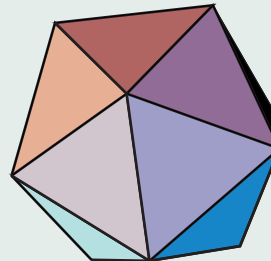
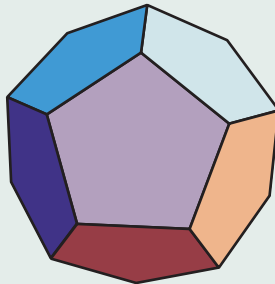
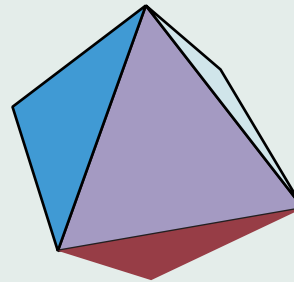
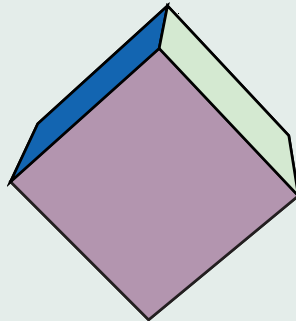
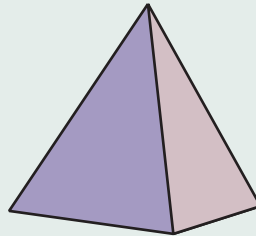




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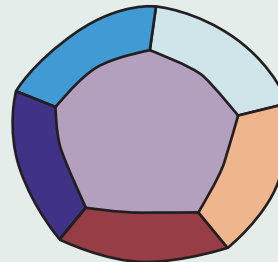
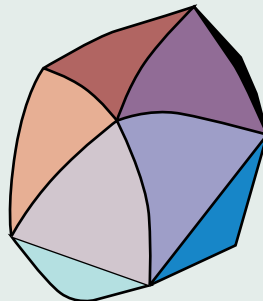
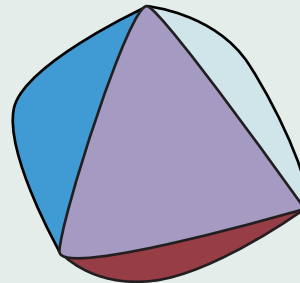
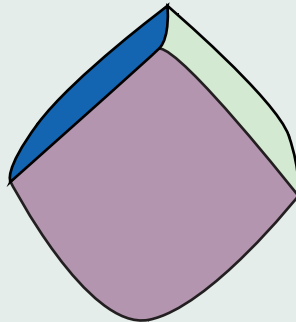
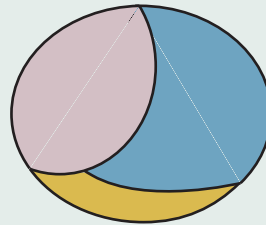
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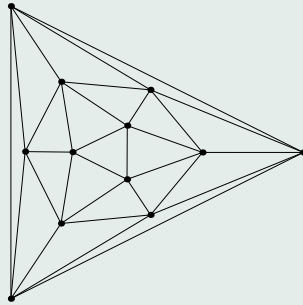
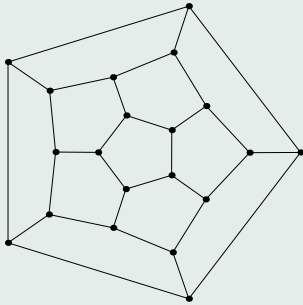
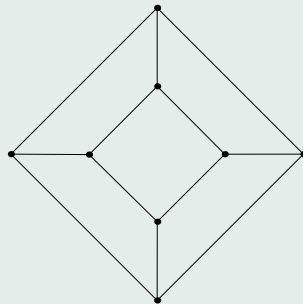
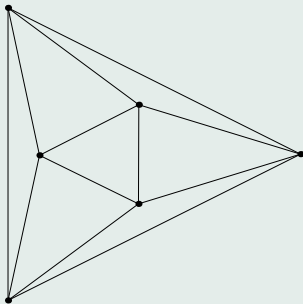
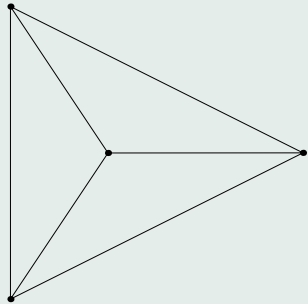
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2. Combinatorics

\mathcal{M} is a regular map on the sphere:
vertices: m -valent
faces: n -valent.

$$2|E| = m|V| = n|F|$$

Euler characteristic: $\chi(\mathcal{M}) = 2$

Thus

$$\frac{2|E|}{m} - |E| + \frac{2|E|}{n} = 2.$$

Integer solutions: $(m, n > 1)$

$$1/n + 1/m > 1/2,$$

$m = 2, n \geq 2$, (an n -cycle separating the sphere into two n -gonal faces)

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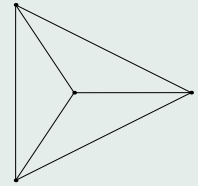
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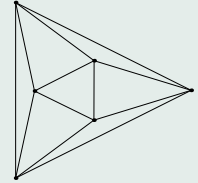
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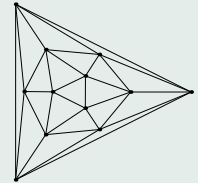
$m = 3, m = 3$ (tetrahedron)



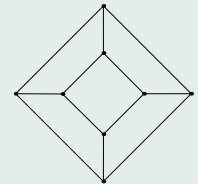
$m = 3, n = 4$ (octahedron),



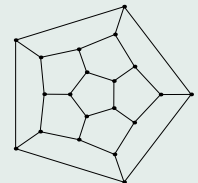
$m = 3$ and $n = 5$ (icosahedron),



$m = 4$ and $n = 3$ (cube),



$m = 5$ and $n = 3$ (dodecahedron),

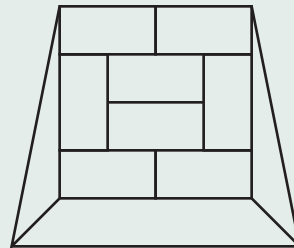


$m \geq 6$ and $n = 2$ (two vertices connected by n edges forming n 2-gons).



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3. How many automorphisms?

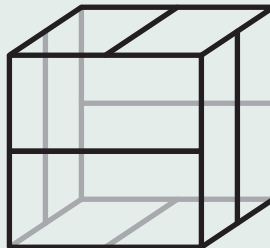


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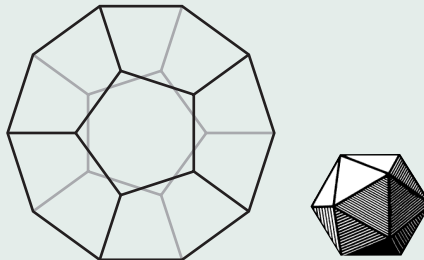
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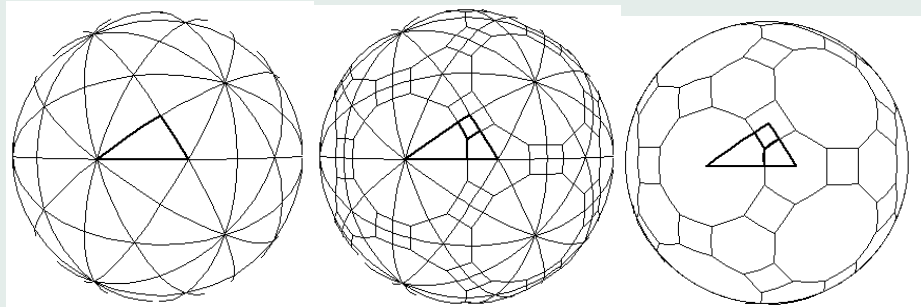
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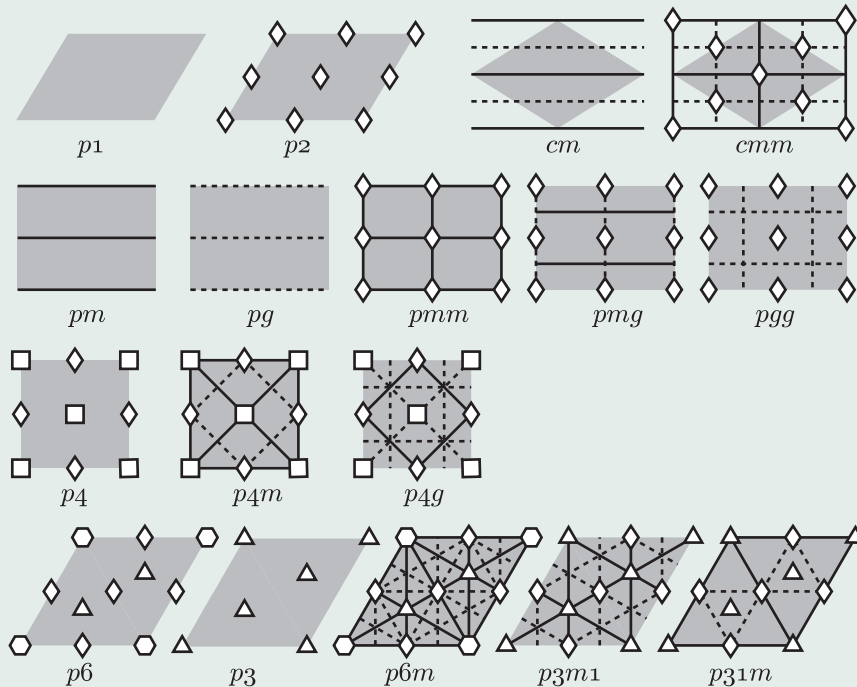
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5. Wallpaper



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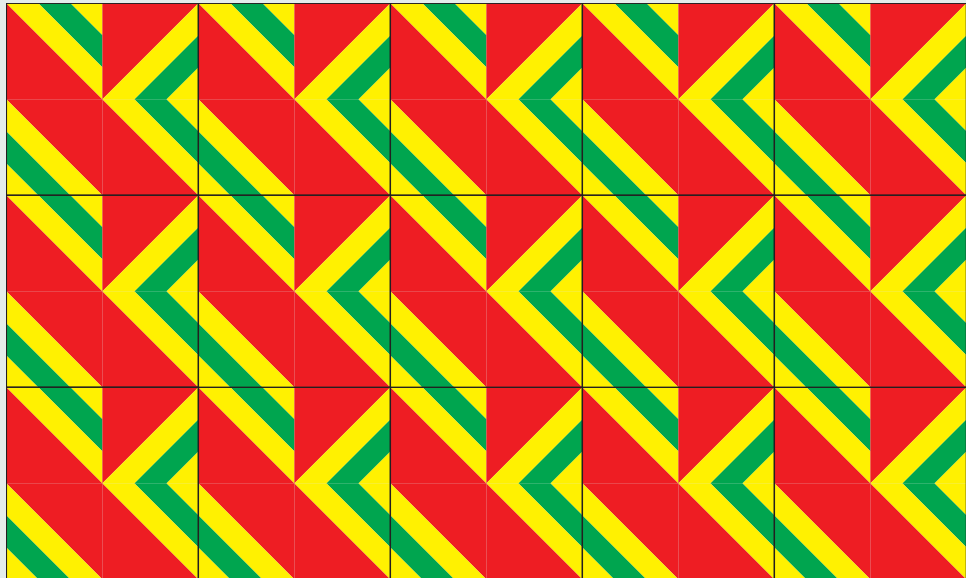
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6. Pattern counting

M. C. Escher laboriously examined multitudes of sketches to determine how many different patterns would result by repeatedly translating a 2×2 square having its four unit squares filled with copies of an asymmetric motif in any of four aspects.



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$bb^* = qq^* = \dots bbbbbb \dots =$	
$bq^* = qb^* = \dots bqbqbq \dots =$	
$bd^* = pq^* = db^* = qp^* = \dots bdbdbd \dots =$	
$bp^* = dq^* = pb^* = qd^* = \dots bpbpbp \dots =$	
$dd^* = pp^* = \dots ddddddd \dots =$	
$dp^* = pd^* = \dots dpdpdp \dots =$	

Rotations and mirrors

$bb^* = qq^* = dd^* = pp^* = \dots bbbbbb \dots =$	
$bq^* = qb^* = dp^* = pd^* = \dots bqbqbq \dots =$	
$bd^* = pq^* = db^* = qp^* = \dots bdbdbd \dots =$	
$bp^* = dq^* = pb^* = qd^* = \dots bpbpbp \dots =$	

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$S = \{w_1, \dots\}$ – signatures

P – permutations

$\langle P \rangle$ – permutation group

different strip patterns = # orbits

Burnside's Lemma: The number of orbits equals the average number of points fixed by the permutations in the group.

N – number of orbits

$\text{fix}(p)$ – the set of signatures fixed by the permutation p .

$$N = \frac{1}{|\langle P \rangle|} \sum_{p \in \langle P \rangle} |\text{fix}(p)|$$

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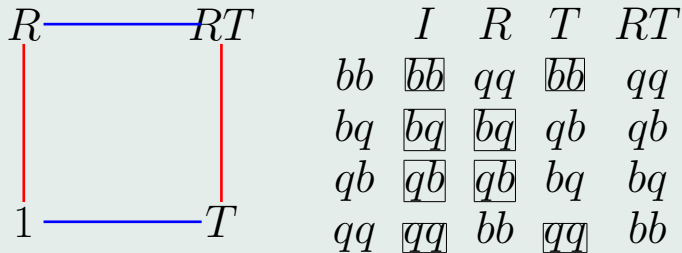
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$$T(XY) = YX,$$

$$R(XY) = R(Y)R(X)$$

$$(R(b) = q, R(q) = b, R(p) = d \text{ and } R(d) = p)$$

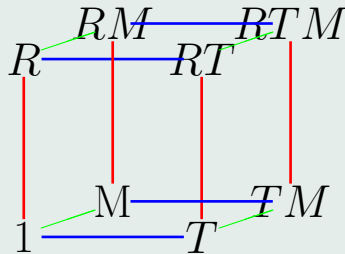


$\langle P \rangle$ and its action on four signatures

Burnside's Lemma: $8/4 = 2$.



$$M(XY) = M(X)M(Y)$$



	I	R	T	RT	M	RM	TM	RTM
bb	\boxed{bb}	qq	\boxed{bb}	qq	pp	dd	pp	dd
bq	\boxed{bq}	\boxed{bq}	qb	qb	pd	pd	dp	dp
qb	\boxed{qb}	\boxed{qb}	bq	bq	dp	dp	pd	pd
qq	\boxed{qq}	bb	\boxed{qq}	bb	dd	pp	dd	pp
bp	\boxed{bp}	dq	pb	qd	pb	qd	\boxed{bp}	dq
bd	\boxed{bd}	pq	db	qp	pq	\boxed{bd}	qp	db
qp	\boxed{qp}	db	pq	bd	db	\boxed{qp}	bd	pq
qd	\boxed{qd}	pb	dq	bp	dq	bp	\boxed{qd}	pb
pb	\boxed{pb}	qd	bp	dq	bp	dq	\boxed{pb}	qd
pq	\boxed{pq}	bd	qp	db	bd	\boxed{pq}	db	qp
db	\boxed{db}	qp	bd	pq	qp	\boxed{dq}	pq	bd
dq	\boxed{dq}	bp	qd	pb	qd	pb	\boxed{dq}	bp
pp	\boxed{pp}	dd	\boxed{pp}	dd	bb	qq	bb	qq
pd	\boxed{pd}	\boxed{pd}	dp	dp	bq	bq	qb	qb
dp	\boxed{dp}	\boxed{dp}	pd	pd	qb	qb	bq	bq
dd	\boxed{dd}	pp	\boxed{dd}	pp	qq	bb	qq	bb

Without reflections - first 4 columns: $24/4 = 6$.

With reflections - $32/8 = 4$

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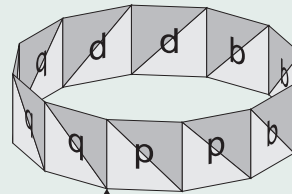
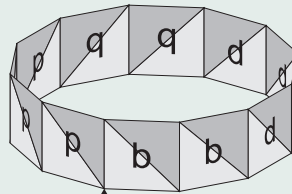
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8. The 1×12 case



The signature $w = bddbbppqqpp$ on a ring, and $T^4M(w)$, and their pattern. $T^6M(w) = w$, so $w \in \text{fix}(T^6M)$.

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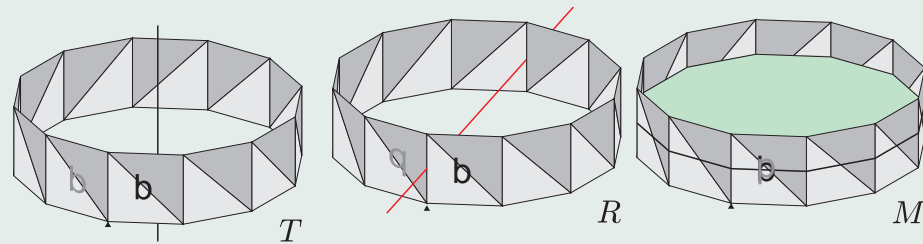
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$$T(a_1 a_2 \dots a_{12}) = a_2 a_3 \dots a_{12} a_1.$$

$$R(a_1 a_2 \dots a_{11} a_{12}) = R(a_{12}) R(a_{11}) \dots R(a_2) R(a_1).$$



$$\langle P \rangle = \langle T, R \mid R^2 = T^{12}, RTR = T^{-1} \rangle$$

$$M(a_1 \dots a_{12}) = M(a_1) \dots M(a_{12}),$$

$$\langle T, R, M \rangle = \langle T, R \rangle \oplus \langle M \rangle$$

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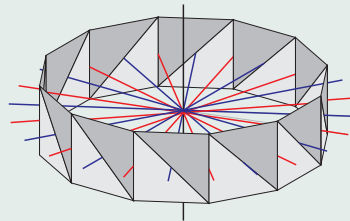
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12 rotations of 180° around axis in the horizontal plane through the center of the ring.

6 have axes passing through the centers of two motifs

6 have axes on the midpoints of motif boundaries, with the motifs being divided into 6 pairs of orbits.

$6 \cdot 4^6$ fixed signatures.

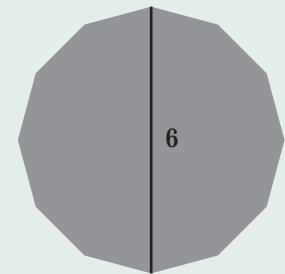
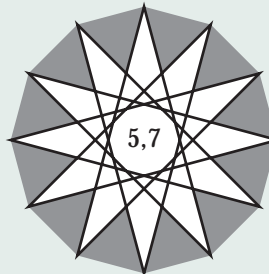
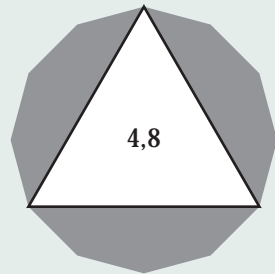
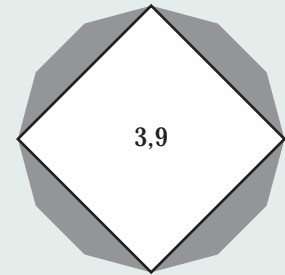
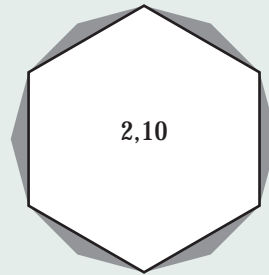
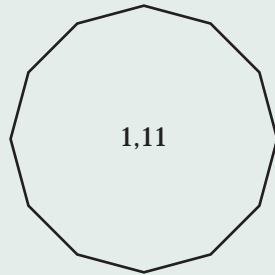


12 rotations about the vertical axis of $\frac{i}{12}360^\circ = i \cdot 30^\circ$, $i = 1 \dots 12$.

i and 12 have a common divisor: k ,

$$i = pk, \quad 12 = qk$$

$q \cdot (i \cdot 30^\circ)$ is a multiple of 360°



$|\text{orbit}(i \cdot 30^\circ)| = 12 / \gcd(i, 12)$ – There are $\gcd(i, 12)$ of them.

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For each divisor k of 12 there are rotations with orbit size k .

Each of these will have $4^{12/k}$ fixed signatures, since we are free to choose any of the four aspects for each orbit.

Twelve has divisors 12, 6, 4, 3, and 2.

$$12 : i = 1, 5, 7, 11,$$

$$6 : i = 2, 10 = 1 \cdot \frac{12}{6}, 5 \cdot \frac{12}{6},$$

$$4 : i = 3, 9 = 1 \cdot \frac{12}{4}, 3 \cdot \frac{12}{4},$$

$$3 : i = 4, 8 = 1 \cdot \frac{12}{3}, 2 \cdot \frac{12}{3},$$

$$2 : i = 6 = 1 \cdot \frac{12}{2}$$

$$\begin{aligned} &\varphi(12) \cdot 4^{12/12} + \varphi(6) \cdot 4^{12/6} + \varphi(4) \cdot 4^{12/4} + \varphi(3) \cdot 4^{12/3} \\ &+ \varphi(2) \cdot 4^{12/2} + \varphi(1) \cdot 4^{12/1} \end{aligned}$$

fixed signatures.

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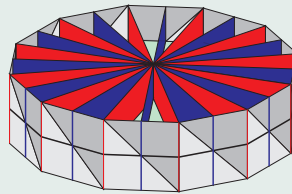
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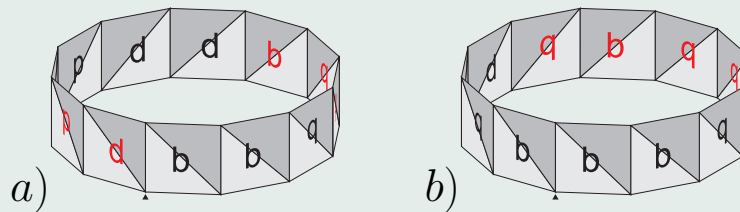
twelve reflections in vertical mirrors,
6 of which pass through the center of a motif,
6 of which pass through the boundaries of the aspects,
 $6 \cdot 4^6$ fixed signatures

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Rotary Reflections



$$\varphi(12) \cdot 4^{12/12} + \varphi(6) \cdot 4^{12/6} + \varphi(4) \cdot 4^{12/4} + \varphi(2) \cdot 4^{12/2}$$

fixed signatures

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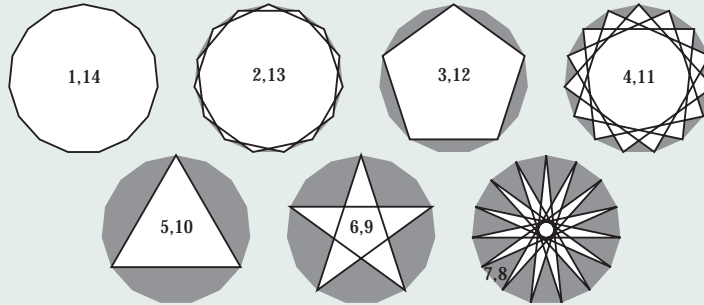
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9. The 1 x 15 case



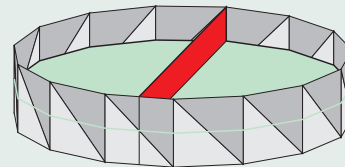
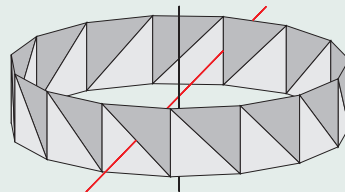
Rotations:

$$\varphi(15) \cdot 4^{15/15} + \varphi(5) \cdot 4^{15/5} + \varphi(3) \cdot 4^{15/3} + \varphi(1) \cdot 4^{15/1}$$

Rotary Reflections: 0

Vertical Mirror Reflections: 0

Horizontal Axis Rotations: 0



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10. The $1 \times n$ case

In the general case, the group $\langle T, R, M \rangle$ has elements:

- Vertical axis rotations T^i
- Horizontal axis rotations $T^i R$
- Vertical reflections $T^i R M$
- Rotary reflections $T^i M$.

and acts on a signature $w = Q_1 Q_2 \cdots Q_n$, $Q_i \in \{b, q, d, p\}$
via

$$\begin{aligned} \text{Translation: } T(Q_1 Q_2 \cdots Q_n) &= Q_2 Q_3 \cdots Q_n Q_1. \\ \text{Rotation: } R(Q_1 Q_2 \cdots Q_n) &= R(Q_n) \cdots R(Q_2) R(Q_1). \\ \text{Mirror: } M(Q_1 Q_2 \cdots Q_n) &= M(Q_1) M(Q_2) \cdots M(Q_n). \end{aligned}$$

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vertical axis rotations:

$$2 \text{ aspects} - p(n) = \sum_{k|n} \varphi(k) 2^{n/k}$$

$$4 \text{ aspects} - P(n) = \sum_{k|n} \varphi(k) 4^{n/k}$$

rotary reflections

$$4 \text{ aspects} - R(n) = \sum_{k|n, 2|k} \varphi(k) 4^{n/k}$$

horizontal axis rotations/vertical mirror reflections:

$$2 \text{ aspects} q(n) = \begin{cases} (n/2) 2^{n/2} & \text{for } n \text{ even and} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$4 \text{ aspects} Q(n) = \begin{cases} (n/2) r^{n/2} & \text{for } n \text{ even and} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

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Two aspects and rotational symmetry

$$f(n) = \frac{p(n) + q(n)}{2n}$$

Four aspects and rotational symmetry

$$F(n) = \frac{P(n) + Q(n)}{2n}$$

Four aspects and general symmetry

$$G(n) = \frac{P(n) + 2Q(n) + R(n)}{4n}$$

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n	2 motifs $f(n)$	4 motifs, oriented $F(n)$	4 motifs, unoriented $G(n)$
1	1	2	1
2	2	6	4
3	2	12	6
4	4	39	23
5	4	104	52
6	9	366	194
7	10	1172	586
8	22	4179	2131
9	30	14572	7286
10	62	52740	26524
11	94	190652	95326
12	192	700274	350738
13	316	2581112	1290556
14	623	9591666	4798174
15	1096	35791472	17895736
16	2122	134236179	67127315
17	3856	505290272	252645136
18	7429	1908947406	954510114
19	13798	7233629132	3616814566
20	26500	27488079132	13744183772

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11. The 1×4 case: Escher revisited

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- 1 *bbbb*
- 2 *bbbq*
- 3 *bbbp*
- 4 *bbbd*
- 5 *bbqq*
- 6 *bbqp*
- 7 *bbqd*
- 8 *bbpq*
- 9 *bbpp*
- 10 *bbpd*

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- 11 *bbdq*
- 12 *bbdp*
- 13 *bbdd*
- 14 *bqbbq*
- 15 *bqbpq*
- 16 *bqbd*
- 17 *bqpd*
- 18 *bqdp*
- 19 *bpbp*
- 20 *bdbp*
- 21 *bdbd*
- 22 *bdqp*
- 23 *bdpq*



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1. $f(n)$ – A053656 in Sloan's On-Line Encyclopedia of integer sequences
2. $G(n) \approx 2F(n)$
3. $G(n) \approx 4^n/(4n)$ ($G(p) = \lceil 4^n/(4n) \rceil$ for $n = p$ prime)
4. Symmetric motifs
5. Over/Under weave motifs
6. Multiple motifs

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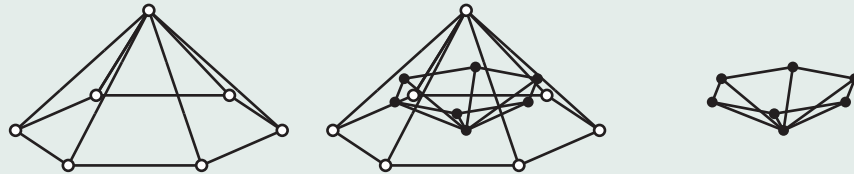
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12. Self-Duality



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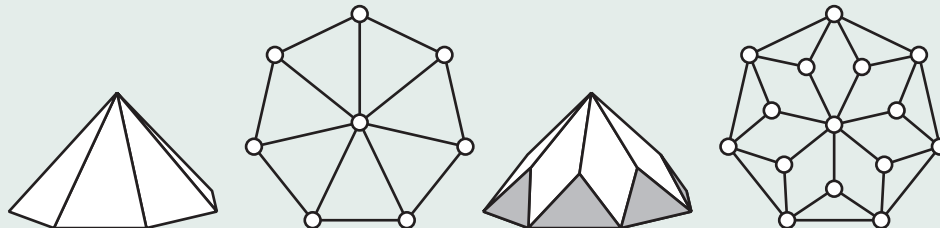
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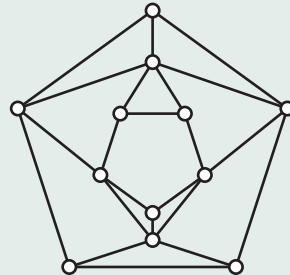
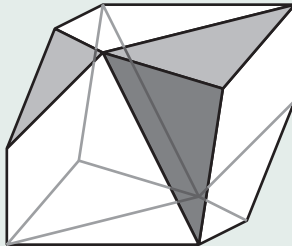


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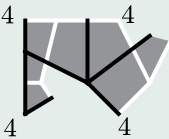
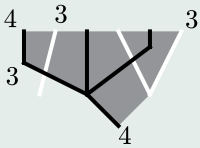
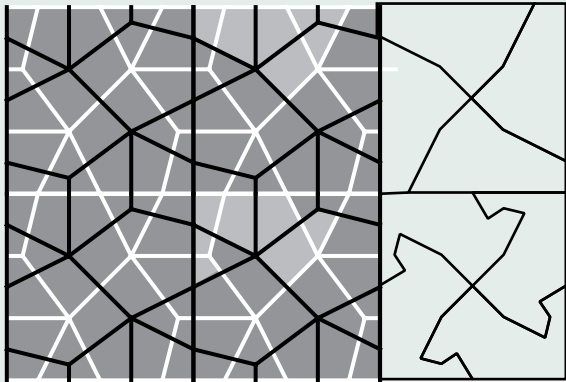
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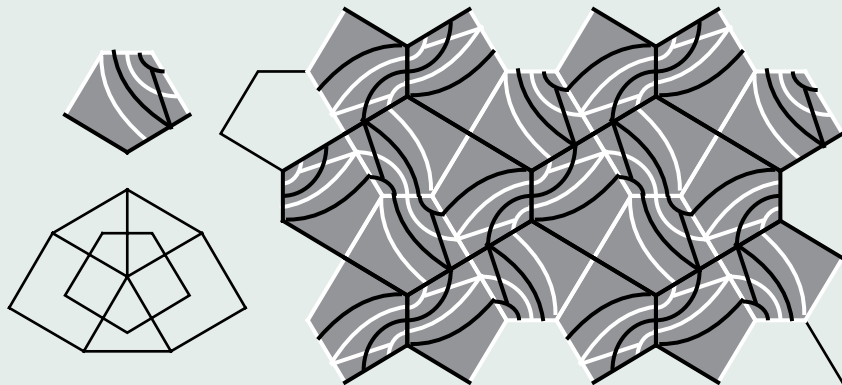
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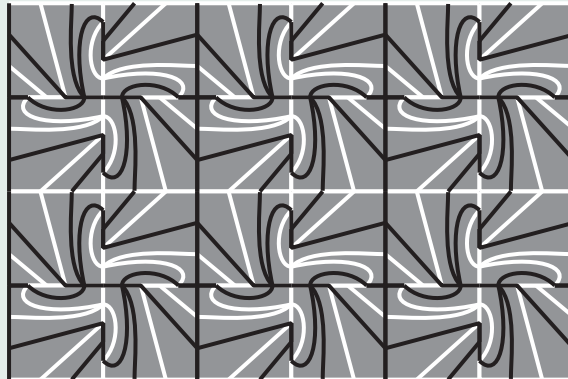
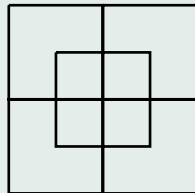
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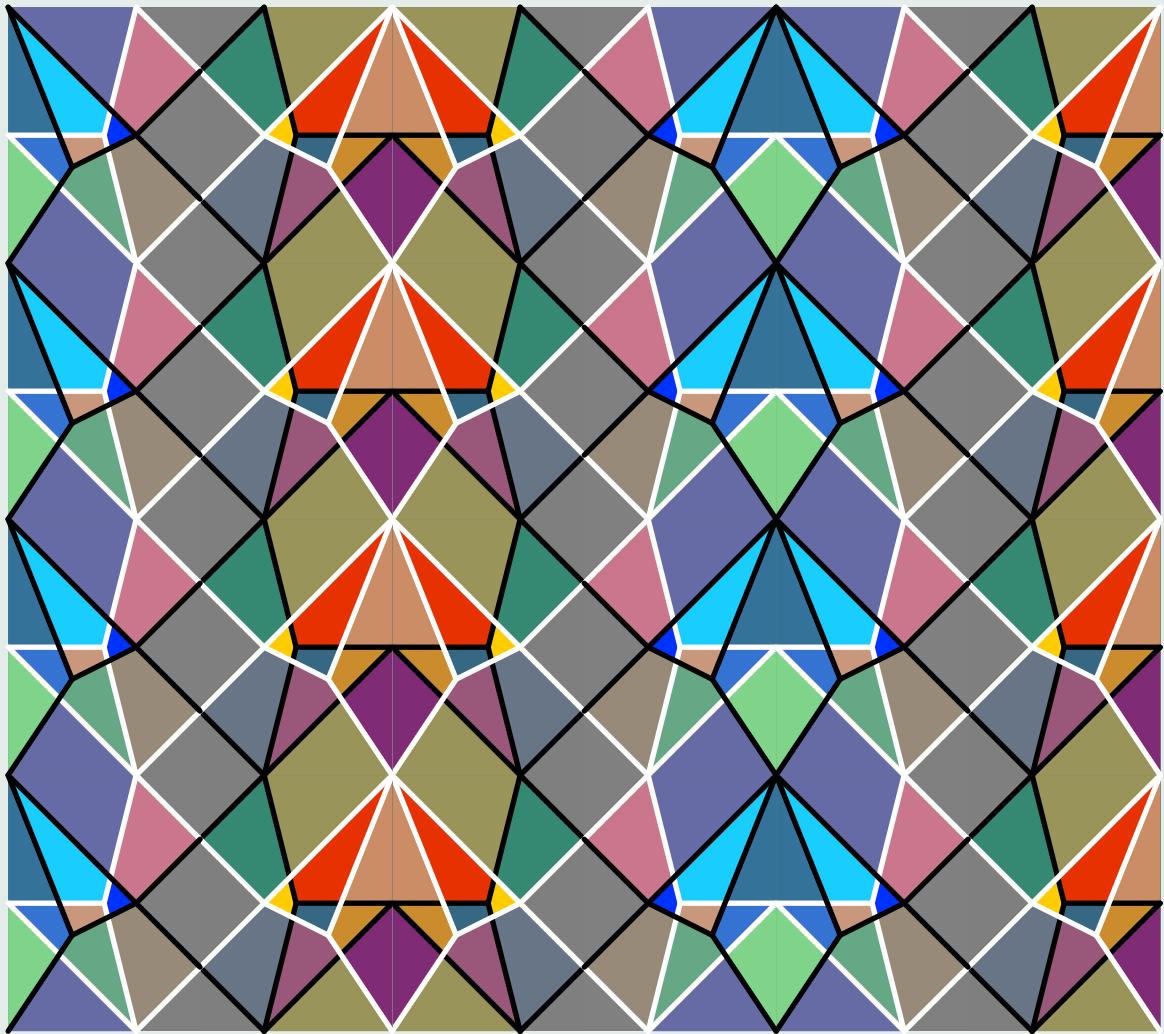
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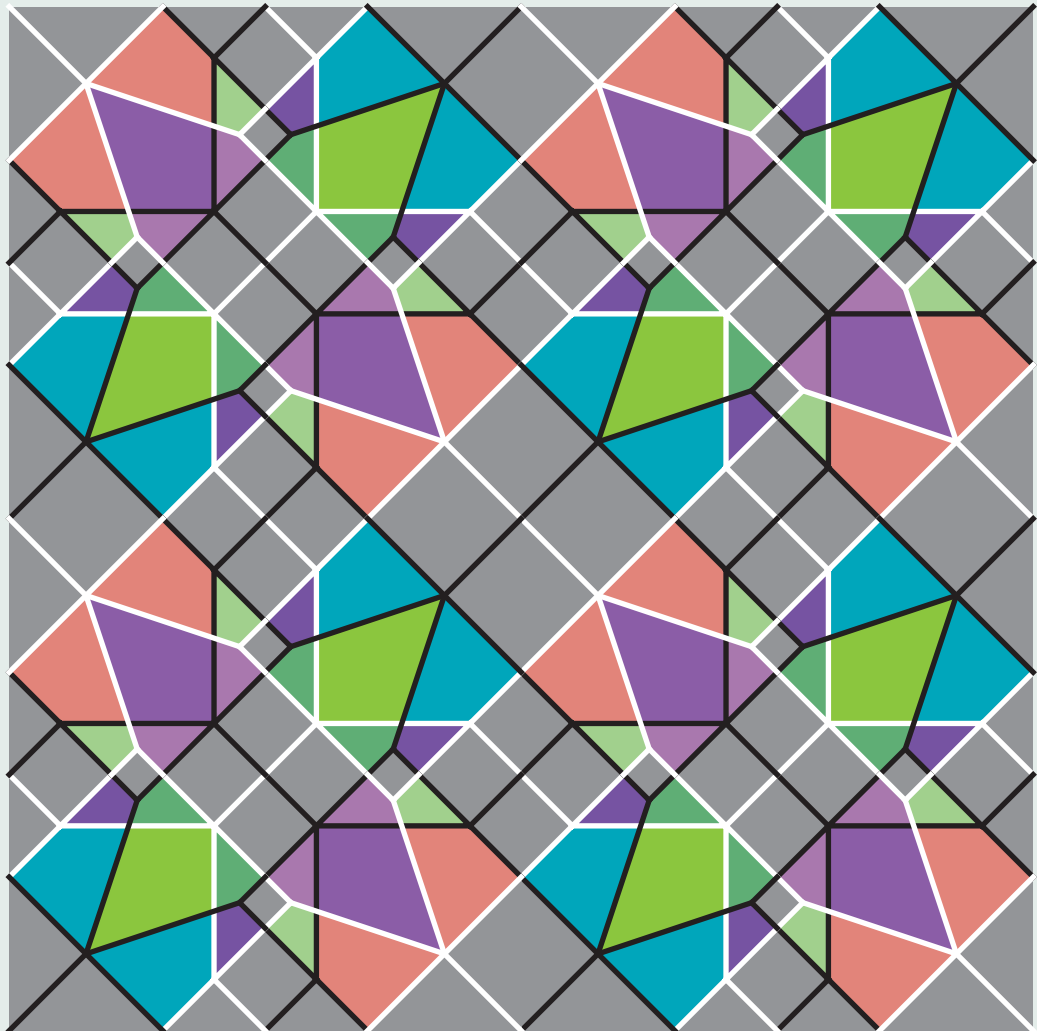
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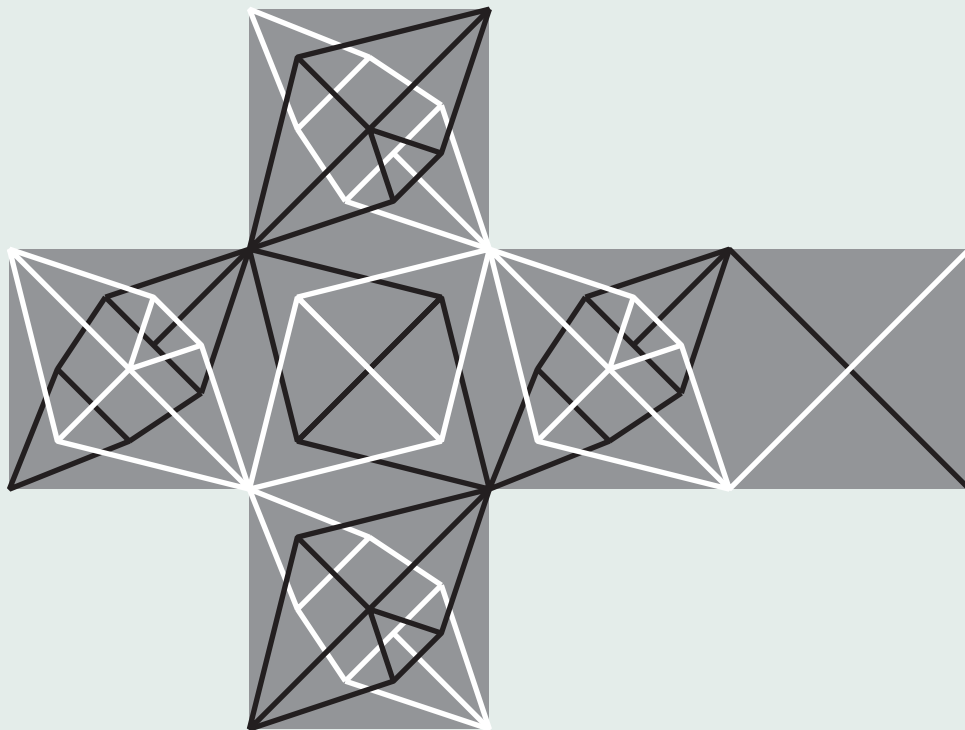
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13. Maps are not enough



A self-dual graph with no corresponding self-dual map.

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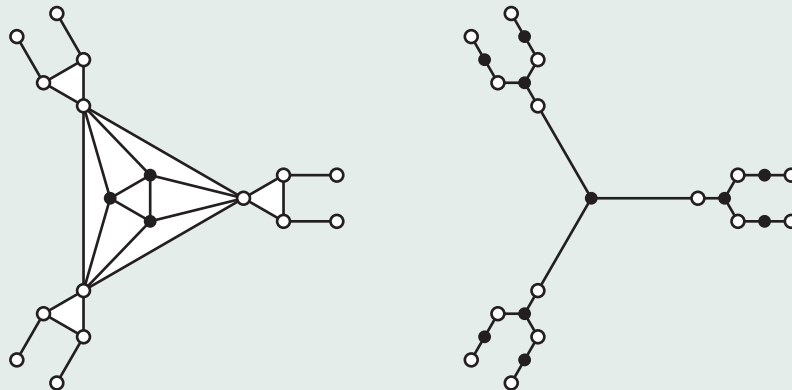
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14. The Block-Cutpoint Tree



For any graph G , the cycle matroid of G is the direct sum over the cycle matroids of the blocks of G :

$$M(G) = \sum M(G_i)$$

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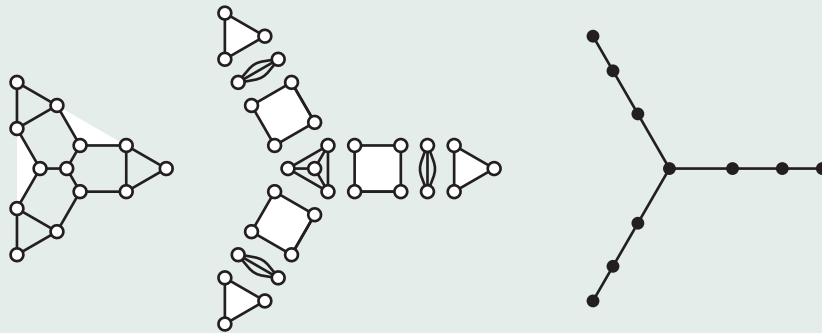
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15. The 3-Block Tree



For a 2-connected graph G , the cycle matroid of G is the 2-sum over the cycle matroids of the 3-blocks of G :

$$M(G) = M(G_0) \bigoplus_{e_1} M(G_1) \dots \bigoplus_{e_k} M(G_k)$$

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16. The Program

Program: Given a planar graph with whose automorphism group has finitely many vertex orbits.

1. If 3-connected – embed and straighten so that the automorphisms are represented by isometries (euclidean, spherical, or hyperbolic).
Apply geometric methods.
2. Else if two-connected – form the 3-block tree and use the program on each block, and merge with data on the automorphisms of the tree.
3. Else if connected – from the block-cutpoint tree and apply the program to each block, merging with the tree automorphisms.
4. Else apply program to each connected component, and merge with permutations of isomorphic components.

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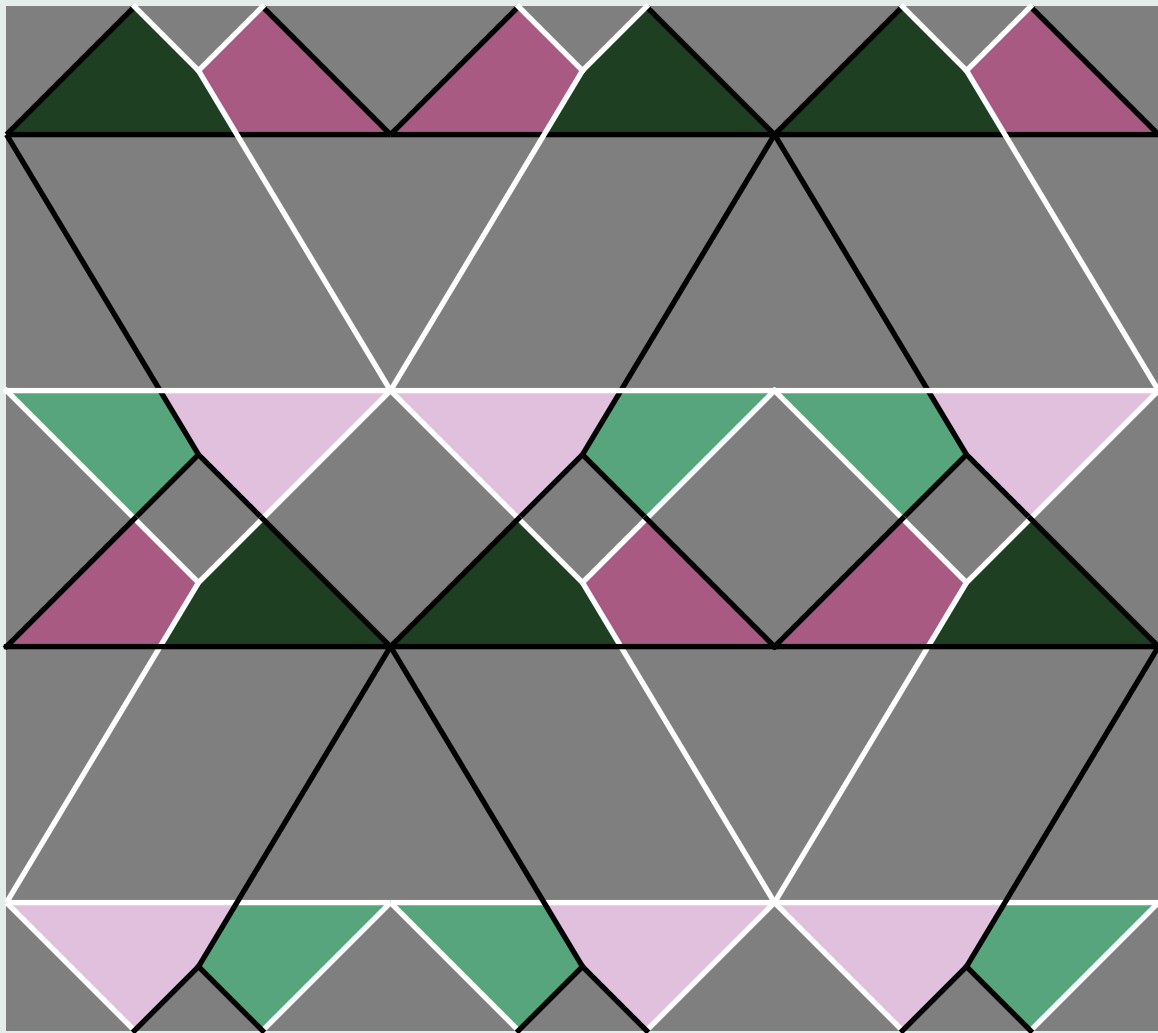
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