge B'C$, $B'' = AC' \wedge A'C$, and $C'' = AB' \wedge A'B$ are collinear. This gives a configuration of 9 points; the first 6 points and the three intersections points; together with 9 lines; the two original lines, the additional lines $\{BC', B'C, AC', A'C, AB', A'B\}$, and the lastly the line given by the theorem. It is a configuration of type 9_3.

Geometrically, there is no particular symmetry in this configuration in the sense that there is no affine transformation of $\mathbb{R}^2$ which permutes the points. Abstractly, however, the Pappus configuration does have symmetry. Moreover, while our construction assigns special roles to certain sets of points and lines, say the original six points and the final line whose existence is guaranteed by the theorem, the combinatorial automorphism group is transitive on both points and lines. That is, any of the six lines could have played the role of the Pascal Line if we had started appropriately.

Let us compute the automorphism group G. It is convenient to assign labels to the 9 points from $\mathbb{Z}_3^2$
and we may easily check that three distinct points are collinear if and only if their vector sum is $(0,0)$ and their second coordinates are distinct.



The Pappus Configuration with point labels in $\mathbb{Z}_3^2$.

Note that in $\mathbb{Z}_3$ the sum of three elements is zero if and only if the three elements are either identical or distinct. Using this observation, we see that we may permute the first and second coordinates of the points independently, so that the automorphism group is transitive on the points and has $S_3 \times S_3$ as a subgroup of order 36.

Since the orbit of $(0,0)$ has 9 elements, we know

$$|G| = 9 \cdot \mathrm{Stab}((0,0)).$$

Since collinear points must have distinct second coordinates, there are three triples of points of the form $\{(0,x),(1,x),(2,x),\}$ of which no two belong to any line.

These three triples are permuted by any automorphism. In particular, the set $\{(0,0),(1,0),(2,0),\}$ is fixed by every element in $\mathrm{Stab}((0,0))$, so the orbit of $(1,0)$ in $\mathrm{Stab}((0,0))$ has size 2 and

$$|G| = 9{\cdot}2{\cdot}|\mathrm{Stab}((0,0)) \cap \mathrm{Stab}((0,1))| = 9{\cdot}2{\cdot}|\mathrm{Stab}((0,0)) \cap \mathrm{Stab}((0,1)$$

The two lines not intersecting $\{(0,0),(0,1),(0,2)\}$ must be permuted in the stabilizer of that line, are are permuted by $(1,2)$ on the first coordinate, so

$$|G| = 9{\cdot}2{\cdot}2{\cdot}|\mathrm{Stab}((0,0)) \cap \mathrm{Stab}((0,1)) \cap \mathrm{Stab}((0,2)) \cap \mathrm{Stab}(\{(1,0),(1,$$

At this point we have exhausted the coordinate permuting automorphism, however the permutation defined by

$$\begin{aligned}(1,0) &\rightarrow (1,1) \rightarrow (1,2) \rightarrow (1,0)\\ (2,0) &\rightarrow (2,2) \rightarrow (2,1) \rightarrow (2,0)\end{aligned}$$

fixes the points $(0,1)$, $(0,2)$, $(0,3)$ as well as the line $\{(1,0),(1,1),(1,2)\}$, and shows that the orbit of $(1,0)$ in the stabilizer is of size 3. Thus

$$|G| = 9\cdot2\cdot2\cdot3\cdot|\mathrm{Stab}((0,0))\cap\mathrm{Stab}((0,1))\cap\mathrm{Stab}((0,2))\cap\mathrm{Stab}((1,0))|.$$

Lastly, it is easy to see that the only automorphism which fixes all of $(0,1)$, $(0,2)$, $(0,3)$ and $(1,0)\}$ must be the identity, so

$$|G| = 9\cdot2\cdot2\cdot3 = 108.$$

Now that we know that there are exactly 108 automorphisms, we can identify the automorphisms group as an affine group of transformations of $\mathbb{Z}_3^2$. Extend the coordinates of the points matrices of the form

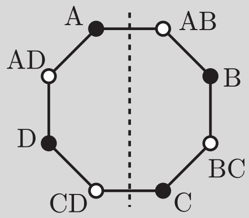
$$\begin{bmatrix} \pm 1 & r & p \\ 0 & \pm 1 & q \\ 0 & 0 & 1 \end{bmatrix}$$

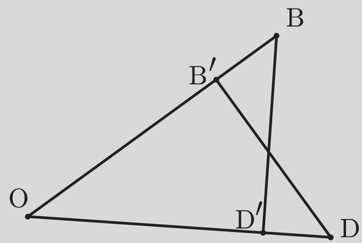
with entries in $\mathbb{Z}_3$. There are exactly $3^3 2^2 = 108$ such matrices. They are all invertible and stabilize the affine plane $\{(x, y, 1)\}$ since the image of $\{(x, y, 1)\}$ is $\{(\pm x + ry + p, \pm y + q, 1)\}$.

Moreover, if there are three points $\{(x_i, y_i, 1)\}$ with the y_i distinct and $\sum x_i = 0$, $\sum y_i = 0$, then the values $\pm y_i + q$ are distinct and $\sum(\pm x_i + ry_i + p) = \pm(\sum x_i) + r(\sum y_i) + 3p = 0$, and $\sum(\pm y_i + q) = \pm(\sum y_i) + 3q = 0$, so all lines are preserved.

2. An Autopolar Quadrilateral

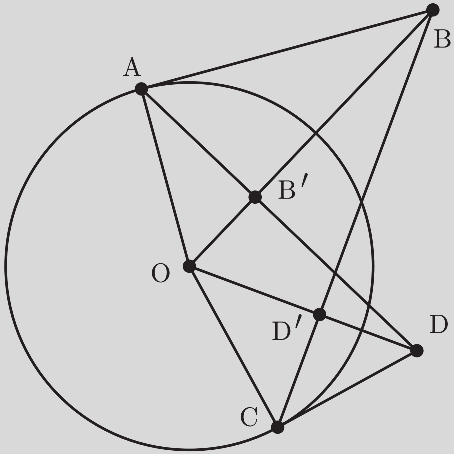
An autopolar quadrilateral must have a color reversing automorphism of it's levi graph which has order 2. This automorphism must be a reflection on two opposite edges of the Levi graph, which is an octagon.





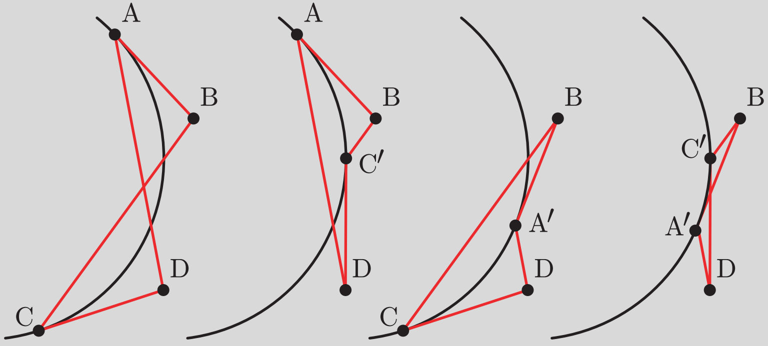
For this reflection, the points A and C are incident to their polar lines, and so must lie on the unit circle, with their polar lines being tangent lines to the circle.

To construct an autopolar quadrilateral, we note that, given angle BOD, and constructing perpendicular B' and D' we have $BO \cdot B'O = D'O \cdot DO$, so, if 0 is the center of the unit circle, and B and B' are inverse points with respect to the unit circle, then so are D and D',



So $ABCD$ is an autopolar quadrilateral.

Alternatively, let B and D be any points outside the unit circle satisfying $B \cdot D = 1$. Let A and C be either of the two points on the circle whose tangents pass through B and D respectively,



Four autopolar quadrilaterals through B and D.

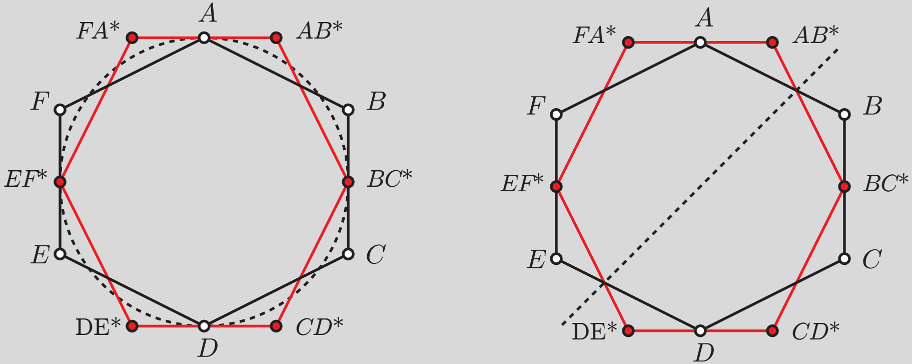
So $A \cdot A = C \cdot C = 1$, and $A \cdot (B - A) = C \cdot (D - C) = 0$. So

$$\begin{array}{l} A \cdot A = 1 \\ A \cdot B = 1 \\ D \cdot B = 1 \\ D \cdot C = 1 \\ C \cdot C = 1 \end{array} \Longrightarrow \begin{array}{l} AB = A^* \\ AD = B^* \\ BC = D^* \\ DC = C^* \end{array}$$

and again $ABCD$ is autopolar, inducing the given reflection on the Levi graph. Moreover, given B and D in the exterior of the unit circle with $B \cdot D = 1$, there are four autopolar quadrilaterals with points B and D with respect to the circle.

3. A Self-polar hexagon

The hexagon of has vertices $A = (0,1)$, $B = (1,1/2)$, $C = (1,-1/2)$, $D = (0,-1)$, $E = (-1,-1/2)$ and $F = (-1,1/2)$. and symmetries given by the Klein four group, D_2.



A non-autopolar but self-polar configuration.

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The lines are given by $BC = (1, 0)$, $AB = (1/2, 1)$, $FA = (-1/2, 1)$, $EF = (-1, 0)$, $DE = (-1/2, -1)$ and $CD = (1/2, -1)$, so the hexagon is not autopolar, since the vertex lists are distinct, nevertheless, there is a transformation

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which interchanges the two sets, and we say that the hexagon is *self-polar* with 4 self-polarities given by $D_4 - D_2$.

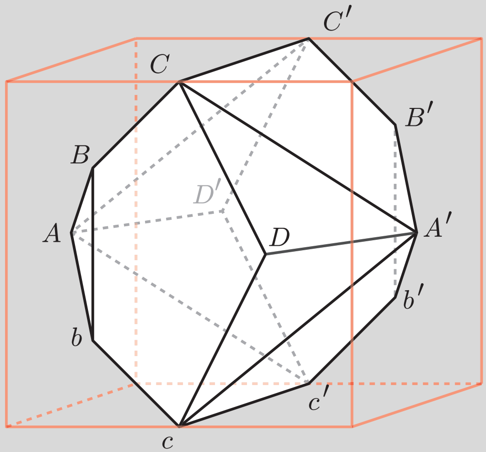
In this case the self-polarities are are isometries. If we require the transformations be be orientation preserving, then the $90°$ rotations are the only self-polarities. On the other hand, we could relax our requirement and allow the self-polarities to be any linear transformation, however in this case we would get no additional structure.

Note that the self polarities permute the vectors encoding both the points and the lines, and that these lists may intersect. In other words, a representation of the group of permutations of the points and lines as matrices need not be faithful. To avoid confusion, we will label the vectors white or black depending on whether it encodes a point or a line. We may now say that a geometric configuration is autopolar, if and only if the identity transformation represents a self-polarity.

Note: Relate the the automorphisms of the Levi graph. Perhaps and exercise.

4. A Selfpolar Dodecahedron

Let p and q, $0 < p, q < 1$ be given. We consider the convex hull of the polyhedron with 12 vertices $\{A, A', B, B', C, C', D, D', b, b', c, c'\}$ with $A = (1, 0, 0)$, $B = (1, p-1, q)$, $C = (1, p-q, 1)$, with $(x, y, z)' = (-x, -y, z)$ and the lower case points being the reflections of the upper case points about the $z = 0$ plane.



There are 12 planar faces, and each face is contained in the polar plane of one of the vertices rotated 90° about the z-axis. For example, the pentagon $bBCDc = (1, 0, 0)^*$ corresponds to vertex A, the quadrilateral $ABCC' = (p - q, -1, 1)^*$ corresponds to the vertex C', the triangle $ABB' = (p, -1, 0)^*$ corresponds to the vertex D', and the triangle $CDA' = (1 - p, 1, q)^*$ corresponds to the vertex B. In fact, the 90° rotation maps this polyhedron onto its polar polyhedron, however, the polyhedron is *not* autopolar.

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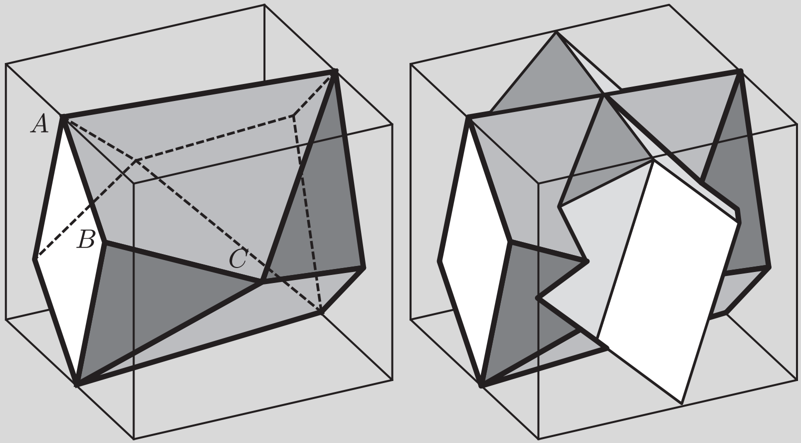
5. Rank of a self-polar polyhedron

Consider the orbit of the points $A = (1, -r, 1)$, $B = (1, p, q)$, $C = (0, 1, 0)$ under the group of transformation generated by $180°$ rotations about the coordinate axes. The convex hull of the orbit is a polyhedron in $\mathbb{R}^3$.



Show that the graph of the polyhedron is self-dual.
What conditions on p, q, and r insure that the polyhedron is self-polar, with a self polarity given by a $90°$ rotation about the z axis?
Find find a self-polarity of the polyhedron which is of order 2. (It is a $180°$ rotation.)
Conclude that the polyhedron is combinatorially autopolar, self-polar, but not autopolar.

6. Answer

Let $\rho = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the 90° counterclockwise rotation about the z axis.

The 10 vertices are

$A = (-r, -1, 1)$, $A_1 = (r, 1, 1)$, $a = (-r, 1, -1)$, $a_1 = (r, -1, -1)$,

$B = (p, -1, q)$, $B_1 = (-p, 1, q)$, $b = (p, 1, -q)$, $b_1 = (-p, -1, -q)$,

$C = (1, 0, 0)$, and $C_1 = (-1, 0, 0)$

There are 6 quadrilateral faces:
$[A, B, a_1, b_1]$, $[A_1, B_1, a, b]$, $[A, A_1, B, C]$, $[A, A_1, B_1, C_1]$, $[a, a_1, b, C]$, and $[a, a_1, b_1, C_1]$,
and there are 4 triangular faces:
$[B, a_1, C]$, $[B_1, a, C_1]$, $[A_1, b, C]$ and $[A, b_1, C_1]$.

To check that the quadrilaterals are in fact planar:
The vertices $[A, B, a_1, b_1]$ each satisfy $\mathbf{x}\cdot(0, -1, 0) = \mathbf{x}\cdot\rho(C_1) = 1$.
The vertices $[A_1, B_1, a, b]$ each satisfy $\mathbf{x}\cdot(0, 1, 0) = \mathbf{x}\cdot\rho(C) = 1$.
The vertices $[A, A_1, B, C]$ each satisfy $\mathbf{x}\cdot(1, -r, 1) = \mathbf{x}\cdot\rho(A) = 1$ provided that $p + q + r = 1$
Then the vertices $[A, A_1, B_1, C_1]$ each satisfy $\mathbf{x}\cdot(-1, r, 1) = \mathbf{x}\cdot\rho(A_1) = 1$,
the vertices $[a, a_1, b, C]$ each satisfy $\mathbf{x}\cdot(1, r, -1) = \mathbf{x}\cdot\rho(a_1) = 1$, and
the vertices $[a, a_1, b_1, C_1]$ each satisfy $\mathbf{x}\cdot(-1, -r, -1) = \mathbf{x}\cdot\rho(a) = 1$.

The triangular faces:

The vertices $[B, a_1, C]$ each satisfy $\mathbf{x} \cdot \rho(b_1) = \mathbf{x} \cdot (1, -p, -q) = 1$, if p and q satisfy $2p - q^2 = 1$.

Then the vertices $[B_1, a, C_1]$ each satisfy $\mathbf{x} \cdot \rho(b) = \mathbf{x} \cdot (-1, p, -q) = 1$,

the vertices $[A_1, b, C]$ each satisfy $\mathbf{x} \cdot \rho(B) = \mathbf{x} \cdot (1, p, q) = 1$, and

the vertices $[A, b_1, C_1]$ each satisfy $\mathbf{x} \cdot \rho(B_1) = \mathbf{x} \cdot (-1, -p, q) = 1$,

So each face is the polar of the image of some vertex under ρ.

Moreover, the images of the vectors representing the quadrilateral faces under ρ are

$$\rho(0,-1,0) = (1,0,0) = C$$
$$\rho(0,1,0) = (-1,0,0) = C_1$$
$$\rho(1,-r,1) = (r,1,1) = A_1$$
$$\rho(-1,r,1) = (-r,-1,1) = A$$
$$\rho(1,r,-1) = (-r,1,-1) = a$$
$$\rho(-1,-r,-1) = (r,-1,-1) = a_1$$

And the images of the vectors representing the triangular faces under ρ are

$$\rho(\ \ 1, -p, -q) = (\ \ p, \ \ 1, -q) = b,$$
$$\rho(-1, \ \ p, -q) = (-p, -1, -q) = b_1,$$
$$\rho(\ \ 1, \ \ p, \ \ q) = (-p, \ \ 1, \ \ q) = B_1$$
$$\rho(-1, -p, \ \ q) = (\ \ p, -1, \ \ q) = B,$$

So each point in the image under ρ of a vector representing a face, and ρ is a self-polarity.

Let $\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rho.$ be ρ followed by an 180° rotation about the y axis. Since the 180° rotation about the y axis is an automorphism of the polyhedron, σ is also a self-polarity. Geometrically, σ is a 180° rotation with fixed subspace generated by $(1, 1, 0)$.

So σ is a self-polarity of order 2, so that the polyhedron is combinatorially autopolar. The polyhedron is nevertheless not autopolar.